Permutations without long decreasing subsequences and random matrices

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Abstract

We study the shape of the Young diagram λ associated via the Robinson–Schensted–Knuth algorithm to a random permutation in S_n such that the length of the longest decreasing subsequence is not bigger than a fixed number d; in other words we study the restriction of the Plancherel measure to Young diagrams with at most d rows. We prove that in the limit $n \to \infty$ the rows of λ behave like the eigenvalues of a certain random matrix (namely the traceless Gaussian Unitary Ensemble random matrix) with d rows and columns. In particular, the length of the longest increasing subsequence of such a random permutation behaves asymptotically like the largest eigenvalue of the corresponding random matrix.

1 Introduction

1.1 Formulation of the problem

Let an integer $d \geq 1$ be fixed. For any integer $n \geq 1$ we consider the set of the permutations $\pi \in S_n$ such that the length of the longest decreasing subsequence of π is not bigger than d; in other words it is the set of the permutations avoiding the pattern $(d+1, d, \dots, 3, 2, 1)$. Let π_n be a random element of this set (probabilities of all elements are equal). In this article we are interested in the following problem:

Problem 1. *Let $\pi_n \in S_n$ be a random permutation with the longest decreasing subsequence of length at most d. What can we say about the asymptotic behavior of the length of the longest increasing subsequence of π_n in the limit $n \to \infty$?*

Let $\lambda_n = (\lambda_{n,1}, \dots, \lambda_{n,d})$ be the (random) Young diagram associated via the Robinson–Schensted–Knuth algorithm to π_n (notice that since the number of the rows of λ_n is equal to the length of the longest decreasing subsequence of π_n, λ_n has at most d rows). In other words, λ_n is a random Young diagram with at most d rows, where the probability of the Young diagram λ is proportional to $(\dim \rho_\lambda)^2$, where $\dim \rho_\lambda$ denotes the dimension of the corresponding irreducible representation of S_n; therefore, if we drop the restriction on the number of the rows of the Young diagrams (which can be alternatively stated as $d \geq n$), then the distribution of λ_n is the celebrated Plancherel measure.

Since $\lambda_{n,1}$ is equal to the length of the longest increasing subsequence in π_n, Problem 1 is a special case of the following more general one:

Problem 2. *What can we say about the asymptotic behavior of the random variables $(\lambda_{n,1}, \dots, \lambda_{n,d})$ in the limit $n \to \infty$?*

1.2 Case $d = 2$

The first non-trivial case $d = 2$ was considered by Deutsch, Hildebrand and Wilf [DHW03]. In this case the random variables $\lambda_{n,1}, \lambda_{n,2}$ are subject to a constraint $\lambda_{n,1} + \lambda_{n,2} = n$ therefore it is enough to study the distribution of $\lambda_{n,1}$. Deutsch, Hildebrand and Wilf proved that the distribution of $\sqrt{\frac{8}{n}}\,(\lambda_{n,1} - \frac{n}{2})$ converges to the distribution of the length of a random Gaussian vector in $\mathbb{R}^3$; in other words $\frac{8}{n}\left(\lambda_{n,1} - \frac{n}{2}\right)^2$ converges to the χ_3^2 distribution with 3 degrees of freedom (a careful reader may notice that the authors of [DHW03] use a non-standard definition of the χ^2 distributions and therefore they claim that $\sqrt{\frac{8}{n}}\,(\lambda_{n,1} - \frac{n}{2})$ itself converges to χ_3^2). Their proof was based on an explicit calculation of the number of the permutations which correspond to a prescribed Young diagram with at most two rows.

1.3 Case $d = \infty$

Another extreme of this problem is to consider $d = \infty$; in other words, not to impose any restrictions on the random permutations π_n. In this case the random Young diagram λ_n is distributed according to the Plancherel measure. The authors of [BDJ99, Oko00, Joh01] proved that the joint distribution of longest rows of λ (after appropriate rescaling) converges to the same distribution (called Tracy–Widom distribution) as the joint distribution of the biggest eigenvalues of a large random matrix from the Gaussian Unitary Ensemble.

1.4 The main result: intermediate values of d

We equip the vector space of $d \times d$ Hermitian matrices with a Gaussian probability measure with a density

$$\frac{1}{Z_d} e^{-\frac{1}{2}\operatorname{Tr} H^2}$$

with respect to the Lebesgue measure, where Z_d is the normalizing constant. We say that a random matrix $(A_{ij})_{1\leq i,j\leq d}$ distributed accordingly to this measure is a Gaussian Unitary Ensemble (GUE) random matrix.

We call $B = A - \frac{1}{d}\operatorname{Tr} A$ a *traceless Gaussian Unitary Ensemble (GUE_0) random matrix*; it corresponds to the Gaussian probability measure on the set of $d \times d$ Hermitian matrices with trace zero and the density

$$\frac{1}{Z'_d} e^{-\frac{1}{2}\operatorname{Tr} H^2}$$

with respect to the Lebesgue measure, where Z'_d is the normalizing constant.

The joint distribution of eigenvalues for GUE is well-known [Meh91], which allows us to find the corresponding distribution for GUE_0; namely, if $x_1 \geq \cdots \geq x_d$ are the eigenvalues of a GUE_0 random matrix, then their joint distribution is supported on the hyperplane $x_1 + \cdots + x_d = 0$ with the density

$$\frac{1}{C_d} e^{-\frac{x_1^2+\cdots+x_d^2}{2}} \prod_{i<j} (x_i - x_j)^2 \tag{1}$$

with respect to the Lebesgue measure, where C_d is the normalization constant.

Theorem 3 (Main theorem). *Let the integer $d \geq 1$ be fixed; for each $n \geq 1$ let $\lambda_n = (\lambda_{n,1}, \dots, \lambda_{n,d})$ be, as in Section 1.1, a random Young diagram with n boxes and with at most d rows.*

Then the joint distribution of the random variables $\left(\sqrt{\frac{2d}{n}}\,(\lambda_{n,i} - \frac{n}{d})\right)_{1\leq i\leq d}$ converges, as $n \to \infty$, to the joint distribution of the eigenvalues of a GUE_0 random matrix.

We postpone its proof to Section 2.

Corollary 4. *Let $d \geq 1$ be fixed, and for each $n \geq 1$ let $\pi_n \in S_n$ be a random permutation with the longest decreasing subsequence of length at most d. We denote by $\lambda_{n,1}$ the length of its longest increasing subsequence. Then the distribution of $\sqrt{\frac{2d}{n}}\,(\lambda_{n,1} - \frac{n}{d})$ converges to the distribution of the largest eigenvalue of the GUE_0 random matrix.*

It should be pointed out that the distibution of eigenvalues of a GUE_0 random matrix appears also in a related asymptotic problem [Joh01] of the distribution of the rows of a Young diagram associated (via RSK algorithm) to a random word consisting of n letters in an alphabet of d symbols in the limit of $n \to \infty$.

1.5 Case $d = 2$ revisited

The set of 2×2 Hermitian matrices with trace zero can be viewed as a 3-dimensional Euclidean space with a scalar product $\langle A, B\rangle = \operatorname{Tr} AB$. A GUE_0 random matrix can be viewed under this correspondence as a Gaussian random vector in $\mathbb{R}^3$ the coordinates of which are independent with mean zero and variance 1. Each 2×2 Hermitian traceless matrix A has two eigenvalues $x_1 = \lambda$, $x_2 = -\lambda$, where $\lambda = \frac{1}{\sqrt{2}}\|A\| = \sqrt{\frac{\langle A,A\rangle}{2}}$. Therefore, for a GUE_0 random matrix the corresponding random variable $2(x_1)^2$ is distributed like the square of the length of a standard Gaussian random vector in $\mathbb{R}^3$, which coincides with the χ_3^2 distribution; thus for $d = 2$ Corollary 4 allows us to recover the result of Deutsch, Hildebrand and Wilf [DHW03].

1.6 Idea of the proof

In Section 2 we will prove Theorem 3, the main result of this article. Our proof will be based on an explicit calculation of the number of standard Young tableaux with a prescribed shape. The standard method to do this would be to use the hook-length formula, which would be not convenient for our purposes. Instead, we will use the determinantal formula of Frobenius and MacMahon. In order to make the connection to random matrices more explicit we shall recall its proof due to Zeilberger [Zei83] which is based on the observation that a Young tableaux with at most d rows can be viewed as a certain trajectory of d non-colliding particles on a line. Thus we will find explicitly the asymptotic joint distribution of the rows of a Young diagram; this distribution turns out to coincide with the distribution (1) of the eigenvalues of a GUE_0 random matrix.

The reader may wonder if the connection between Young diagrams and random matrices given by Theorem 3 might be purely accidental. In the following paragraph we will argue why it is not the case and how deep connections between Young diagrams and random matrices may be seen in our proof of Theorem 3.

In the above discussion we treated the distribution (1) of the eigenvalues of a GUE_0 random matrix as granted; now let us think for a moment about its derivation. GUE_0 is a Gaussian matrix; for this reason (up to a simple scaling factor) it can be viewed as a value at some fixed time of a matrix-valued Brownian bridge. It is known [Dys62, Gra99] that the eigenvalues of a matrix-valued Brownion motion behave like Brownian motions conditioned not to collide. Since a matrix-valued Brownian bridge is a matrix-valued Brownian motion conditioned to be zero at time 1, it follows that its eigenvalues form Brownian motions conditioned not to collide and to be zero at time 1; in other words these eigenvalues form Brownian bridges conditioned not to collide. In this way the determinantal formula of Karlin and McGregor [KM59] can be applied. In the conditioning procedure we assume that the original positions of d non-colliding particles are all different and we consider the limit as these initial positions converge to zero; in this way their final distribution is given by a continuous analogue of the formulas (10) and (8) which give the square of the number of Young tableaux of a given shape, with the transition probabilities replaced by the Gaussian kernels. One can easily check that such a derivation of the distribution of eigenvalues of a GUE_0 random matrix follows (8) very closely.

To summarize: our proof of the main result will be based on the observation that both Young tableaux and the eigenvalues of matrix-valued Brownian motions can be interpreted as non-colliding particles and applying the determinantal formula of Karlin and McGregor [KM59].

1.7 Final remarks

We can see that both the case when d is finite and the case considered in Section 1.3 corresponding to $d = \infty$ are asymptotically described by GUE random matrices. It would be very interesting to find a direct link between these two cases.

2 Proof of the main result

For a function $f : \mathbb{R} \to \mathbb{R}$ we define its difference $\Delta_n f : \mathbb{R} \to \mathbb{R}$ by

$$\Delta_n f(y) = \frac{f\left(y + \sqrt{\frac{d}{n}}\right) - f(y)}{\sqrt{\frac{d}{n}}}.$$

By iterating we define $\Delta_n^\alpha f$ for any integer $\alpha \geq 0$. We also define its shift $S_n f : \mathbb{R} \to \mathbb{R}$ by

$$S_n f(y) = f\left(y + \sqrt{\frac{d}{n}}\right).$$

Notice that $S_n^\alpha f$ is well-defined for any integer α.

Lemma 5. *For each n we define a function $f_n : \mathbb{R} \to \mathbb{R}$ which is constant on each interval of the form $\left[\frac{k-\frac{n}{d}}{\sqrt{\frac{n}{d}}}, \frac{k+1-\frac{n}{d}}{\sqrt{\frac{n}{d}}}\right)$ for each integer k and such that*

$$f_n\left(\frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}\right) = \begin{cases} \sqrt{\frac{n}{d}} \frac{\left(\frac{n}{d}\right)^k e^{-\frac{n}{d}}}{k!} & \text{if } k \text{ is a non-negative integer,} \\ 0 & \text{if } k \text{ is a negative integer.} \end{cases} \tag{2}$$

Then for each integer $\alpha \geq 0$ and $y \in \mathbb{R}$

$$\lim_{n\to\infty} \Delta_n^\alpha f_n(y) = \frac{d^\alpha}{dy^\alpha} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}. \tag{3}$$

Furthermore, for each $\alpha \geq 0$ there exists a polynomial P_α such that

$$\left|\Delta_n^\alpha f_n(y)\right| < P_\alpha(y) e^{-|y|} \tag{4}$$

holds true for all n and y.

Proof. Before presenting the proof we notice that f_n is a density of a probability measure arising as follows: we normalize the Poisson distribution with the parameter $\nu = \frac{n}{d}$ in order to have mean 0 and variance 1 and we convolve it with a uniform distribution on the interval $\left[0, \sqrt{\frac{d}{n}}\right]$; therefore (3) states for $\alpha = 0$ that the suitably rescaled probabilities of the Poisson distribution converge to the density of the normal distribution. The case $\alpha \geq 1$ shows that this convergence holds true also for differences (respectively, derivatives).

The proof of (3) in the case $\alpha = 0$ is a straightforward application of the Stirling approximation $\log z! = \left(z + \frac{1}{2}\right) \log z - z + \frac{\log 2\pi}{2} + O(z^{-1})$, namely for $y = \frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}$ such that k is an integer we denote $c = \frac{n}{d}$. Then

$$\log f_n(y) = \left(c + y\sqrt{c} + \frac{1}{2}\right) \log c - c - \log \left(c + y\sqrt{c}\right)! =$$

$$-\left(c + y\sqrt{c} + \frac{1}{2}\right) \log \left(1 + \frac{y}{\sqrt{c}}\right) + y\sqrt{c} - \frac{\log 2\pi}{2} + O\left(c^{-1}\right) = -\frac{y^2}{2} - \frac{\log 2\pi}{2} + O\left(c^{-\frac{1}{2}}\right),$$

where the above equalities hold true asymptotically for y bounded and $c \to \infty$.

In order to treat the case $\alpha \geq 1$ we observe that the iterated derivative on the right-hand side of (3) can be calculated by using the following three rules:

$$\frac{d}{dy} e^{\frac{-y^2}{2}} = -y e^{\frac{-y^2}{2}}; \qquad \frac{d}{dy} y = 1; \qquad \frac{d}{dy}(\phi\psi) = \left(\frac{d}{dy}\phi\right)\psi + \phi \frac{d}{dy}\psi.$$

Similarly, the iterated difference on the left-hand side of (3) can be calculated using the following three rules:

$$\Delta_n f_n = -g_n S f_n; \qquad \Delta_n g_n = 1; \qquad \Delta_n(ab) = (\Delta_n a)b + (S_n a)\Delta_n b$$

where $g_n : \mathbb{R} \to \mathbb{R}$ is a function which is constant on each interval of the form $\left[\frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}, \frac{k+1-\frac{n}{d}}{\sqrt{\frac{n}{d}}}\right)$ for each integer k and such that

$$g_n\left(\frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}\right) = \frac{k + 1 - \frac{n}{d}}{\sqrt{\frac{n}{d}}}.$$

For each integer β we have $\lim_{n\to\infty}(S_n^\beta f_n)(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ and $\lim_{n\to\infty}(S_n^\beta g_n)(y) = y$ therefore each term contributing to the left-hand side of (3) converges to its counterpart on the right-hand side of (3), which finishes the proof of (3).

We consider $y = \frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}$; then

$$\frac{\log f_n\left(\frac{k - \frac{n}{d}}{\sqrt{\frac{n}{d}}}\right) - \log f_n\left(\frac{k-1-\frac{n}{d}}{\sqrt{\frac{n}{d}}}\right)}{\sqrt{\frac{n}{d}}} = -\log\left(1 + \frac{y}{\sqrt{\frac{n}{d}}}\right)\sqrt{\frac{n}{d}}. \tag{5}$$

There is a constant $C_1 < 0$ with a property that if $y < C_1$ then the right-hand side of (5) is greater than 1 for any value of n. It follows that if $y_i = \frac{k_i - \frac{n}{d}}{\sqrt{\frac{n}{d}}}$ for $i \in \{1, 2\}$ and $y_1 < y_2 \leq C_1$ then

$$f_n(y_1) \leq f_n(y_2) e^{y_1 - y_2}. \tag{6}$$

Similarly we find a constant $C_2 > 0$ with a property that if $C_2 \leq y_1 < y_2$ then

$$f_n(y_2) \leq f_n(y_1) e^{y_1 - y_2}. \tag{7}$$

For $\alpha = 0$ inequality (4) holds true for y in a small neighborhood of the interval $[C_1, C_2]$ for P_α being a sufficiently big constant which follows from (3) and compactness argument. Inequality (4) holds true outside of the interval $[C_1, C_2]$ by inequalities (6) and (7).

The case $\alpha \geq 1$ can be proved in an analogous way to the above proof of (3): we show that $\Delta_n^\alpha f_n$ is a sum of the terms of the form $(S_n^{\beta_1} g_n) \cdots (S_n^{\beta_l} g_n)(S_n^\beta f_n)$ and the absolute value of each such a term can be easily bounded by $P(y)e^{-|y|}$, where P is a suitably chosen polynomial. □

Proof of Theorem 3. The following discussion is based on the work of Zeilberger [Zei83]. Every Young tableau T with at most d rows and n boxes can be interpreted as a trajectory of d non-colliding particles $x_1(t), \dots, x_d(t)$ on the real line as follows. We set

$$x_i(t) = d + 1 - i + (\text{number of boxes of } T \text{ in row } i \text{ which are not bigger than } t).$$

In other words: the initial positions of the particles are given by $\big(x_1(0), \dots, x_d(0)\big) = (d, d-1, \dots, 1)$. In each step one of the particles jumps to the right; the number of the particle which jumps in step t is equal to the number of the row of the Young diagram T which carries the box with a label t. The condition that T is a standard Young tableau is equivalent to $x_1(t) > \cdots > x_d(t)$ for every value of $0 \leq t \leq n$.

Thus the results of Karlin and McGregor [KM59] can be applied and the number of standard Young tableaux of the shape $\lambda_1, \dots, \lambda_d$, where $|\lambda| = \lambda_1 + \cdots + \lambda_d = n$, is equal to the determinant

$$\begin{aligned}
N_{\lambda_1,\dots,\lambda_n} &= n! \begin{vmatrix} \frac{1}{\lambda_1!} & \frac{1}{(\lambda_1+1)!} & \cdots & \frac{1}{(\lambda_1+d-1)!} \\ \frac{1}{(\lambda_2-1)!} & \frac{1}{\lambda_2!} & \cdots & \frac{1}{(\lambda_2+d-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(\lambda_d-d+1)!} & \frac{1}{(\lambda_d-d+2)!} & \cdots & \frac{1}{\lambda_d!} \end{vmatrix} \\
&= \frac{n! e^n}{\left(\frac{n}{d}\right)^{n+\frac{d}{2}}} \begin{vmatrix} f_n(y_1) & S_n f_n(y_1) & \cdots & S_n^{d-1} f_n(y_1) \\ S_n^{-1} f_n(y_2) & f_n(y_2) & \cdots & S_n^{d-2} f_n(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_n^{-d+1} f_n(y_d) & S_n^{-d+2} f_n(y_d) & \cdots & f_n(y_d) \end{vmatrix} \\
&= \frac{n! e^n}{\left(\frac{n}{d}\right)^{n+\frac{d(d+1)}{4}}} \begin{vmatrix} f_n(y_1) & \Delta_n f_n(y_1) & \cdots & \Delta_n^{d-1} f_n(y_1) \\ S_n^{-1} f_n(y_2) & \Delta_n S_n^{-1} f_n(y_2) & \cdots & \Delta_n^{d-1} S_n^{-1} f_n(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_n^{-d+1} f_n(y_d) & \Delta_n S_n^{-d+1} f_n(y_d) & \cdots & \Delta_n^{d-1} S_n^{-d+1} f_n(y_d) \end{vmatrix}, \qquad (8)
\end{aligned}$$

where

$$y_i = \frac{\lambda_i - \frac{n}{d}}{\sqrt{\frac{n}{d}}}. \qquad (9)$$

We are interested in a probability distribution on Young diagrams with n boxes with the probability of $(\lambda_1, \dots, \lambda_d)$ equal to

$$\frac{1}{C_{n,d}} (N_{\lambda_1,\dots,\lambda_d})^2, \qquad (10)$$

where $C_{n,d}$ is the suitably chosen normalizing constant. Clearly,

$$C_{n,d} \frac{\left(\frac{n}{d}\right)^{2n+\frac{d^2+2d-1}{2}}}{(n!)^2 e^{2n}} =$$

$$\sum_{\lambda_1,\dots,\lambda_{d-1}} \left(\sqrt{\frac{n}{d}}\right)^{d-1} \begin{vmatrix} f_n(y_1) & \Delta_n f_n(y_1) & \cdots & \Delta_n^{d-1} f_n(y_1) \\ S_n^{-1} f_n(y_2) & \Delta_n S_n^{-1} f_n(y_2) & \cdots & \Delta_n^{d-1} S_n^{-1} f_n(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_n^{-d+1} f_n(y_d) & \Delta_n S_n^{-d+1} f_n(y_d) & \cdots & \Delta_n^{d-1} S_n^{-d+1} f_n(y_d) \end{vmatrix}^2,$$

where the sum runs over $\lambda_1, \dots, \lambda_{d-1}$ such that for $\lambda_d = n - (\lambda_1 + \cdots + \lambda_{d-1})$ we have that $\lambda_1, \dots, \lambda_d$ is a Young diagram with n boxes. The right-hand side can be viewed as a Riemann sum; Lemma 5 shows that the dominated convergence theorem can be applied (with the dominating function of the form $P(y_1, \dots, y_d) e^{-2(|y_1|+\cdots+|y_d|)}$, where P is some polynomial) and

$$\lim_{n\to\infty} C_{n,d} \frac{\left(\frac{n}{d}\right)^{2n+\frac{d^2+2d-1}{2}}}{(n!)^2 e^{2n}} = \int_{y_1,\dots,y_{d-1}} \begin{vmatrix} e^{-\frac{y_1^2}{2}} & \frac{d}{dy_1} e^{-\frac{y_1^2}{2}} & \cdots & \frac{d^{d-1}}{dy_1^{d-1}} e^{-\frac{y_1^2}{2}} \\ e^{-\frac{y_2^2}{2}} & \frac{d}{dy_2} e^{-\frac{y_2^2}{2}} & \cdots & \frac{d^{d-1}}{dy_2^{d-1}} e^{-\frac{y_2^2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{y_d^2}{2}} & \frac{d}{dy_d} e^{-\frac{y_d^2}{2}} & \cdots & \frac{d^{d-1}}{dy_d^{d-1}} e^{-\frac{y_d^2}{2}} \end{vmatrix}^2 dy_1 \cdots dy_{d-1},$$

where the integral runs over $(y_1, \dots, y_{d-1})$ such that for $y_d = -(y_1 + \cdots + y_{d-1})$ we have $y_1 \geq \cdots \geq y_d$.

Since the limit density defines a probability measure, in the limit $n \to \infty$ the random variables $(y_1, \dots, y_{d-1})$ (please notice that due to the constraint $y_1 + \cdots + y_d = 0$ the value of y_d is uniquely determined by $y_1, \dots, y_{d-1}$) converge in distribution to the probability measure on the set $y_1 \geq y_2 \geq \cdots \geq y_{d-1} \geq -(y_1 + \cdots + y_{d-1})$ with a density

$$\frac{1}{C'_d} \begin{vmatrix} e^{-\frac{y_1^2}{2}} & \frac{d}{dy_1} e^{-\frac{y_1^2}{2}} & \cdots & \frac{d^{d-1}}{dy_1^{d-1}} e^{-\frac{y_1^2}{2}} \\ e^{-\frac{y_2^2}{2}} & \frac{d}{dy_2} e^{-\frac{y_2^2}{2}} & \cdots & \frac{d^{d-1}}{dy_2^{d-1}} e^{-\frac{y_2^2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{y_d^2}{2}} & \frac{d}{dy_d} e^{-\frac{y_d^2}{2}} & \cdots & \frac{d^{d-1}}{dy_d^{d-1}} e^{-\frac{y_d^2}{2}} \end{vmatrix}^2 = \frac{1}{C'_d} \begin{vmatrix} p_0(y_1) e^{-\frac{y_1^2}{2}} & p_1(y_1) e^{-\frac{y_1^2}{2}} & \cdots & p_{d-1}(y_1) e^{-\frac{y_1^2}{2}} \\ p_0(y_2) e^{-\frac{y_2^2}{2}} & p_1(y_2) e^{-\frac{y_2^2}{2}} & \cdots & p_{d-1}(y_2) e^{-\frac{y_2^2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ p_0(y_d) e^{-\frac{y_d^2}{2}} & p_1(y_d) e^{-\frac{y_d^2}{2}} & \cdots & p_{d-1}(y_d) e^{-\frac{y_d^2}{2}} \end{vmatrix}^2$$

for a suitably chosen normalizing constant C'_d, where $\frac{d^k}{dz^k} e^{-\frac{z^2}{2}} = p_k(z) e^{-\frac{z^2}{2}}$ for some polynomial p_k (related to Hermite polynomials). Since $p_k(z) = (-z)^k +$ (summands of lower degree) the above expression takes a simpler form:

$$\frac{1}{C'_d}\begin{vmatrix} e^{-\frac{y_1^2}{2}} & (-y_1)e^{-\frac{y_1^2}{2}} & \cdots & (-y_1)^{d-1}e^{-\frac{y_1^2}{2}} \\ e^{-\frac{y_2^2}{2}} & (-y_2)e^{-\frac{y_2^2}{2}} & \cdots & (-y_2)^{d-1}e^{-\frac{y_2^2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{y_d^2}{2}} & (-y_d)e^{-\frac{y_d^2}{2}} & \cdots & (-y_d)^{d-1}e^{-\frac{y_d^2}{2}} \end{vmatrix}^2 = \frac{1}{C'_d} e^{-(y_1^2+\cdots+y_d^2)} \prod_{1\le i<j\le d} (y_i-y_j)^2.$$

When we set $x_i = \sqrt{2} y_i = \sqrt{\frac{2d}{n}}\,(\lambda_{n,i} - \frac{n}{d})$ it becomes clear that the limit distribution of $(x_1,\dots,x_d)$ coincides with the distribution (1) of the eigenvalues of a GUE_0 random matrix, which finishes the proof. □

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