On the genus distribution of (p, q, n)-dipoles

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Abstract

There are many applications of the enumeration of maps in surfaces to other areas of mathematics and the physical sciences. In particular, in quantum field theory and string theory, there are many examples of occasions where it is necessary to sum over all the Feynman graphs of a certain type. In a recent paper of Constable et al. on pp-wave string interactions, they must sum over a class of Feynman graphs which are equivalent to what we call (p, q, n)-dipoles. In this paper we perform a combinatorial analysis that gives an exact formula for the number of (p, q, n)-dipoles in the torus (genus 1) and double torus (genus 2).

1 Introduction

A *map* is an embedding of a graph in an orientable surface such that the deletion of the edges decomposes the surface into regions homeomorphic to open disks. These regions are the faces of the map. If a map has i vertices, j faces and k edges, then the genus g of the underlying surface is determined by the Euler-Poincaré formula, $2 - 2g = i - k + j$. If $\mathcal{M}$ is a set containing m_g maps of genus g for every $g \geq 0$, then

$$M(u) = \sum_{g \geq 0} m_g u^g$$

is called the *genus series* for $\mathcal{M}$. A map is *rooted* by distinguishing a mutually incident vertex and edge, and the *root edge* is indicated in diagrams by a directed edge whose origin is the *root vertex*. Since our maps are in orientable surfaces, for any vertex v of the

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map, we can specify a cyclic list of the edges incident with v that would be encountered in traversing the boundary of a small disk, centered at v, in the sense of the orientation of the surface. We will assume throughout that this sense is anticlockwise. For example, consider the map in Figure 1 with edges labelled $e_0, \ldots, e_6$. The edges incident with the root vertex occur in the cyclic order $(e_0, e_1, e_2, e_3, e_4, e_5, e_6)$ about the root vertex. The edges incident with the nonroot vertex occur in the cyclic order $(e_0, e_6, e_4, e_5, e_2, e_3, e_1)$ about the nonroot vertex.

To define a (p, q, n)-*dipole*, let $\mathfrak{m}$ be a rooted map with 2 vertices of degree n (with no loops) and one other distinguished edge (indicated by a dashed edge in diagrams). Let E^- be the set of edges occurring after the root edge, but before the dashed edge, in the list of edges ordered about the root vertex (in an anticlockwise sense as described above). Let E^+ be the set of edges occurring after the root edge, but before the dashed edge, in the list of edges ordered about the nonroot vertex. For example, for the map in Figure 1, $E^- = \{e_1, e_2, e_3, e_4\}$ and $E^+ = \{e_6, e_4\}$. If $|E^-| = p - 1$ and $|E^+| = q - 1$, then $\mathfrak{m}$ is a (p, q, n)-*dipole*. Equivalently, we could say that the dashed edge is the pth edge after the root edge when considering the edges ordered about the root vertex, but is the qth edge after the root edge when considering the edges ordered about the nonroot vertex. Figure 1 gives an example of a $(5, 3, 7)$-dipole. Since this map has 3 faces, its genus is $g = (2 + e - v - f)/2 = (2 + 7 - 2 - 3)/2 = 2$.

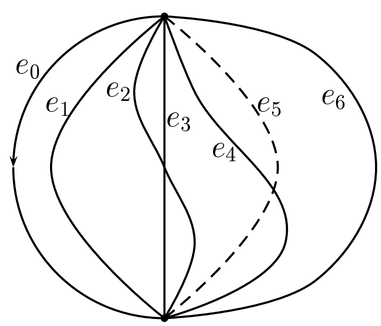

Figure 1: A $(5, 3, 7)$-dipole.

Let $D_{p,q,n}(u)$ be the genus series for (p, q, n)-dipoles. By considering the symmetry between the two vertices and the symmetry between the two distinguished edges, respectively, we obtain the following relationships:

$$D_{p,q,n}(u) = D_{q,p,n}(u), \quad D_{p,q,n}(u) = D_{n-p,n-q,n}(u).$$

So we need only find $D_{p,q,n}(u)$ for $1 \leq p \leq q \leq n - p$.

Let $D_n(u)$ be the genus series for all rooted dipoles (with no other distinguished edge). Since there are $n - 1$ nonroot edges which could be distinguished we observe that

$$D_n(u) = \frac{1}{n-1} \sum_{1 \leq p,q < n} D_{p,q,n}(u).$$

Appendix A lists $D_{p,q,n}(u)$ and $D_n(u)$ for small values of n.

Our interest in computing $D_{p,q,n}(u)$ comes out of an application to string theory. In [1], (p, q, n)-dipoles are equivalent to a set of Feynman graphs used in the computation of a certain two-point correlation function in free Yang-Mills theory. The purpose of their calculation is to draw some connections between Yang-Mills theory and string theory in a pp-wave background. In their paper, the authors obtain asymptotic results for the number of (p, q, n)-dipoles of small genera. Here we will give an exact enumeration of (p, q, n)-dipoles of genus 0, 1 and 2. Results for the sphere and torus are given in Section 2, and for the double torus in Section 3. This section will conclude with a few definitions and some pertinent background about dipoles.

Let $\mathfrak{S}_n$ represent the symmetric group on n symbols. The cycle-type of a permutation $\pi \in \mathfrak{S}_n$ will be described by a *partition* $\theta = [1^{a_1}2^{a_2}\cdots]$ to indicate a permutation with a_i cycles of length i for $i = 1, 2, \ldots$. Since the lengths of all the cycles of π sum to n, θ is a partition of n, and we write $\theta \vdash n$ to denote this. For this reason, the partitions of n index the conjugacy classes of $\mathfrak{S}_n$. If $\mathfrak{m}$ is a map on n edges, we describe the *face-type* of $\mathfrak{m}$ by a partition of $2n$, $[1^{f_1}2^{f_2}\cdots]$, indicating that $\mathfrak{m}$ has f_i faces of degree i for $i = 1, 2, \ldots$.

It is well-known that a map can be encoded by a pair of permutations (ν, ϵ) describing the vertex and edge structure of the map in such a way that the cycle-type of the product $\nu\epsilon$ gives the face-type of the map. This axiomatization for combinatorial maps is described in great detail in [5]. Jackson [3] used the character theory of the symmetric group to count the number of ways of representing a fixed cycle of length n in $\mathfrak{S}_n$ as a product of permutations from specified conjugacy classes. This immediately gave the genus series for monopoles [2] (maps with 1 vertex) and the character theoretic tools developed in [3] led to a new approach to map enumeration that gave results for arbitrary genus [4].

Kwak and Lee [6] (and independently Rieper) used similar methods to obtain the genus series for rooted dipoles. They accomplish this by giving a combinatorial argument that the number of rooted dipoles on n edges with k faces is equal to the number of full cycles $\pi \in \mathfrak{S}_n$ such that $\omega\pi$ has k cycles where ω is a fixed n-cycle. They then appeal to Jackson's result to obtain

$$D_n(u) = \sum_{g\geq 0} \frac{\left[{n+1 \atop n-2g}\right]}{\binom{n+1}{2}}\, u^g,$$

where $\left[{n \atop k}\right]$ is the (unsigned) Stirling number of the first kind and is equal to the number of permutations in $\mathfrak{S}_n$ with k cycles.

Using Kwak and Lee's encoding of a dipole and the character theoretic methods of Jackson, we can give a slightly more detailed formula for rooted dipoles. Let χ^θ denote the irreducible character associated with the conjugacy class of permutations with cycle-type θ and let χ^θ_ϕ denote its value at any element with cycle-type ϕ. Let f^θ denote the degree of χ^θ and h^ϕ be the number of permutations with cycle-type ϕ. Then the number of rooted dipoles with n edges and face-type $2\phi = [2^{a_1}\,4^{a_2}\ldots]$ is

$$\frac{h^{[n]}h^\phi}{n!}\sum_{\theta\vdash n}\frac{1}{f^\theta}\left(\chi^\theta_{[n]}\right)^2\chi^\theta_\phi = \frac{h^\phi}{n}\sum_{k\geq 0}\chi_\phi^{[n-k,1^k]}/\binom{n-1}{k}.$$

In [3], Jackson shows that if $\phi = [1^{a_1} 2^{a_2} \cdots]$, then $\chi_{\phi}^{[n-k,1^k]}$ is the coefficient of x^k in

$$(1+x)^{-1} \prod_{i=1}^{n} (1-(-x)^i)^{a_i},$$

making it very convenient to use the above formula for computation.

2 Results for the sphere and torus

The number of (p, q, n)-dipoles on the sphere is easily seen to be 1 if $p + q = n$ and 0 otherwise. Determining the number of (p, q, n)-dipoles on the torus is more interesting, and still involves considering these two cases separately. We begin with the case $p+q < n$ and then look at $p + q = n$. Of course, $p + q > n$ is determined by symmetry.

Proposition 2.1 *If $p + q < n$, then the number of (p, q, n)-dipoles of genus 1 is*

$$pq(n-p-q) + \sum_{b=1}^{p-1} (p-b) \binom{q-b}{1}.$$

Proof: Genus 1 dipoles have $2 + e - v - 2g = n - 2$ faces. Since for any dipole every face has an even number of edges, this implies that genus 1 dipoles must have face-type $[2^{n-3}\, 6]$ or $[2^{n-4}\, 4^2]$.

Figure 2(a) shows a dipole on n edges with face-type $[2^{n-3}\, 6]$, where the gray region labelled 'a' represents a edges that do not cross each other, the gray region labelled 'b' represents b edges that do not cross each other, and so on. That is, if $a \geq 2$ then the gray region labelled 'a' represents $a-1$ digons, etc. Notice that $a+b+c+d = n-1$, and that $a, d \geq 0$ and $b, c \geq 1$.

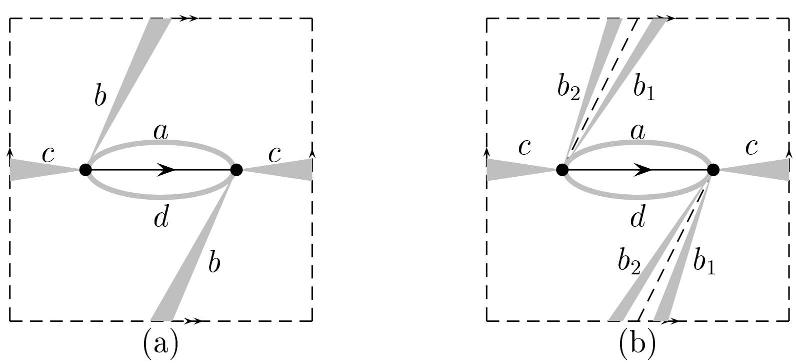

Figure 2: Rooted dipole, (p, q, n)-dipole with face-type $[2^{n-3}\, 6]$.

If the distinguished (dashed) edge of a (p, q, n)-dipole is one of the edges that are represented by the gray region labelled 'a' or 'd', we have $p + q = n$. Also, if the distinguished edge is one of the edges that are represented by the gray region labelled 'c', we have $p+q > n$. Hence we need only look at the case where the distinguished edge is one of the edges represented by the gray region labelled 'b'. Figure 2(b) shows a (p,q,n)-dipole with face-type $[2^{n-3}\,6]$ and the property that $p+q<n$.

Notice that here $a+b_1+b_2+c+d=n-2$, and that $a,b_1,b_2,d\geq 0$ and $c\geq 1$. Since $a+b_1=p-1$ and $d+b_2=q-1$, there are p choices for (a,b_1) and q choices for (b_2,d) with c then being determined. So the number of (p,q,n)-dipoles of this form is pq.

Now let us consider dipoles with face-type $[2^{n-4}\,4^2]$, as shown in Figure 3(a). Here we must have $b,c,d\geq 1$. Furthermore, the only locations for the distinguished edge that produce $p+q<n$ are among the edges represented by the gray regions labelled 'b' and 'c', and the latter only if $b<d$.

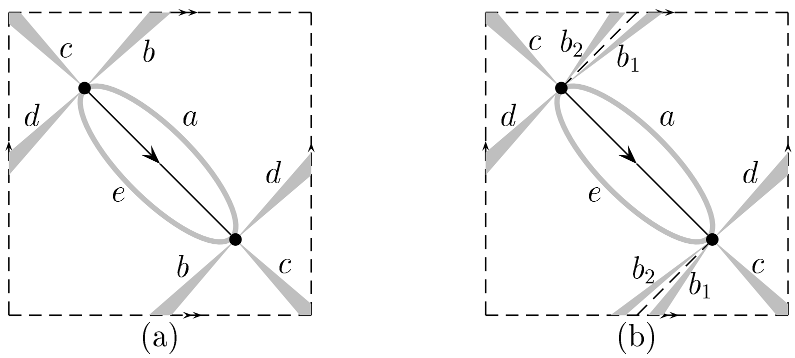

Figure 3: Rooted dipole, (p,q,n)-dipole with face-type $[2^{n-3}\,4^2]$.

First suppose that the distinguished edge is located as depicted in Figure 3(b). Then $a+b_1+b_2+c+d+e=n-2$ with $a,b_1,b_2,e\geq 0$ and $c,d\geq 1$. Since $a+b_1=p-1$ and $e+b_2=q-1$, there are p choices for (a,b_1), q choices for (b_2,e), and $c+d=n-p-q$. Hence there are $n-p-q-1$ choices for (c,d). So the number of (p,q,n)-dipoles of this form is $pq(n-p-q-1)$.

If the distinguished edge is one of the edges that are represented by the gray region labelled 'c' in Figure 3(a), then suppose that c_1 of the edges in this region precede the distinguished edge and c_2 edges follow it as we travel in an anticlockwise direction about the root vertex. (The is in analogy with the definition of b_1,b_2 in Figure 3(b).) Then $a+b+c_1+c_2+d+e=n-2$ with $a,c_1,c_2,e\geq 0$ and $b,d\geq 1$. Since $a+b+c_1=p-1$ and $e+b+c_2=q-1$, there are $p-b$ choices for (a,c_1) and $q-b$ choices for (c_2,e), for any fixed choice of b. Since b must be less than both p and q, the number of (p,q,n)-dipoles of this form is

$$\sum_{b=1}^{p-1}(p-b)\binom{q-b}{1}.$$

Therefore, there are a total of

$$pq+pq(n-p-q-1)+\sum_{b=1}^{p-1}(p-b)\binom{q-b}{1}=pq(n-p-q)+\sum_{b=1}^{p-1}(p-b)\binom{q-b}{1}$$

(p,q,n)-dipoles with the property that $p+q<n$. □

For the $p+q=n$ case, we will make use of the following well-known lemma.

Lemma 2.2 *For positive integers n, k, there are $\binom{n}{k}$ solutions to the equation*

$$x + y_1 + \cdots + y_{k-1} + z = n - 1,$$

where x, z are nonnegative integers and y_i is a positive integer for $i = 1, \ldots, k$.

Proposition 2.3 *If $p + q = n$, then the number of (p, q, n)-dipoles of genus 1 is*

$$\binom{p+1}{4} + \binom{q+1}{4} + \sum_{b=1}^{p-1} (p-b) \binom{q-b}{1}.$$

Proof: Again, we need only consider those dipoles that have face-type $[2^{n-3}\, 6]$ or $[2^{n-4}\, 4^2]$. Let us begin with face-type $[2^{n-3}\, 6]$. As seen in Figure 2(a), if the distinguished edge is one of the edges that are represented by the gray regions labelled 'a' or 'd', we have $p + q = n$.

If the distinguished edge is located among the edges represented by the gray region labelled 'a' in Figure 2(a), then suppose that a_1 of the edges in this region precede the distinguished edge and a_2 edges follow it as we travel in an anticlockwise direction about the root vertex. Then $a_1 = p - 1$ and $d + b + c + a_2 = q - 1$ with $a_1, a_2, d \geq 0$ and $b, c \geq 1$. So there are $\binom{q}{3}$ choices for (d, b, c, a_2), by Lemma 2.2 and the choice of a_1 is fixed. Hence there are $\binom{q}{3}$ (p, q, n)-dipoles of this type. Similarly, there are $\binom{p}{3}$ (p, q, n)-dipoles such that the distinguished edge is located among the edges represented by the gray region labelled 'd' in Figure 2(a).

For face-type $[2^{n-4}\, 4^2]$, we see in Figure 3(a) that there are 3 possible locations for the distinguished edge that yield $p + q = n$, namely among the edges represented by the gray regions labelled 'a', 'e', or 'c', and the latter only if $b = d$.

If the distinguished edge is located among the edges represented by the gray region labelled 'a' in Figure 3(a), then we can define a_1, a_2 just as we did in the previous case and find the number of (p, q, n)-dipoles of this type is given by the number of solutions to the equation $e + b + c + d + a_2 = q - 1$ where $b, c, d \geq 1$ and $e, a_2 \geq 0$. By Lemma 2.2, this is $\binom{q}{4}$. Similarly, there are $\binom{p}{4}$ (p, q, n)-dipoles such that the distinguished edge is located among the edges represented by the gray region labelled 'e' in Figure 3(a).

Finally, if the distinguished edge is located among the edges represented by the gray region labelled 'c' in Figure 3(a), then we find that there

$$\sum_{b=1}^{p-1} (p-b) \binom{q-b}{1}$$

(p, q, n)-dipoles of this type using the exact same reasoning as for the $p + q < n$ case.

Therefore there are a total of

$$\binom{q}{3} + \binom{p}{3} + \binom{q}{4} + \binom{p}{4} + \sum_{b=1}^{p-1} (p-b) \binom{q-b}{1}$$

$$= \binom{p+1}{4} + \binom{q+1}{4} + \sum_{b=1}^{p-1} (p-b)\binom{q-b}{1}$$

genus 1 (p,q,n)-dipoles such that $p+q=n$. $\square$

By symmetry, we need only consider $p \le q$, so we can simplify $\sum_{b=1}^{p-1}(p-b)\binom{q-b}{1}$ and conclude that for $p \le q$, the number of (p,q,n)-dipoles on the torus is

$$\begin{cases} pq(n-p-q) + \dfrac{p(p-1)(3q-p-1)}{6}, & \text{if } p+q<n; \\ \dbinom{p+1}{4} + \dbinom{q+1}{4} + \dfrac{p(p-1)(3q-p-1)}{6}, & \text{if } p+q=n. \end{cases}$$

3 Results for the double torus

The method used to count (p,q,n)-dipoles of genus 1 can be used, in principle, to obtain formulae for genus 2, 3, ..., but it soon becomes unfeasible to try to deal with all of the cases that arise. To convince ourselves that our procedure generalizes in a natural way, we carried it out to solve the genus 2 case. In this section we present our results for the double torus followed by a description of the steps taken to obtain them. Since the method required dealing with over 100 similar cases, we look at only one typical case and conclude with an interesting observation about some of the other cases.

Proposition 3.1 *If $p+q<n$, then the number of (p,q,n)-dipoles of genus 2 is*

$$\frac{1}{12} pq\,(n-p-q)\left(\binom{n-p-1}{2}\left(2\binom{n-p-q}{2} + (q-2)\,(q+1)\right)\right.$$

$$\left.+\binom{n-q-1}{2}\left(2\binom{n-p-q}{2} + (p-2)\,(p+1)\right)\right) + \sum_{i=1}^{p-1}(p-i)(q-i)\binom{n-i}{4}$$

$$+\frac{1}{4}pq\binom{n-p-q+1}{3}(2\ (p-1)\,(q-1) - (n-p-q)\,(n-p-q-2))$$

$$-\binom{p}{3}\left(3\,n\binom{q}{3} + \tfrac{1}{280}p^4 - \tfrac{11}{280}p^3 + \tfrac{1}{120}p^3q + \tfrac{1}{40}p^2 - \tfrac{1}{20}p^2q^2 + \tfrac{1}{10}p^2q\right.$$

$$\left.+\tfrac{17}{20}pq^2 - \tfrac{91}{120}pq - \tfrac{1}{4}pq^3 + \tfrac{43}{280}p + \tfrac{7}{30}q^2 + \tfrac{3}{4}q^3 - \tfrac{17}{20}q - \tfrac{1}{3}q^4 + \tfrac{3}{35}\right).$$

Proof: Since genus 2 dipoles on n edges have $2+e-v-2g = n-4$ faces, we look at the five possible face-types for rooted dipoles that have $n-4$ faces, namely $[10\,2^{n-5}]$, $[8\,4\,2^{n-6}]$, $[6^2\,2^{n-6}]$, $[6\,4^2\,2^{n-7}]$ and $[4^4\,2^{n-8}]$. Table 1 lists the number of distinct rooted dipoles with no digons for each of these face-types.

Since the procedure we used is the same for each face-type, we only look at one. Let us consider face-type $[6^2\,2^{n-6}]$. When we multiply all cycles of length 6 in $\mathfrak{S}_6$ with (123456),

Face-type	No. of rooted dipoles
[10]	8
[8 4]	24
$[6^2]$	12
$[6\,4^2]$	49
$[4^4]$	21

Table 1: Genus 2 dipoles with no digons.

we find that there are 12 that yield a product that consists of two 3-cycles. These 12 permutations represent the only rooted dipoles on 6 edges with 2 faces of degree 6. A dipole on n edges of face-type $[6^2\,2^{n-6}]$ can now be constructed by replacing the edges of these dipoles with face-type $[6^2]$ with multiple non-crossing edges. Because of the awkwardness of the polygonal representation of the double torus, we return to the type of diagram used in Figure 1. Figure 4 shows a rooted dipole on 6 edges with face-type $[6^2]$, and Figure 5 shows the same dipole, now on n edges, where the gray edges represent groupings of edges that do not cross each other. As in Proposition 2.1, we distinguish the rooted edge from the others by adding an edge on both sides of the rooted edge for the digons that follow this edge.

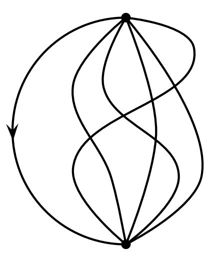

Figure 4: Rooted dipole with face-type $[6^2]$.

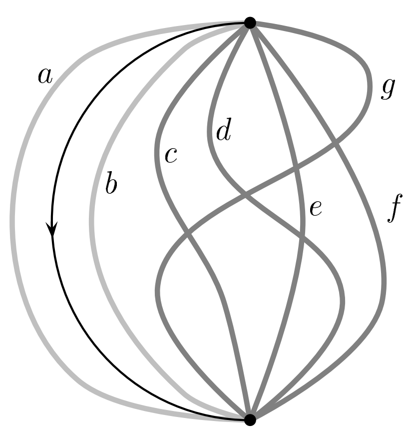

Figure 5: Rooted dipole with face-type $[6^2\,2^{n-6}]$.

Notice that it is possible for the two (light) gray edges adjacent to the rooted edge to represent no edges at all, as in the case $n = 6$, but all the other gray edges must represent at least one edge. Of course, the total number of edges represented by the gray edges must be $n - 1$. Now, since this gives us seven possible groupings of digons in which the second distinguished, or dashed, edge could be found, we must consider these seven cases separately. Of course, we consider only the cases that yield $p + q < n$. As with the rooted edge, we want to distinguish the dashed edge from the others, so we add a gray edge on both sides of the dashed edge for the digons that follow this edge. Again, it is possible for these two gray edges to represent no edges at all. For each of the applicable cases, we use the same techniques we used in the proof of Proposition 2.1 to calculate the number of (p, q, n)-dipoles that are equivalent to our underlying case, that is, the number of (p, q, n)-dipoles that, when each grouping of digons is replaced by one edge, are identical to our given case.

For the dipole in Figure 5, if the distinguished edge is among the group of edges labelled 'a' or 'b', we have the case $p + q = n$, and if it is among the edges labelled 'g', we have the case $p + q > n$. So for (p, q, n)-dipoles with $p + q < n$, we look at the cases where the distinguished edge is among the edges labelled 'c','d','e', or 'f'. Figure 6 shows the case where the distinguished edge is among the edges labelled 'c'.

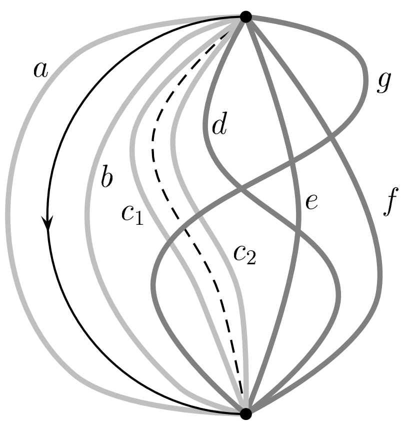

Figure 6: (p, q, n)-dipole with face-type $[6^2\, 2^{n-6}]$.

Here we have $b + c_1 = p - 1$ and $a + f + d + e + c_2 = q - 1$, and since we know that $a + b + c_1 + c_2 + d + e + f + g = n - 2$, we also have $g = n - p - q$. Since $a, b, c_1, c_2 \geq 0$ and $d, e, f, g \geq 1$, Lemma 2.2 gives the number of (p, q, n)-dipoles such that the distinguished edge is among the edges labelled 'c' is

$$p\binom{q}{4}.$$

Using a similar analysis, we calculate that there are

$$\binom{p}{2}\binom{q}{2}\binom{n-p-q-1}{1}$$

(p, q, n)-dipoles such that the distinguished edge is among the edges labelled 'd', and

$$\binom{p}{4} q$$

(p, q, n)-dipoles such that the distinguished edge is among the edges labelled 'f'.

The case where the distinguished edge is among the edges labelled 'e' is a little different because here the group of edges labelled 'd' contributes to both p and q. We account for this by summing over the possible number of edges labelled d. Labelling 'e_1' and 'e_2' analogous to 'c_1' and 'c_2' in Figure 6, we have that $b + c + e_1 = p - d - 1$ and $a + f + e_2 = q - d - 1$ and $g = n - p - q$. If we let $i = d$, then since $c, f \geq 1$ and $a, b, e_1, e_2 \geq 0$, we appeal once more to Lemma 2.2 to get that there are

$$\sum_{i \geq 1} \binom{p-i}{2} \binom{q-i}{2}$$

(p, q, n)-dipoles such that the distinguished edge is among the edges labelled 'e'.

A very similar calculation is performed for each of the 114 rooted dipoles referred to in Table 1. To arrive at our final result, we added the equations obtained for each such case, and then simplified that sum. □

It is important to note that each case must be handled separately. The relevant calculations don't just depend on face-type. For example, consider the two dipoles in Figure 7. Although both of these dipoles have face-type $[6^2\, 2^{n-6}]$ and if we let $n = 6$, both

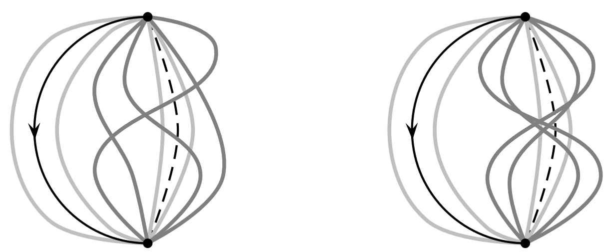

Figure 7: Two (p, q, n)-dipoles with face-type $[6^2\, 2^{n-6}]$.

have $p = 3$ and $q = 3$, they yield different equations. There are

$$\sum_{i \geq 1} \binom{p-i}{2} \binom{q-i}{2}$$

(p, q, n)-dipoles of the first type, and

$$\sum_{i \geq 2} \binom{p-i}{1} \binom{q-i}{1} \binom{n-p-q-1+i}{1} (i-1)$$

(p, q, n)-dipoles of the second type. In fact, even when considering different rootings of the same dipole, we can obtain different expressions for the number of (p, q, n)-dipoles obtained by adding digons in all possible ways.

Proposition 3.2 *If $p+q=n$, then the number of (p,q,n)-dipoles of genus 2 is*

$$\frac{1}{336}\binom{p+1}{5}\left(21\,p^3-155\,p^2+338\,p+456\right)+\frac{1}{8}\binom{q+1}{6}\left(3\,q^2-q-6\right)$$

$$+\frac{1}{2880}\,pq\,(p-1)\left(84\,p^3q-132\,p^3-281\,p^2q+5\,p^2q^3+10\,p^2q^2+458\,p^2-102\,p\right.$$

$$\left.-210\,pq^2+209\,pq+55\,pq^3+214\,q+440\,q^2-692-310\,q^3+60\,q^4\right).$$

We obtain this result using the same method as that used for Proposition 3.1, except that now we look at cases that yield $p+q=n$.

It should be noted that all of the results in this paper have been carefully checked. This is especially important for Propositions 3.1 and 3.2, where many detailed calculations were performed and then summed over a large number of cases. The final simplification was done using Maple. Appendix A has a list of the genus series for all (p,q,n)-dipoles on up to 10 edges. These were created by generating each (p,q,n)-dipole by exhaustive search using the permutation encoding of [6]. Our results were then checked against each entry in these tables.

References

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A Tables

Finally, we list tables giving $D_{p,q,n}(u)$ and $D_n(u)$ for small values of n.

2 edges

p	q	$D_{p,q,n}(u)$
1	1	1

$D_2(u) = 1$

3 edges

p	q	$D_{p,q,n}(u)$
1	1	u
1	2	1
2	1	1
2	2	u

$D_3(u) = u + 1$

4 edges

p	q	$D_{p,q,n}(u)$
1	1	$2\,u$
1	2	$2\,u$
1	3	$u + 1$
2	1	$2\,u$
2	2	$u + 1$
2	3	$2\,u$
3	1	$u + 1$
3	2	$2\,u$
3	3	$2\,u$

$D_4(u) = 5\,u + 1$

5 edges

p	q	$D_{p,q,n}(u)$
1	1	$3\,u^2 + 3\,u$
1	2	$2\,u^2 + 4\,u$
1	3	$3\,u^2 + 3\,u$
1	4	$5\,u + 1$
2	1	$2\,u^2 + 4\,u$
2	2	$u^2 + 5\,u$
2	3	$2\,u^2 + 3\,u + 1$
2	4	$3\,u^2 + 3\,u$
3	1	$3\,u^2 + 3\,u$
3	2	$2\,u^2 + 3\,u + 1$
3	3	$u^2 + 5\,u$
3	4	$2\,u^2 + 4\,u$
4	1	$5\,u + 1$
4	2	$3\,u^2 + 3\,u$
4	3	$2\,u^2 + 4\,u$
4	4	$3\,u^2 + 3\,u$

$D_5(u) = 8\,u^2 + 15\,u + 1$

6 edges

p	q	$D_{p,q,n}(u)$
1	1	$20\,u^2 + 4\,u$
1	2	$18\,u^2 + 6\,u$
1	3	$18\,u^2 + 6\,u$
1	4	$20\,u^2 + 4\,u$
1	5	$8\,u^2 + 15\,u + 1$
2	1	$18\,u^2 + 6\,u$
2	2	$15\,u^2 + 9\,u$
2	3	$16\,u^2 + 8\,u$
2	4	$15\,u^2 + 8\,u + 1$
2	5	$20\,u^2 + 4\,u$
3	1	$18\,u^2 + 6\,u$
3	2	$16\,u^2 + 8\,u$
3	3	$16\,u^2 + 7\,u + 1$
3	4	$16\,u^2 + 8\,u$
3	5	$18\,u^2 + 6\,u$
4	1	$20\,u^2 + 4\,u$
4	2	$15\,u^2 + 8\,u + 1$
4	3	$16\,u^2 + 8\,u$
4	4	$15\,u^2 + 9\,u$
4	5	$18\,u^2 + 6\,u$
5	1	$8\,u^2 + 15\,u + 1$
5	2	$20\,u^2 + 4\,u$
5	3	$18\,u^2 + 6\,u$
5	4	$18\,u^2 + 6\,u$
5	5	$20\,u^2 + 4\,u$

$D_6(u) = 84\,u^2 + 35\,u + 1$

7 edges

p	q	$D_{p,q,n}(u)$
1	1	$40\,u^3 + 75\,u^2 + 5\,u$
1	2	$32\,u^3 + 80\,u^2 + 8\,u$
1	3	$36\,u^3 + 75\,u^2 + 9\,u$
1	4	$32\,u^3 + 80\,u^2 + 8\,u$
1	5	$40\,u^3 + 75\,u^2 + 5\,u$
1	6	$84\,u^2 + 35\,u + 1$
2	1	$32\,u^3 + 80\,u^2 + 8\,u$
2	2	$24\,u^3 + 83\,u^2 + 13\,u$
2	3	$28\,u^3 + 78\,u^2 + 14\,u$
2	4	$24\,u^3 + 85\,u^2 + 11\,u$
2	5	$32\,u^3 + 68\,u^2 + 19\,u + 1$
2	6	$40\,u^3 + 75\,u^2 + 5\,u$
3	1	$36\,u^3 + 75\,u^2 + 9\,u$
3	2	$28\,u^3 + 78\,u^2 + 14\,u$
3	3	$32\,u^3 + 74\,u^2 + 14\,u$
3	4	$28\,u^3 + 77\,u^2 + 14\,u + 1$
3	5	$24\,u^3 + 85\,u^2 + 11\,u$
3	6	$32\,u^3 + 80\,u^2 + 8\,u$
4	1	$32\,u^3 + 80\,u^2 + 8\,u$
4	2	$24\,u^3 + 85\,u^2 + 11\,u$
4	3	$28\,u^3 + 77\,u^2 + 14\,u + 1$
4	4	$32\,u^3 + 74\,u^2 + 14\,u$
4	5	$28\,u^3 + 78\,u^2 + 14\,u$
4	6	$36\,u^3 + 75\,u^2 + 9\,u$
5	1	$40\,u^3 + 75\,u^2 + 5\,u$
5	2	$32\,u^3 + 68\,u^2 + 19\,u + 1$
5	3	$24\,u^3 + 85\,u^2 + 11\,u$
5	4	$28\,u^3 + 78\,u^2 + 14\,u$
5	5	$24\,u^3 + 83\,u^2 + 13\,u$
5	6	$32\,u^3 + 80\,u^2 + 8\,u$
6	1	$84\,u^2 + 35\,u + 1$
6	2	$40\,u^3 + 75\,u^2 + 5\,u$
6	3	$32\,u^3 + 80\,u^2 + 8\,u$
6	4	$36\,u^3 + 75\,u^2 + 9\,u$
6	5	$32\,u^3 + 80\,u^2 + 8\,u$
6	6	$40\,u^3 + 75\,u^2 + 5\,u$

$D_7(u) = 180\,u^3 + 469\,u^2 + 70\,u + 1$

8 edges

p	q	$D_{p,q,n}(u)$
1	1	$504\,u^3+210\,u^2+6\,u$
1	2	$460\,u^3+250\,u^2+10\,u$
1	3	$468\,u^3+240\,u^2+12\,u$
1	4	$468\,u^3+240\,u^2+12\,u$
1	5	$460\,u^3+250\,u^2+10\,u$
1	6	$504\,u^3+210\,u^2+6\,u$
1	7	$180\,u^3+469\,u^2+70\,u+1$
2	1	$460\,u^3+250\,u^2+10\,u$
2	2	$408\,u^3+295\,u^2+17\,u$
2	3	$420\,u^3+280\,u^2+20\,u$
2	4	$416\,u^3+285\,u^2+19\,u$
2	5	$416\,u^3+290\,u^2+14\,u$
2	6	$420\,u^3+259\,u^2+40\,u+1$
2	7	$504\,u^3+210\,u^2+6\,u$
3	1	$468\,u^3+240\,u^2+12\,u$
3	2	$420\,u^3+280\,u^2+20\,u$
3	3	$428\,u^3+269\,u^2+23\,u$
3	4	$428\,u^3+272\,u^2+20\,u$
3	5	$424\,u^3+268\,u^2+27\,u+1$
3	6	$416\,u^3+290\,u^2+14\,u$
3	7	$460\,u^3+250\,u^2+10\,u$
4	1	$468\,u^3+240\,u^2+12\,u$
4	2	$416\,u^3+285\,u^2+19\,u$
4	3	$428\,u^3+272\,u^2+20\,u$
4	4	$420\,u^3+275\,u^2+24\,u+1$
4	5	$428\,u^3+272\,u^2+20\,u$
4	6	$416\,u^3+285\,u^2+19\,u$
4	7	$468\,u^3+240\,u^2+12\,u$

p	q	$D_{p,q,n}(u)$
5	1	$460\,u^3+250\,u^2+10\,u$
5	2	$416\,u^3+290\,u^2+14\,u$
5	3	$424\,u^3+268\,u^2+27\,u+1$
5	4	$428\,u^3+272\,u^2+20\,u$
5	5	$428\,u^3+269\,u^2+23\,u$
5	6	$420\,u^3+280\,u^2+20\,u$
5	7	$468\,u^3+240\,u^2+12\,u$
6	1	$504\,u^3+210\,u^2+6\,u$
6	2	$420\,u^3+259\,u^2+40\,u+1$
6	3	$416\,u^3+290\,u^2+14\,u$
6	4	$416\,u^3+285\,u^2+19\,u$
6	5	$420\,u^3+280\,u^2+20\,u$
6	6	$408\,u^3+295\,u^2+17\,u$
6	7	$460\,u^3+250\,u^2+10\,u$
7	1	$180\,u^3+469\,u^2+70\,u+1$
7	2	$504\,u^3+210\,u^2+6\,u$
7	3	$460\,u^3+250\,u^2+10\,u$
7	4	$468\,u^3+240\,u^2+12\,u$
7	5	$468\,u^3+240\,u^2+12\,u$
7	6	$460\,u^3+250\,u^2+10\,u$
7	7	$504\,u^3+210\,u^2+6\,u$

$$D_8(u)=3044\,u^3+1869\,u^2+126\,u+1$$

9 edges

p	q	$D_{p,q,n}(u)$
1	1	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$
1	2	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
1	3	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
1	4	$1104\,u^4 + 3304\,u^3 + 616\,u^2 + 16\,u$
1	5	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
1	6	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
1	7	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$
1	8	$3044\,u^3 + 1869\,u^2 + 126\,u + 1$
2	1	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
2	2	$900\,u^4 + 3309\,u^3 + 810\,u^2 + 21\,u$
2	3	$960\,u^4 + 3254\,u^3 + 800\,u^2 + 26\,u$
2	4	$924\,u^4 + 3303\,u^3 + 786\,u^2 + 27\,u$
2	5	$960\,u^4 + 3246\,u^3 + 810\,u^2 + 24\,u$
2	6	$900\,u^4 + 3353\,u^3 + 770\,u^2 + 17\,u$
2	7	$1080\,u^4 + 2994\,u^3 + 889\,u^2 + 76\,u + 1$
2	8	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$
3	1	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
3	2	$960\,u^4 + 3254\,u^3 + 800\,u^2 + 26\,u$
3	3	$1020\,u^4 + 3193\,u^3 + 795\,u^2 + 32\,u$
3	4	$984\,u^4 + 3238\,u^3 + 786\,u^2 + 32\,u$
3	5	$1020\,u^4 + 3199\,u^3 + 795\,u^2 + 26\,u$
3	6	$960\,u^4 + 3250\,u^3 + 779\,u^2 + 50\,u + 1$
3	7	$900\,u^4 + 3353\,u^3 + 770\,u^2 + 17\,u$
3	8	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
4	1	$1104\,u^4 + 3304\,u^3 + 616\,u^2 + 16\,u$
4	2	$924\,u^4 + 3303\,u^3 + 786\,u^2 + 27\,u$
4	3	$984\,u^4 + 3238\,u^3 + 786\,u^2 + 32\,u$
4	4	$948\,u^4 + 3295\,u^3 + 767\,u^2 + 30\,u$
4	5	$984\,u^4 + 3220\,u^3 + 795\,u^2 + 40\,u + 1$
4	6	$1020\,u^4 + 3199\,u^3 + 795\,u^2 + 26\,u$
4	7	$960\,u^4 + 3246\,u^3 + 810\,u^2 + 24\,u$
4	8	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$

9 edges (cont'd)

p	q	$D_{p,q,n}(u)$
5	1	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
5	2	$960\,u^4 + 3246\,u^3 + 810\,u^2 + 24\,u$
5	3	$1020\,u^4 + 3199\,u^3 + 795\,u^2 + 26\,u$
5	4	$984\,u^4 + 3220\,u^3 + 795\,u^2 + 40\,u + 1$
5	5	$948\,u^4 + 3295\,u^3 + 767\,u^2 + 30\,u$
5	6	$984\,u^4 + 3238\,u^3 + 786\,u^2 + 32\,u$
5	7	$924\,u^4 + 3303\,u^3 + 786\,u^2 + 27\,u$
5	8	$1104\,u^4 + 3304\,u^3 + 616\,u^2 + 16\,u$
6	1	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
6	2	$900\,u^4 + 3353\,u^3 + 770\,u^2 + 17\,u$
6	3	$960\,u^4 + 3250\,u^3 + 779\,u^2 + 50\,u + 1$
6	4	$1020\,u^4 + 3199\,u^3 + 795\,u^2 + 26\,u$
6	5	$984\,u^4 + 3238\,u^3 + 786\,u^2 + 32\,u$
6	6	$1020\,u^4 + 3193\,u^3 + 795\,u^2 + 32\,u$
6	7	$960\,u^4 + 3254\,u^3 + 800\,u^2 + 26\,u$
6	8	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
7	1	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$
7	2	$1080\,u^4 + 2994\,u^3 + 889\,u^2 + 76\,u + 1$
7	3	$900\,u^4 + 3353\,u^3 + 770\,u^2 + 17\,u$
7	4	$960\,u^4 + 3246\,u^3 + 810\,u^2 + 24\,u$
7	5	$924\,u^4 + 3303\,u^3 + 786\,u^2 + 27\,u$
7	6	$960\,u^4 + 3254\,u^3 + 800\,u^2 + 26\,u$
7	7	$900\,u^4 + 3309\,u^3 + 810\,u^2 + 21\,u$
7	8	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
8	1	$3044\,u^3 + 1869\,u^2 + 126\,u + 1$
8	2	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$
8	3	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
8	4	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
8	5	$1104\,u^4 + 3304\,u^3 + 616\,u^2 + 16\,u$
8	6	$1140\,u^4 + 3255\,u^3 + 630\,u^2 + 15\,u$
8	7	$1080\,u^4 + 3318\,u^3 + 630\,u^2 + 12\,u$
8	8	$1260\,u^4 + 3283\,u^3 + 490\,u^2 + 7\,u$

$$D_9(u) = 8064\,u^4 + 26060\,u^3 + 5985\,u^2 + 210\,u + 1$$

10 edges

p	q	$D_{p,q,n}(u)$
1	1	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$
1	2	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
1	3	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
1	4	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
1	5	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
1	6	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
1	7	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
1	8	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$
1	9	$8064\,u^4 + 26060\,u^3 + 5985\,u^2 + 210\,u + 1$
2	1	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
2	2	$20604\,u^4 + 17815\,u^3 + 1876\,u^2 + 25\,u$
2	3	$20968\,u^4 + 17388\,u^3 + 1932\,u^2 + 32\,u$
2	4	$20860\,u^4 + 17535\,u^3 + 1890\,u^2 + 35\,u$
2	5	$20896\,u^4 + 17486\,u^3 + 1904\,u^2 + 34\,u$
2	6	$20908\,u^4 + 17451\,u^3 + 1932\,u^2 + 29\,u$
2	7	$20784\,u^4 + 17780\,u^3 + 1736\,u^2 + 20\,u$
2	8	$21308\,u^4 + 16127\,u^3 + 2751\,u^2 + 133\,u + 1$
2	9	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$
3	1	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
3	2	$20968\,u^4 + 17388\,u^3 + 1932\,u^2 + 32\,u$
3	3	$21284\,u^4 + 16999\,u^3 + 1996\,u^2 + 41\,u$
3	4	$21200\,u^4 + 17116\,u^3 + 1960\,u^2 + 44\,u$
3	5	$21212\,u^4 + 17089\,u^3 + 1978\,u^2 + 41\,u$
3	6	$21272\,u^4 + 17068\,u^3 + 1948\,u^2 + 32\,u$
3	7	$21088\,u^4 + 17092\,u^3 + 2051\,u^2 + 88\,u + 1$
3	8	$20784\,u^4 + 17780\,u^3 + 1736\,u^2 + 20\,u$
3	9	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
4	1	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
4	2	$20860\,u^4 + 17535\,u^3 + 1890\,u^2 + 35\,u$
4	3	$21200\,u^4 + 17116\,u^3 + 1960\,u^2 + 44\,u$
4	4	$21100\,u^4 + 17239\,u^3 + 1935\,u^2 + 46\,u$
4	5	$21136\,u^4 + 17230\,u^3 + 1914\,u^2 + 40\,u$
4	6	$21100\,u^4 + 17159\,u^3 + 1994\,u^2 + 66\,u + 1$
4	7	$21272\,u^4 + 17068\,u^3 + 1948\,u^2 + 32\,u$
4	8	$20908\,u^4 + 17451\,u^3 + 1932\,u^2 + 29\,u$
4	9	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
5	1	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
5	2	$20896\,u^4 + 17486\,u^3 + 1904\,u^2 + 34\,u$
5	3	$21212\,u^4 + 17089\,u^3 + 1978\,u^2 + 41\,u$
5	4	$21136\,u^4 + 17230\,u^3 + 1914\,u^2 + 40\,u$
5	5	$21160\,u^4 + 17090\,u^3 + 2009\,u^2 + 60\,u + 1$

10 edges (cont'd)

p	q	$D_{p,q,n}(u)$
5	6	$21136\,u^4 + 17230\,u^3 + 1914\,u^2 + 40\,u$
5	7	$21212\,u^4 + 17089\,u^3 + 1978\,u^2 + 41\,u$
5	8	$20896\,u^4 + 17486\,u^3 + 1904\,u^2 + 34\,u$
5	9	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
6	1	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
6	2	$20908\,u^4 + 17451\,u^3 + 1932\,u^2 + 29\,u$
6	3	$21272\,u^4 + 17068\,u^3 + 1948\,u^2 + 32\,u$
6	4	$21100\,u^4 + 17159\,u^3 + 1994\,u^2 + 66\,u + 1$
6	5	$21136\,u^4 + 17230\,u^3 + 1914\,u^2 + 40\,u$
6	6	$21100\,u^4 + 17239\,u^3 + 1935\,u^2 + 46\,u$
6	7	$21200\,u^4 + 17116\,u^3 + 1960\,u^2 + 44\,u$
6	8	$20860\,u^4 + 17535\,u^3 + 1890\,u^2 + 35\,u$
6	9	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
7	1	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
7	2	$20784\,u^4 + 17780\,u^3 + 1736\,u^2 + 20\,u$
7	3	$21088\,u^4 + 17092\,u^3 + 2051\,u^2 + 88\,u + 1$
7	4	$21272\,u^4 + 17068\,u^3 + 1948\,u^2 + 32\,u$
7	5	$21212\,u^4 + 17089\,u^3 + 1978\,u^2 + 41\,u$
7	6	$21200\,u^4 + 17116\,u^3 + 1960\,u^2 + 44\,u$
7	7	$21284\,u^4 + 16999\,u^3 + 1996\,u^2 + 41\,u$
7	8	$20968\,u^4 + 17388\,u^3 + 1932\,u^2 + 32\,u$
7	9	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
8	1	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$
8	2	$21308\,u^4 + 16127\,u^3 + 2751\,u^2 + 133\,u + 1$
8	3	$20784\,u^4 + 17780\,u^3 + 1736\,u^2 + 20\,u$
8	4	$20908\,u^4 + 17451\,u^3 + 1932\,u^2 + 29\,u$
8	5	$20896\,u^4 + 17486\,u^3 + 1904\,u^2 + 34\,u$
8	6	$20860\,u^4 + 17535\,u^3 + 1890\,u^2 + 35\,u$
8	7	$20968\,u^4 + 17388\,u^3 + 1932\,u^2 + 32\,u$
8	8	$20604\,u^4 + 17815\,u^3 + 1876\,u^2 + 25\,u$
8	9	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
9	1	$8064\,u^4 + 26060\,u^3 + 5985\,u^2 + 210\,u + 1$
9	2	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$
9	3	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
9	4	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
9	5	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
9	6	$22800\,u^4 + 16100\,u^3 + 1400\,u^2 + 20\,u$
9	7	$22872\,u^4 + 16002\,u^3 + 1428\,u^2 + 18\,u$
9	8	$22568\,u^4 + 16366\,u^3 + 1372\,u^2 + 14\,u$
9	9	$24352\,u^4 + 14952\,u^3 + 1008\,u^2 + 8\,u$

$$D_{10}(u) = 193248\,u^4 + 152900\,u^3 + 16401\,u^2 + 330\,u + 1$$