Combinatorial Interpretations for Rank-Two Cluster Algebras of Affine Type

Gregg Musiker

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112
gmusiker@math.ucsd.edu

James Propp

Department of Mathematical Sciences
University of Massachusetts Lowell
Lowell, MA 01854
propp@jamespropp.org.ignorethis

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Abstract

Fomin and Zelevinsky [6] show that a certain two-parameter family of rational recurrence relations, here called the (b, c) family, possesses the Laurentness property: for all b, c, each term of the (b, c) sequence can be expressed as a Laurent polynomial in the two initial terms. In the case where the positive integers b, c satisfy $bc < 4$, the recurrence is related to the root systems of finite-dimensional rank 2 Lie algebras; when $bc > 4$, the recurrence is related to Kac-Moody rank 2 Lie algebras of general type [9]. Here we investigate the borderline cases $bc = 4$, corresponding to Kac-Moody Lie algebras of affine type. In these cases, we show that the Laurent polynomials arising from the recurence can be viewed as generating functions that enumerate the perfect matchings of certain graphs. By providing combinatorial interpretations of the individual coefficients of these Laurent polynomials, we establish their positivity.

1 Introduction

In [5, 6], Fomin and Zelevinsky prove that for all positive integers b and c, the sequence of rational functions x_n $(n \geq 0)$ satisfying the "(b, c)-recurrence"

$$x_n = \begin{cases} (x_{n-1}^b + 1)/x_{n-2} & \text{for } n \text{ odd} \\ (x_{n-1}^c + 1)/x_{n-2} & \text{for } n \text{ even} \end{cases}$$

is a sequence of Laurent polynomial in the variables x_1 and x_2; that is, for all $n \geq 2$, x_n can be written as a sum of Laurent monomials of the form $ax_1^i x_2^j$, where the coefficient a is an integer and i and j are (not necessarily positive) integers. In fact, Fomin and Zelevinsky conjecture that the coefficients are always *positive* integers.

It is worth mentioning that variants of this recurrence typically lead to rational functions that are not Laurent polynomials. For instance, if one initializes with x_1, x_2 and defines rational functions

$$x_n = \begin{cases} (x_{n-1}^b + 1)/x_{n-2} & \text{for } n = 2 \\ (x_{n-1}^c + 1)/x_{n-2} & \text{for } n = 3 \\ (x_{n-1}^d + 1)/x_{n-2} & \text{for } n = 4 \\ (x_{n-1}^e + 1)/x_{n-2} & \text{for } n = 5 \end{cases}$$

with b, c, d, e all integers larger than 1, then it appears that x_5 is not a Laurent polynomial (in x_1 and x_2) unless $b = d$ and that x_6 is not a Laurent polynomial unless $b = d$ and $c = e$. (This has been checked by computer in the cases where b, c, d, e are all between 2 and 5.)

One reason for studying (b, c)-recurrences is their relationship with root systems associated to rank two Kac-Moody Lie algebras. Furthermore, algebras generated by a sequence of elements satisfying a (b, c)-recurrence provide examples of rank two cluster algebras, as defined in [6, 7] by Fomin and Zelevinsky. The property of being a sequence of Laurent polynomials, *Laurentness*, is in fact proven for all cluster algebras [6] as well as a class of examples going beyond cluster algebras [5]. In this context, the positivity of the coefficients is no mere curiosity, but is related to important (albeit still conjectural) total-positivity properties of dual canonical bases [13].

The cases $bc < 4$ correspond to finite-dimensional Lie algebras (that is, semisimple Lie algebras), and these cases have been treated in great detail by Fomin and Zelevinsky [6, 12]. For example, the cases $(1, 1)$, $(1, 2)$, and $(1, 3)$ correspond respectively to the Lie algebras A_2, B_2, and G_2. In these cases, the sequence of Laurent polynomials x_n is periodic. More specifically, the sequence repeats with period 5 when $(b, c) = (1, 1)$, with period 6 when $(b, c) = (1, 2)$ or $(2, 1)$, and with period 8 when $(b, c) = (1, 3)$ or $(3, 1)$. For each of these cases, one can check that each x_n has positive integer coefficients.

Very little is known about the cases $bc > 4$, which should correspond to Kac-Moody Lie algebras of general type. It can be shown that for these cases, the sequence of Laurent polynomials x_n is non-periodic.

This article gives a combinatorial approach to the intermediate cases $(2, 2)$, $(1, 4)$ and $(4, 1)$, corresponding to Kac-Moody Lie algebras of affine type; specifically algebras of types $A_1^{(1)}$ and $A_2^{(2)}$. Work of Sherman and Zelevinsky [12] has also focused on the rank two affine case. In fact, they are able to prove positivity of the $(2, 2)$-, $(1, 4)$- and $(4, 1)$-cases, as well as a complete description of the positive cone. They prove both cases simultaneously by utilizing a more general recurrence which specializes to either case. By using Newton polygons, root systems and algebraic methods analogous to those used in the finite type case [7], they are able to construct the dual canonical bases for these cluster algebras explicitly.

Our method is intended as a complement to the purely algebraic method of Sherman and Zelevinsky [12]. In each of the cases $(2,2)$, $(1,4)$ and $(4,1)$ we show that the positivity conjecture of Fomin and Zelevinsky is true by providing (and proving) a combinatorial interpretation of all the coefficients of x_n. That is, we show that the coefficient of $x_1^i x_2^j$ in x_n is actually the cardinality of a certain set of combinatorial objects, namely, the set of those perfect matchings of a particular graph that contain a specified number of "x_1-edges" and a specified number of "x_2-edges". This combinatorial description provides a different way of understanding the cluster variables, one where the binomial exchange relations are visible geometrically.

The reader may already have guessed that the cases $(1,4)$ and $(4,1)$ are closely related. One way to think about this relationship is to observe that the formulas

$$x_n = (x_{n-1}^b + 1)/x_{n-2} \text{ for } n \text{ odd}$$

and

$$x_n = (x_{n-1}^c + 1)/x_{n-2} \text{ for } n \text{ even}$$

can be re-written as

$$x_{n-2} = (x_{n-1}^b + 1)/x_n \text{ for } n \text{ odd}$$

and

$$x_{n-2} = (x_{n-1}^c + 1)/x_n \text{ for } n \text{ even;}$$

these give us a canonical way of recursively defining rational functions x_n with $n < 0$, and indeed, it is not hard to show that

$$x_{-n}^{(b,c)}(x_1, x_2) = x_{n+3}^{(c,b)}(x_2, x_1) \quad \text{for} \quad n \in \mathbb{Z}. \tag{1}$$

So the $(4,1)$ sequence of Laurent polynomials can be obtained from the $(1,4)$ sequence of Laurent polynomials by running the recurrence in reverse and switching the roles of x_1 and x_2. Henceforth we will not consider the $(4,1)$ recurrence; instead, we will study the $(1,4)$ recurrence and examine x_n for all integer values of n, the negative together with the positive.

Our approach to the $(1,4)$ case will be the same as our approach to the simpler $(2,2)$ case: in both cases, we will utilize perfect matchings of graphs as studied in [2, 10, 11, et al.].

Definition 1. For a graph $G = (V, E)$, which has an assignment of weights $w(e)$ to its edges $e \in E$, a *perfect matching* of G is a subset $S \subset E$ of the edges of G such that each vertex $v \in V$ belongs to exactly one edge in S. We define the *weight* of a perfect matching S to be the product of the weights of its constituent edges,

$$w(S) = \prod_{e \in S} w(e).$$

With this definition in mind, the main result of this paper is the construction of a family of graphs $\{G_n\}$ indexed by $n \in \mathbb{Z} \setminus \{1, 2\}$ with weights on their edges such that the terms of the $(2, 2)$- (resp. $(1, 4)$-) recurrence, x_n, satisfy $x_n = p_n(x_1, x_2)/m_n(x_1, x_2)$; where $p_n(x_1, x_2)$ is the polynomial

$$\sum_{S \subset E \text{ is a perfect matching of } G_n} w(S),$$

and m_n is the monomial $x_1^{c_1} x_2^{c_2}$ where c_1 and c_2 characterize the 2-skeleton of G_n. These constructions appear as Theorem 1 and Theorem 4 in sections 2 and 3, for the $(2, 2)$- and $(1, 4)$- cases, respectively.

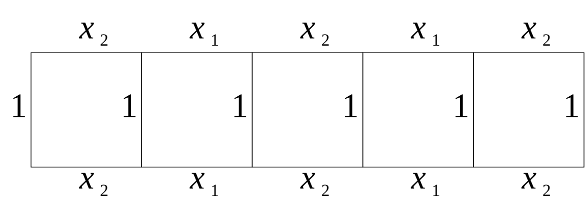
Thus $p_n(x_1, x_2)$ may be considered a two-variable generating function for the perfect matchings of G_n, and $x_n(x_1, x_2)$ may be considered a generating function as well, with a slightly different definition of the weight that includes "global" factors (associated with the structure of G_n) as well as "local" factors (associated with the edges of a particular perfect matching). We note that the families G_n with $n > 0$ and G_n with $n < 0$ given in this paper are just one possible pair of families of graphs with the property that the $x_n(x_1, x_2)$'s serve as their generating functions.

The plan of this article is as follows. In Section 2, we treat the case $(2, 2)$; it is simpler than $(1, 4)$, and makes a good warm-up. In Section 3, we treat the case $(1, 4)$ (which subsumes the case $(4, 1)$, since we allow n to be negative). Section 4 gives comments and open problems arising from this work.

2 The $(2, 2)$ case

Here we study the sequence of Laurent polynomials $x_1, x_2, x_3 = (x_2^2 + 1)/x_1$, etc. If we let $x_1 = x_2 = 1$, then the first few terms of sequence $\{x_n(1, 1)\}$ for $n \geq 3$ are $2, 5, 13, 34, 89, \ldots$. It is not too hard to guess that this sequence consists of every other Fibonacci number (and indeed this fact follows readily from Lemma 1 given on the next page).

For all $n \geq 1$, let H_n be the (edge-weighted) graph shown below for the case $n = 6$.



The graph $G_5 = H_6$.

That is, H_n is a 2-by-n grid in which every vertical edge has been assigned weight 1 and the horizontal edges alternate between weight x_2 and weight x_1, with the two leftmost horizontal edges having weight x_2, the two horizontal edges adjoining them having weight

x_1, the two horizontal edges adjoining *them* having weight x_2, and so on (ending at the right with two edges of weight x_2 when n is even and with two edges of weight x_1 when n is odd). Let $G_n = H_{2n-4}$ (so that for example the above picture shows G_5), and let $p_n(x_1, x_2)$ be the sum of the weights of all the perfect matchings of G_n. Also let $m_n(x_1, x_2) = x_1^{n-2} x_2^{n-3}$ for $n \geq 3$. We note the following combinatorial interpretation of this monomial: $m_n(x_1, x_2) = x_1^i x_2^j$ where i is the number of square cells of G_n with horizontal edges having weight x_2 and j is the number of square cells with horizontal edges having weight x_1. Using these definitions we obtain

Theorem 1. *For the case $(b, c) = (2, 2)$, the Laurent polynomials x_n satisfy*

$$x_n(x_1, x_2) = p_n(x_1, x_2)/m_n(x_1, x_2) \text{ for } n \neq 1, 2$$

where p_n and m_n are given combinatorially as in the preceding paragraph.

E.g., for $n = 3$, the graph $G_3 = H_2$ has two perfect matchings with respective weights x_2^2 and 1, so $x_3(x_1, x_2) = (x_2^2 + 1)/x_1$. For $n = 4$, the graph $G_4 = H_4$ has five perfect matchings with respective weights x_2^4, x_2^2, x_2^2, 1, and x_1^2, so $p_4(x_1, x_2) = 1 + 2x_2^2 + x_2^4 + x_1^2$; since $m_4(x_1, x_2) = x_1^2 x_2$, we have $p_4(x_1, x_2)/m_4(x_1, x_2) = (x_2^4 + 2x_2^2 + 1 + x_1^2)/x_1^2 x_2$, as required.

Proof. We will have proved the claim if we can show that the Laurent polynomials $p_n(x_1, x_2)/m_n(x_1, x_2)$ satisfy the same quadratic recurrence as the Laurent polynomials $x_n(x_1, x_2)$; that is,

$$\frac{p_n(x_1, x_2)}{m_n(x_1, x_2)} \, \frac{p_{n-2}(x_1, x_2)}{m_{n-2}(x_1, x_2)} = \left(\frac{p_{n-1}(x_1, x_2)}{m_{n-1}(x_1, x_2)} \right)^2 + 1. \tag{2}$$

Proposition 1. *The polynomials $p_n(x_1, x_2)$ satisfy the recurrence*

$$p_n(x_1, x_2) p_{n-2}(x_1, x_2) = (p_{n-1}(x_1, x_2))^2 + x_1^{2n-6} x_2^{2n-8} \text{ for } n \geq 5. \tag{3}$$

Proof. To prove (3) we let $q_n(x_1, x_2)$ be the sum of the weights of the perfect matchings of the graph H_n, so that $p_n(x_1, x_2) = q_{2n-4}(x_1, x_2)$ for $n \geq 3$. Each perfect matching of H_n is either a perfect matching of H_{n-1} with an extra vertical edge at the right (of weight 1) or a perfect matching of H_{n-2} with two extra horizontal edges at the right (of weight x_1 or weight x_2, according to whether n is odd or even, respectively). We thus have

$$\begin{aligned} q_{2n} &= q_{2n-1} + x_2^2 q_{2n-2} && (4) \\ q_{2n-1} &= q_{2n-2} + x_1^2 q_{2n-3} && (5) \\ q_{2n-2} &= q_{2n-3} + x_2^2 q_{2n-4}. && (6) \end{aligned}$$

Solving the first and third equations for q_{2n-1} and q_{2n-3}, respectively, and substituting the resulting expressions into the second equation, we get $(q_{2n} - x_2^2 q_{2n-2}) = q_{2n-2} + x_1^2(q_{2n-2} - x_2^2 q_{2n-4})$ or $q_{2n} = (x_1^2 + x_2^2 + 1)q_{2n-2} - x_1^2 x_2^2 q_{2n-4}$, so that we obtain

Lemma 1.

$$p_{n+1} = (x_1^2 + x_2^2 + 1)p_n - x_1^2 x_2^2 p_{n-1}. \tag{7}$$

It is easy enough to verify that

$$p_5 p_3 = ((x_2^2+1)^3 + x_1^4 + 2x_1^2(x_2^2+1)\cdot(x_2^2+1) = \left((x_2^2+1)^2 + x_1^2\right)^2 + x_1^4 x_2^2 = p_4^2 + x_1^4 x_2^2$$

so for induction we assume that

$$p_{n-1}p_{n-3} = p_{n-2}^2 + x_1^{2n-8}x_2^{2n-10} \text{ for } n \geq 5. \tag{8}$$

Using Lemma 1 and (8) we are able to verify that polynomials p_n satisfy the quadratic recurrence relation (3):

$$\begin{aligned} p_n p_{n-2} &= \\ (x_1^2+x_2^2+1)p_{n-1}p_{n-2} - x_1^2x_2^2p_{n-2}^2 &= \\ (x_1^2+x_2^2+1)p_{n-1}p_{n-2} - x_1^2x_2^2(p_{n-1}p_{n-3} - x_1^{2n-8}x_2^{2n-10}) &= \\ p_{n-1}((x_1^2+x_2^2+1)p_{n-2} - x_1^2x_2^2p_{n-3}) + x_1^{2n-6}x_2^{2n-8} &= p_{n-1}^2 + x_1^{2n-6}x_2^{2n-8}. \end{aligned}$$

□

Since $m_n(x_1, x_2) = x_1^{n-2}x_2^{n-3}$ we have that recurrence (2) reduces to recurrence (3) of Proposition 1. Thus $p_n(x_1,x_2)/m_n(x_1,x_2)$ satisfy the same initial conditions and recursion as the x_n's, and we have proven Theorem 1 . □

An explicit formula has recently been found for the $x_n(x_1, x_2)$'s by Caldero and Zelevinsky using the geometry of quiver representations:

Theorem 2. *[4, Theorem 4.1], [14, Theorem 2.2]*

$$x_{-n} = \left(x_1^{2n+2} + \sum_{q+r\leq n} \binom{n+1-r}{q}\binom{n-q}{r} x_1^{2q}x_2^{2r}\right) \Big/ x_1^n x_2^{n+1} \tag{9}$$

$$x_{n+3} = \left(x_2^{2n+2} + \sum_{q+r\leq n} \binom{n-r}{q}\binom{n+1-q}{r} x_1^{2q}x_2^{2r}\right) \Big/ x_1^{n+1} x_2^{n} \tag{10}$$

for all $n \geq 0$.

They also present expressions (Equations (5.16) of [4]) for the x_n's in terms of Fibonacci polynomials, as defined in [8], which can easily seen to be equivalent to the combinatorial interpretation of Theorem 1. Subsequently, Zelevinsky has obtained a short elementary proof of these two results [14].

2.1 Direct combinatorial proof of Theorem 2

Here we provide yet a third proof of Theorem 2: instead of using induction as in Zelevinsky's elementary proof, we use a direct bijection. This proof was found after Zelevinsky's result came to our attention. First we make precise the connection between the combinatorial interpretation of [4] and our own.

Lemma 2. *The number of ways to choose a perfect matching of H_m with $2q$ horizontal edges labeled x_2 and $2r$ horizontal edges labeled x_1 is the number of ways to choose a subset $S \subset \{1, 2, \ldots, m-1\}$ such that S contains q odd elements, r even elements, and no consecutive elements.*

Notice that in the case $m = 2n+2$, this number is the coefficient of $x_1^{2r-n-1} x_2^{2q-n}$ in x_{n+3} (for $n \geq 0$), and when $m = 2n+1$, this number is the coefficient of $x_1^{2r-n} x_2^{2q-n}$ in s_n, as defined in [4, 12, 14].

Proof. There is a bijection between perfect matchings of H_m and subsets $S \subset \{1, 2, \ldots, m-1\}$, with no two elements consecutive. We label the top row of edges of H_m from 1 to $m-1$ and map a horizontal edge in the top row to the label of that edge. Since horizontal edges come in parallel pairs and span precisely two vertices, we have an inverse map as well. $\square$

With this formulation we now prove

Theorem 3. *The number of ways to choose a subset $S \subset \{1, 2, \ldots, N\}$ such that S contains q odd elements, r even elements, and no consecutive elements is*

$$\binom{n+1-r}{q}\binom{n-q}{r}$$

if $N = 2n+1$ and

$$\binom{n-r}{q}\binom{n-q}{r}$$

if $N = 2n$.

Proof. List the parts of S in order of size and reduce the smallest by 0, the next smallest by 2, the next smallest by 4, and so on (so that the largest number gets reduced by $2(q+r-1)$).

This will yield a multiset consisting of q not necessarily distinct odd numbers between 1 and $2n+1-2(q+r-1) = 2(n-q-r+1)+1$ if $N = 2n+1$, and between 1 and $2(n-q-r+1)$ if $N = 2n$, as well as r not necessarily distinct even numbers between 1 and $2(n-q-r+1)$, regardless of whether N is $2n+1$ or $2n$.

Conversely, every such multiset, when you apply the bijection in reverse, you get a set consisting of q odd numbers and r even numbers in $\{1, 2, \ldots, 2n\}$ (resp. $\{1, 2, \ldots, 2n+1\}$), no two of which differ by less than 2.

The number of such multisets is clearly $\left(\!\!\binom{n-q-r+1}{q}\!\!\right) \times \left(\!\!\binom{n-q-r+1}{r}\!\!\right)$ in the first case and $\left(\!\!\binom{n-q-r+2}{q}\!\!\right) \times \left(\!\!\binom{n-q-r+1}{r}\!\!\right)$ in the second case (since $2n+1$ is an additional odd number), where $\left(\!\binom{n}{k}\!\right) =$"$n$ multichoose k" $= \binom{n+k-1}{k}$. Since $\left(\!\!\binom{n-q-r+1}{q}\!\!\right) \times \left(\!\!\binom{n-q-r+1}{r}\!\!\right) = \binom{n-r}{q}\binom{n-q}{r}$ $\left(\text{resp. } \left(\!\!\binom{n-q-r+2}{q}\!\!\right) \times \left(\!\!\binom{n-q-r+1}{r}\!\!\right) = \binom{n+1-r}{q}\binom{n-q}{r}\right)$, the claim follows. $\square$

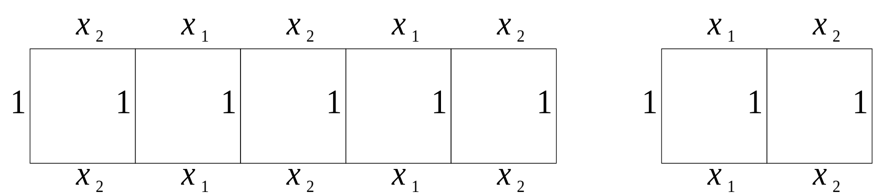
Once we know this interpretation for the coefficients of x_{n+3} (s_n), we obtain a proof of the formula for the entire sum, i.e. Theorem 2 and Theorem 5.2 of [4] (Theorem 2.2 of [14]).

It is worth remarking that the extra terms x_1^{2n+2} and x_2^{2n+2} in Theorem 2 correspond to the extreme case in which one's subset of $\{1, 2, \ldots, N\}$ consists of all the odd numbers in that range.

2.2 Bijective proof of Lemma 1

The recurrence of Lemma 1 can also be proven bijectively by computing in two different ways the sum of the weights of the perfect matchings of the graph $G_n \sqcup H_3$ (the disjoint union of G_n and H_3, which has all the vertices and edges of graphs G_n and H_3 and no identifications). We provide this proof since this method will be used later on in the $(1, 4)$ case.

On the one hand, the sum of the weights of the perfect matchings of G_n is the polynomial p_n and the sum of the weights of the perfect matchings of H_3 is $x_1^2 + x_2^2 + 1$, so the sum of the weights of the perfect matchings of $G_n \sqcup H_3$ is $(x_1^2 + x_2^2 + 1)p_n$.



The sum of the weights of all perfect matchings of $G_5 \sqcup H_3$ is $(x_1^2 + x_2^2 + 1)p_5$.

On the other hand, observe that the graph G_{n+1} can be obtained from $G_n \sqcup H_3$ by identifying the rightmost vertical edge of G_n with the leftmost vertical edge of H_3. Furthermore, there is a weight-preserving bijection ϕ between the set of perfect matchings of the graph G_{n+1} and the set of perfect matchings of $G_n \sqcup H_3$ that do not simultaneously contain the two rightmost horizontal edges of G_n and the two leftmost horizontal edges of H_3 (a set that can also be described as the set of perfect matchings of $G_n \sqcup H_3$ that contain either the rightmost vertical edge of G_n or the leftmost vertical edge of H_3 or both). It is slightly easier to describe the inverse bijection ϕ^{-1}: given a perfect matching of $G_n \sqcup H_3$ that contains either the rightmost vertical edge of G_n or the leftmost vertical edge of H_3 or both, view the matching as a set of edges and push it forward by the gluing map from $G_n \sqcup H_3$ to G_{n+1}. We obtain a multiset of edges of G_{n+1} that contains either 1 or 2 copies of the third vertical edge from the right, and then delete 1 copy of this edge, obtaining a *set* of edges that contains either 0 or 1 copies of that edge. It is not hard to see that this set of edges is a perfect matching of G_{n+1}, and that every perfect matching of G_{n+1} arises from this operation in a unique fashion. Furthermore, since the vertical edge that we have deleted has weight 1, the operation is weight-preserving.

The perfect matchings of $G_n \sqcup H_3$ that are not in the range of the bijection ϕ are those that consist of a perfect matching of G_n that contains the two rightmost horizontal edges of G_n and a perfect matching of H_3 that contains the two leftmost horizontal edges of H_3. Removing these edges yields a perfect matching of G_{n-1} and a perfect matching of H_1. Moreover, every pair consisting of a perfect matching of G_{n-1} and a perfect matching of H_1 occurs in this fashion. Since the four removed edges have weights that multiply to $x_1^2x_2^2$, and H_1 has just a single matching (of weight 1), we see that the perfect matchings excluded from ϕ have total weight $x_1^2x_2^2p_{n-1}$ [3, c.f.].

Remark 1. We can also give a bijective proof of the quadratic recurrence relation (3) by using a technqiue known as graphical condensation which was developed by Eric Kuo [10]. He even gives the unweighted version of this example in his write-up.

Remark 2. As we showed via equation (1), there is a reciprocity that allows us to relate the cluster algebras for the (b, c)- and (c, b)-cases by running the recurrence backwards. For the $(2, 2)$-case, $b = c$ so we do not get anything new when we run it backwards; we only switch the roles of x_1 and x_2. This reciprocity is a special case of the reciprocity that occurs not just for 2-by-n grid graphs, but more generally in the problem of enumerating (not necessarily perfect) matchings of m-by-n grid graphs, as seen in [1] and [11]. For the $(1, 4)$-case, we will also encounter a type of reciprocity.

Remark 3. We have seen that the sequence of polynomials $q_n(x_1, x_2)$ satisfies the relation

$$q_{2n-4}q_{2n-8} = q_{2n-6}^2 + x_1^{2n-6}x_2^{2n-8}.$$

It is worth mentioning that the odd-indexed terms of the sequence satisfy an analogous relation

$$q_{2n-3}q_{2n-7} = q_{2n-5}^2 - x_1^{2n-6}x_2^{2n-8}.$$

This relation can be proven via Theorem 2.3 of [10]. In fact the sequence of Laurent polynomials $\{q_{2n+1}/x_1^nx_2^n : n \geq 0\}$ are the collection of elements of the semicanoncial basis which are not cluster monomials, i.e. not of the form $x_n^px_{n+1}^q$ for $p, q \geq 0$. These are denoted as s_n in [4] and [14] and are defined as $\overline{S}_n(s_1)$ where $\overline{S}_n(x)$ is the normalized Chebyshev polynomial of the second kind, $S_n(x/2)$, and $s_1 = (x_1^2 + x_2^2 + 1)/x_1x_2$. We are thankful to Andrei Zelevinsky for alerting us to this fact. We describe an analogous combinatorial interpretation for the s_n's in the $(1, 4)$-case in subsection 3.4.

3 The $(1, 4)$ case

In this case we let

$$
\begin{aligned}
x_n &= \frac{x_{n-1}+1}{x_{n-2}} \quad \text{for } n \text{ odd} \\
&= \frac{x_{n-1}^4+1}{x_{n-2}} \quad \text{for } n \text{ even}
\end{aligned}
$$

for $n \geq 3$. If we let $x_1 = x_2 = 1$, the first few terms of $\{x_n(1,1)\}$ for $n \geq 3$ are: $2, 17, 9, 386, 43, 8857, 206, 203321, 987, 4667522, 4729, \ldots$

Splitting this sequence into two increasing subsequences, we get for $n \geq 1$:

$$x_{2n+1} = a_n = 2, 9, 43, 206, 987, 4729 \tag{11}$$
$$x_{2n+2} = b_n = 17, 386, 8857, 203321, 4667522. \tag{12}$$

Furthermore, we can run the recurrence backwards and continue the sequence for negative values of n:
$\ldots, 386, 9, 17, 2, 1, 1, 2, 3, 41, 14, 937, 67, 21506, 321, 493697, 1538, 11333521, 7369 \ldots$
whose negative terms split into two increasing subsequences (for $n \geq 1$)

$$x_{-2n+2} = c_n = 2, 41, 937, 21506, 493697, 11333521 \tag{13}$$
$$x_{-2n+1} = d_n = 3, 14, 67, 321, 1538, 7369. \tag{14}$$

As in the $(2,2)$-case, it turns out that this sequence $\{x_n(1,1)\}$ (respectively $\{x_n\}$) has a combinatorial interpretation as the number (sum of the weights) of perfect matchings in a sequence of graphs. We prove that these graphs, which we again denote as G_n, have the x_n's as their generating functions in the later subsections. We first give the unweighted version of these graphs where graph G_n contains $x_n(1,1)$ perfect matchings. We describe how to assign weights to yield the appropriate Laurent polynomials x_n in the next subsection, deferring proof of correctness until the ensuing two subsections. The proof of two recurrences, in sections 3.2 and 3.3, will conclude the proof of Theorem 4. The final subsection provides a combinatorial interpretation for elements of the semicanonical basis that are distinct from cluster monomials.

Definition 2. We will have four types of graphs G_n, one for each of the above four sequences (i.e. for a_n, b_n, c_n, and d_n). Graphs in all four families are built up from squares (consisting of two horizontal and two vertical edges) and octagons (consisting of two horizontal, two vertical, and four diagonal edges), along with some extra arcs. We describe each family of graphs by type.

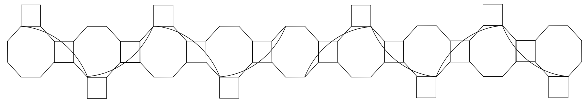
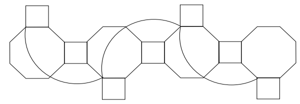
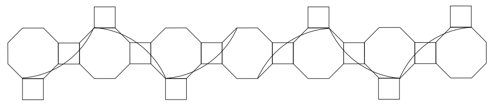
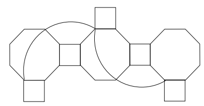
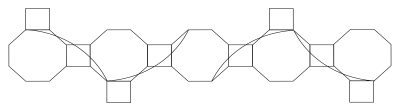
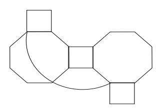
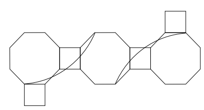
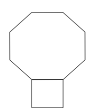
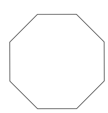
Firstly, G_3 (a_1) is a single square, and G_5 (a_2) is an octagon surrounded by three squares. While the orientation of this graph will not affect the number of perfect matchings, for convenience of describing the rest of the sequence G_{2n+3}, we assume the three squares of G_5 are attached along the eastern, southern, and western edges of the octagon and identify G_3 with the eastern square. For $n \geq 3$ the graph associated to a_{n+1}, G_{2n+3}, can be inductively built from the graph for a_n, G_{2n+1} by attaching a complex consisting of one octagon with two squares attached at its western edge and northern/southern edge (depending on parity). We attach this complex to the western edge of G_{2n+1}, and additionally adjoin one arc between the northeast (resp. southeast) corner of the southern (resp. northern) square of $G_{2n+3} \setminus G_{2n+1}$ and the southeast (resp. northeast) corner of the northern (resp. southern) square of $G_{2n+1} \setminus G_{2n-1}$.

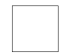
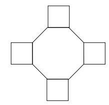
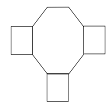
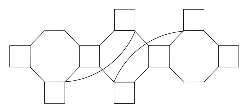
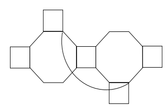
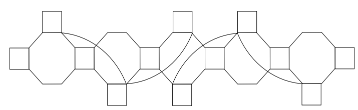
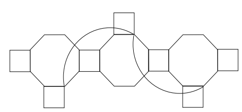
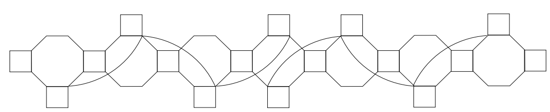
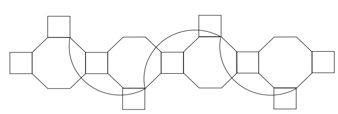
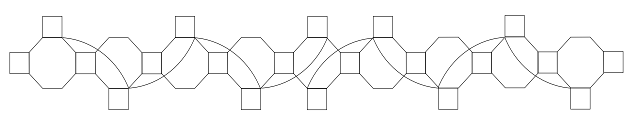
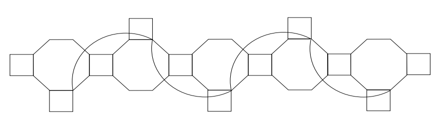
We can inductively build up the sequence of graphs corresponding to the d_n's, G_{-2n-1}, analogously. Here G_{-1}, consists of a single octagon with a single square attached along its northern edge. We attach the same complex (one octagon and two squares) except this time we orient it so that the squares are along the northern/southern and *eastern* edges of the octagon. We then attach the complex so that the eastern square attaches to G_{-2n+1}. Lastly we adjoin one arc between the north*west* (resp. south*west*) corner of the southern (resp. northern) square of $G_{-2n-1} \setminus G_{-2n+1}$ and the south*west* (resp. north*west*) corner of the northern (resp. southern) square of $G_{-2n+1} \setminus G_{-2n+3}$.

The graph corresponding to b_1, G_4, is one octagon surrounded by four squares while the graph corresponding to c_1, G_0, consists of a single octagon. As in the case of a_n or d_n, for $n \geq 1$, the graphs for b_{n+1} (G_{2n+4}) and c_{n+1} (G_{-2n}) are constructed from b_n and c_n (resp.), but this time we add a complex of an octagon, two squares and an arc on *both sides*. Note that this gives these graphs symmetry with respect to rotation by 180°.

The graphs G_{2n+2} (b_n) consist of a structure of octagons and squares such that there are squares on the two ends, four squares around the center octagon, and additional arcs shifted towards the center (the vertices joined by an arc lie on the vertical edges closest to the central octagon). On the other hand, the graphs G_{-2n+2} (c_n) have a structure of octagons and squares such that the central and two end octagons have only two squares surrounding them, and the additional arcs are shifted towards the outside.

For the reader's convenience we illustrate these graphs for small n. In our pictures, the extra arcs are curved, and the other edges are line segments. It should be noted that all these graphs are planar, even though it is more convenient to draw them in such a way that the (curved) extra arcs cross the other (straight) edges.

n	$x_n(1,1)$	Type	Graph G_n
-8	493697	c_5	
-7	321	d_4	
-6	21506	c_4	
-5	67	d_3	
-4	937	c_3	
-3	14	d_2	
-2	41	c_2	
-1	3	d_1	
0	2	c_1	
1	1		x_1
2	1		x_2

3	2	a_1	
4	17	b_1	
5	9	a_2	
6	386	b_2	
7	43	a_3	
8	8857	b_3	
9	206	a_4	
10	203321	b_4	
11	987	a_5	
12	4667522	b_5	
13	4729	a_6	

Remark 4. As described above, the sequence of graphs corresponding to the a_n's and the d_n's can both be built up inductively. In fact, if one assumes that the graphs associated with d_n are "negative" then one can even construct a_1 from d_1 by "adding" two squares and an octagon. The negative square and octagon cancels with the positive square and octagon, leaving only a square for the graph of a_1. (When we construct the graph associated to a_2 from the graph for a_1, we do not add an arc as in the $n \geq 2$ case. Similarly, we omit an extra arc when we construct the graph for a_1 from the graph for d_1. We do not have a "principled" explanation for these exceptions, but we do note that in these two cases, the graph is sufficiently small that there are no candidate vertices to connect by such an arc.)

Comparing graphs with equal numbers of octagons, we find a nice reciprocity between the graph for x_{2n+3} and the graph for x_{-2n+1} for $n \geq 1$. Namely, the two graphs are isomorphic (up to horizontal and vertical reflection) except for the fact that the graph G_{2n+3} contains two squares on the left and right ends while graph G_{-2n+1} lacks these squares. Notice that graph G_3 lies outside this pattern since it contains no octagons and instead contains a single square.

Studying the other types of graphs, i.e. those corresponding to b_n and c_n, we see that in each pair of graphs containing the same number of octagons, there is a nice reciprocity between the two. (Compare the definitions of the graphs associated to the b_n's with those associated to the c_n's.) These two reciprocal relationships reduce into one, i.e. G_{-n} and G_{n+4} are reciprocal graphs for all integers $n \geq 0$. Recall that a reciprocity also exists (between graphs G_{-n} and G_{n+3}) for the (2,2) case, so perhaps these reciprocities are signs of a more general phenomenon.

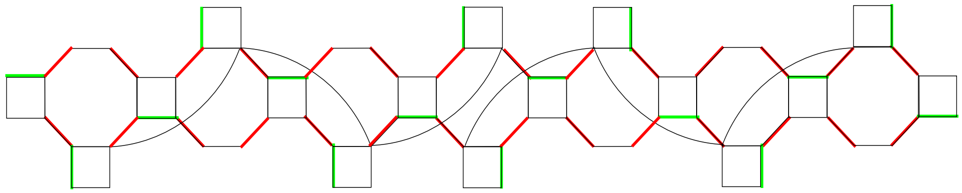
3.1 Weighted versions of the graphs

We now turn to the analysis of the sequence of Laurent polynomials $x_n(x_1, x_2)$, and give the graphs G_n weights on the various edges. In this case, the denominator depends on the number of faces in the graph, ignoring extra arcs. The exponent of x_1 in the denominator will equal the number of squares while the exponent of x_2 will equal the number of octagons. Because of this interpretation, for $n \neq 1$ or 2 we will rewrite x_n as $\frac{p_n(x_1,x_2)}{x_1^{sq(n)} x_2^{oct(n)}}$ where $sq(n)$ and $oct(n)$ are both nonnegative integers. By the description of graphs G_n, we find

$$sq(n) = \begin{cases} |n-1| - 1 & \text{for } n \text{ odd} \\ |2n-2| - 2 & \text{for } n \text{ even} \end{cases} \tag{15}$$

$$oct(n) = \begin{cases} |\frac{n}{2} - 1| - \frac{1}{2} & \text{for } n \text{ odd} \\ |n-2| - 1 & \text{for } n \text{ even.} \end{cases} \tag{16}$$

To construct these weighted graphs, we take the graphs G_n and assign weights such that each of the squares has one edge of weight x_2 and three edges of weight 1 while the octagons have weights alternating between x_1 and 1. As an example, consider the following close-up of the graph associated to x_{10}.



The graph associated to x_{10}.

The vertical and horizontal edges colored in green are given weight x_2, and the diagonal edges marked in red are given weight x_1. All other edges are given weight 1. Notice that the vertical edges weighted x_2 lie furthest away from the arcs and the horizontal edges weighted x_2 alternate between top and bottom, starting with bottom on the righthand side.

Table of x_n for small n:

n	x_n
-3	$\frac{(x_2+1)^3+2x_1^4+3x_1^4x_2+x_1^8}{x_1^3x_2^2}$
-2	$\frac{(x_2+1)^4+3x_1^4+8x_1^4x_2+6x_1^4x_2^2+3x_1^8+4x_1^8x_2+x_1^{12}}{x_1^4x_2^3}$
-1	$\frac{(x_2+1)+x_1^4}{x_1x_2}$
0	$\frac{x_1^4+1}{x_2}$
1	x_1
2	x_2
3	$\frac{x_2+1}{x_1}$
4	$\frac{(x_2+1)^4+x_1^4}{x_1^4x_2}$
5	$\frac{(x_2+1)^3+x_1^4}{x_1^3x_2}$
6	$\frac{(x_2+1)^8+3x_1^4+16x_1^4x_2+34x_1^4x_2^2+36x_1^4x_2^3+19x_1^4x_2^4+4x_1^4x_2^5+3x_1^8+8x_1^8x_2+6x_1^8x_2^2+x_1^{12}}{x_1^8x_2^3}$
7	$\frac{(x_2+1)^5+2x_1^4+5x_1^4x_2+3x_1^4x_2^2+x_1^8}{x_1^5x_2^2}$

We now wish to prove the following.

Theorem 4. *For the case $(b, c) = (1, 4)$, the Laurent polynomials x_n satisfy*

$$x_n(x_1, x_2) = p_n(x_1, x_2)/m_n(x_1, x_2) \text{ for } n \neq 1, 2$$

where p_n is the weighted sum over all perfect matchings in G_n given in definition 2, with weighting as in the preceding paragraph; and $m_n = x_1^{sq(n)} x_2^{oct(n)}$ with $sq(n)$ given by (15) and $oct(n)$ given by (16).

In the above table, we see that Theorem 4 is true for small values of n, thus it suffices to prove that $p_n(x_1, x_2)/x_1^{sq(n)} x_2^{oct(n)}$ satisfies the same periodic quadratic recurrences as Laurent polynomials x_n. By the definition of $m_n(x_1, x_2)$, it suffices to verify the following two recurrences:

$$p_{2n+1}p_{2n+3} = p_{2n+2} + x_1^{|4n+2|-2} x_2^{|2n|-1} \tag{17}$$

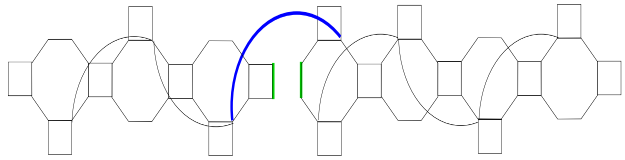
$$p_{2n}p_{2n+2} = p_{2n+1}^4 + x_1^{|8n|-4} x_2^{|4n-2|-2}. \tag{18}$$

3.2 Proof of the first recurrence

We use a decomposition of superimposed graphs to prove equality (17). Unlike Kuo's technique of graphical condensation, we will not use a central graph containing multi-matchings, but will instead use superpositions that only overlap on one edge, as in the bijective proof of Lemma 1. First, let G_{2n+1} be defined as in the previous subsection, and let H_{2n+3} (resp. H_{2n-1}) be constructed by taking graph G_{2n+3} (resp. G_{2n-1}), reflecting it horizontally, and then rotating the leftmost square upwards. For convenience of notation we will henceforth let $M = 2n + 1$ so that we can abbreviate these two cases as $H_{M\pm 2}$ (Throughout this section we choose the sign of $H_{M\pm 2}$, $G_{M\pm 2}$ and $G_{M\pm 1}$ by using H_{2n+3}, G_{2n+3}, and G_{2n+2} if $n \geq 1$ and H_{2n-1}, G_{2n-1} and G_{2n} if $n \leq -1$.) This reflection and rotation will not change the number (or sum of the weights) of perfect matchings. Thus the sum of the weights of perfect matchings in the graph $H_{M\pm 2}$ also equals $p_{M\pm 2}$. Also, we will let K_2 denote the graph consisting of two vertices and a single edge connecting them.

The graph $G_{M\pm 1}$ can be decomposed as the union of the graphs G_M, $H_{M\pm 2}$ and K_2 where G_M and $H_{M\pm 2}$ are joined together on an overlapping edge (the rightmost edge of G_M and the leftmost edge of $H_{M\pm 2}$). The graph K_2 is joined to these graphs so that it connects to the bottom-right (bottom-left) vertex of the rightmost octagon of G_M and the top-right (top-left) vertex of the leftmost octagon of H_{2n+3} (H_{2n-1}). See the picture below for an example with G_{10}. The blue arc in the middle represents K_2. We will denote this decomposition as

$$G_{M\pm 1} \quad = \quad G_M \cup H_{M\pm 2} \cup K_2.$$



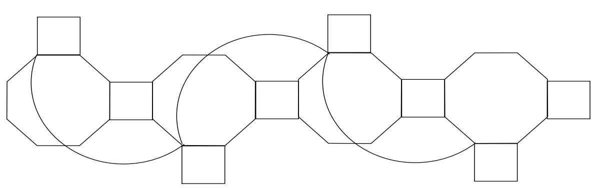
As in subsection 2.2, we let $G \sqcup H$ be the graph formed by the disjoint union of graph G and graph H. A perfect matching of G_M and a perfect matching of $H_{M\pm 2}$ will meet at the edge of incidence in one of four ways (verticals meeting, horizontals meeting vertical, verticals meeting diagonals, or horizontals meeting diagonals).

In three of the cases, edges of weight 1 are utilized, and we can bijectively associate a perfect matching of $G_M \sqcup H_{M\pm 2}$ to a perfect matching of $G_{M\pm 1}$ by removing an edge of weight 1 on the overlap; though it is impossible to map to a perfect matching of $G_{M\pm 1}$ that uses the edge of K_2 in this way. This bijection is analogous to the one discussed in subsection 2.2. Thus we have a weight-preserving bijection between {perfect matchings of $G_{M\pm 1} - K_2$ } and the set {perfect matchings of $G_M \sqcup H_{M\pm 2}$} − {pairs with nontrivial incidence (horizontals meeting diagonals) } where by abuse of notation we here and henceforth let $G - K_2$ refer to the subgraph of G with K_2's edge deleted (without deleting any vertices). Thus proving $p_{2n+1}p_{2n+3} - p_{2n+2} = p_M p_{M+2} - p_{M+1} = x_1^{|4n+2|-2} x_2^{|2n|-1}$ reduces to proving the following claim.

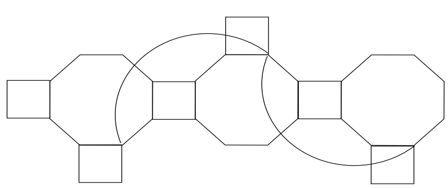
Proposition 2. *The sum of the weights of all perfect matchings of $G_{M\pm 1}$ that contains K_2 is $x_1^{|4n+2|-2} x_2^{|2n|-1}$ less than the sum of the weights of all perfect matchings of $G_M \sqcup H_{M\pm 2}$ that have nontrivial incidence.*

Before giving the proof of this Proposition we introduce a new family of graphs that will allow us to write out several steps of this proof more elegantly. For $n \geq 1$, we let $\tilde{G}_{2n+1}$ be the graph obtained from G_{2n+1} by *deleting* the outer square on the extreme right. If $n \leq -1$, we let $\tilde{G}_{2n+1}$ be the graph obtained from G_{2n+1} by *adjoining* an outer square on the extreme right. For example, $\tilde{G}_9$ and $\tilde{G}_{-7}$ are shown below. We let $\tilde{p}_{2n+1}$ be the sum of the weights of perfect matchings in $\tilde{G}_{2n+1}$. Notice that this construction creates a reciprocity such that graphs $\tilde{G}_{-M} = \tilde{G}_{-2n-1}$ and $\tilde{G}_{M+4} = \tilde{G}_{2n+5}$ are isomorphic.

$\tilde{G}_{-7}$



$\tilde{G}_9$



The polynomials p_{2n+1} and $\tilde{p}_{2n+1}$ are related in a very simple way.

Lemma 3.

$$p_{2n+1} = (x_2+1)\tilde{p}_{2n+1} - x_1^4 x_2 \tilde{p}_{2n-1} \qquad \text{for } n \geq 2, \tag{19}$$
$$p_{2n+1} = (x_1^4+x_2+1)\tilde{p}_{2n+3} - x_1^4 x_2^2 \tilde{p}_{2n+5} \qquad \text{for } n \leq -2. \tag{20}$$

Proof. The proof of Lemma 3 follows the same logic as the inclusion-exclusion argument of section 2.2 that proved Lemma 1. In this case, we use the fact that we can construct G_{2n+1} by adjoining the graph G_3 (resp. G_{-1}) to $\tilde{G}_{2n+1}$ (resp. $\tilde{G}_{2n+3}$) on the right for $n \geq 2$ (resp. $n \leq -2$). It is clear that $p_3 = x_2 + 1$ (resp. $p_{-1} = x_1^4 + x_2 + 1$) and the only perfect matchings we must exclude are those that contain a pair of diagonals meeting a pair of horizontals. For the $n \leq -2$ case, we must also add in those perfect matchings that use the rightmost arc of G_{2n+1}. □

Proof. (Prop. 2) By analyzing how the inclusion of certain key edges in a perfect matching dictates how the rest of the perfect matching must look, we arrive at the expressions

$$x_1^4 x_2(x_2+1)p_{2n-1}\tilde{p}_{2n+1} + x_1^8 x_2^2 \tilde{p}_{2n-3}\tilde{p}_{2n+1} \quad \text{for } n \geq 2 \tag{21}$$
$$x_1^4 x_2 p_{2n+3}\tilde{p}_{2n+1} + x_1^4 x_2^2 \tilde{p}_{2n+5}\tilde{p}_{2n+1} \quad \text{for } n \leq -2 \tag{22}$$

for the sum of the weights of all perfect matchings of $G_M \sqcup H_{M\pm 2}$ with nontrivial incidence (horizontals meeting diagonals), and expressions

$$x_1^4 x_2(x_2+1)\tilde{p}_{2n-1}p_{2n+1} + x_1^8 x_2^2 \tilde{p}_{2n-1}^2 \quad \text{for } n \geq 2 \tag{23}$$
$$x_1^4 x_2 \tilde{p}_{2n+3}p_{2n+1} + x_1^4 x_2^2 \tilde{p}_{2n+3}^2 \quad \text{for } n \leq -2 \tag{24}$$

for the sum of the weights of all perfect matchings of $G_{M\pm 1}$ using the K_2's edge.

To prove Proposition 2, it suffices to prove (21) = (23) + $x_1^{4n}x_2^{2n-1}$ and (22) = (24) + $x_1^{-4n-4}x_2^{-2n-1}$. To do so, we prove equalities

$$x_1^4 x_2(x_2+1)(p_{2n-1}\tilde{p}_{2n+1} - p_{2n+1}\tilde{p}_{2n-1}) = x_1^{4n}x_2^{2n-2} + x_1^{4n}x_2^{2n-1} \tag{25}$$
$$x_1^8 x_2^2(\tilde{p}_{2n-1}^2 - \tilde{p}_{2n+1}\tilde{p}_{2n-3}) = x_1^{4n}x_2^{2n-2} \tag{26}$$

for $n \geq 2$ and equalities

$$x_1^4 x_2(p_{2n+3}\tilde{p}_{2n+1} - p_{2n+1}\tilde{p}_{2n+3}) = x_1^{-4n}x_2^{-2n} + x_1^{-4n}x_2^{-2n+1} \tag{27}$$
$$x_1^4 x_2^2(\tilde{p}_{2n+3}^2 - \tilde{p}_{2n+1}\tilde{p}_{2n+5}) = x_1^{-4n}x_2^{-2n} \tag{28}$$

for $n \leq -2$ and then subtract (25) – (26) and (27) – (28). After shifting indices and dividing both sides of equations (25) through (28) to normalize, we obtain that it suffices to prove

Lemma 4.

$$\begin{aligned}
p_{2n-1}\tilde{p}_{2n+1} - p_{2n+1}\tilde{p}_{2n-1} &= x_1^{4n-4}x_2^{2n-3} && \textit{for } n \geq 2 && (29)\\
&= -x_1^{-4n}x_2^{-2n+1}(x_2+1) && \textit{for } n \leq -2 && (30)\\
\tilde{p}_{2n+1}^2 - \tilde{p}_{2n-1}\tilde{p}_{2n+3} &= x_1^{4n-4}x_2^{2n-2} && \textit{for } n \geq 2 && (31)\\
&= x_1^{-4n}x_2^{-2n} && \textit{for } n \leq -2. && (32)
\end{aligned}$$

We prove equations (31) and (32) simultaneously since $\tilde{p}_{2n+1} = \tilde{p}_{-2n+3}$. We prove (31) by making superimposed graphs involving $\tilde{G}_{2n-1}$ and $\tilde{G}_{2n+3}$ and comparing it to the superimposed graph of $\tilde{G}_{2n+1}$ with itself. Perhaps Kuo's technique could be adapted to prove (31), but instead we consider a superposition overlapping over one edge, as we did earlier in this proof. Our superimposed graph thus resembles $\tilde{G}_{4n-1}$, with a double edge somewhere in the middle and a missing arc. Analogous to the analysis that allowed us to reduce from recurrence (17) to Proposition 2, we reduce our attention to the cases where gluing together $\tilde{G}_{2n-1}$ and $\tilde{G}_{2n+3}$ and decomposing back into two copies of $\tilde{G}_{2n+1}$ would not be allowed (or vice versa). This entails focusing on cases where horizontals meet diagonals at the double edge, or the arc appearing exclusively in that decomposition (and not the other) appears in the matching.

After accounting for the possible perfect matchings, we find the following two expressions

$$\tilde{p}_{2n-1}^2 x_1^4 x_2(x_2+1) + \tilde{p}_{2n-3}\tilde{p}_{2n+1}x_1^4x_2 \tag{33}$$

$$\tilde{p}_{2n-3}\tilde{p}_{2n+1}x_1^4x_2(x_2+1) + \tilde{p}_{2n-1}^2x_1^4x_2 \tag{34}$$

which represent the sum of the weights of nontrivial perfect matchings in the superpositions of $\tilde{G}_{2n+1}$ and $\tilde{G}_{2n+1}$ (resp. $\tilde{G}_{2n-1}$ and $\tilde{G}_{2n+3}$). Taking the difference of these two expressions, we find that

$$\tilde{p}_{2n+1}^2 - \tilde{p}_{2n-1}\tilde{p}_{2n+3} = x_1^4x_2^2(\tilde{p}_{2n-1}^2 - \tilde{p}_{2n-3}\tilde{p}_{2n+1}).$$

So after a simple check of the base case ($\tilde{p}_5^2 - \tilde{p}_3\tilde{p}_7 = x_1^4x_2^2$) we get equation (31) by induction.

We easily derive (29) from equations (19) and (31):

$$\begin{aligned}
p_{2n-1}\tilde{p}_{2n+1} - p_{2n+1}\tilde{p}_{2n-1} &= \\
((x_2+1)\tilde{p}_{2n-1} - x_1^4x_2\tilde{p}_{2n-3})\tilde{p}_{2n+1} - ((x_2+1)\tilde{p}_{2n+1} - x_1^4x_2\tilde{p}_{2n-1})\tilde{p}_{2n-1} &= \\
x_1^4x_2(\tilde{p}_{2n-1}^2 - \tilde{p}_{2n-3}\tilde{p}_{2n+1}) &= x_1^{4n-4}x_2^{2n-3}.
\end{aligned}$$

We can also derive (30) from (32) but we first need to prove the following Lemma:

Lemma 5.

$$\tilde{p}_{2n+1}\tilde{p}_{2n-1} - \tilde{p}_{2n-3}\tilde{p}_{2n+3} = x_1^{4n-8}x_2^{2n-4}(x_1^4 + (x_2+1)^2) \quad \text{for } n \geq 2, \tag{35}$$

$$\tilde{p}_{2n+1}\tilde{p}_{2n-1} - \tilde{p}_{2n-3}\tilde{p}_{2n+3} = x_1^{-4n}x_2^{-2n}(x_1^4 + (x_2+1)^2) \quad \text{for } n \leq -2. \tag{36}$$

Proof. We prove this Lemma by using the the same technique that we used to prove equation (31). Analogously, they can be proven simultaneously by proving (35) because of the reciprocity $\tilde{p}_{2n+1} = \tilde{p}_{-2n+3}$. In this case, the superimposed graph resembles $\tilde{G}_{4n-1}$ and by considering nontrivial perfect matchings, we obtain that

$$\tilde{p}_{2n+1}\tilde{p}_{2n-1} - \tilde{p}_{2n-3}\tilde{p}_{2n+3} = x_1^4 x_2^2(\tilde{p}_{2n-1}\tilde{p}_{2n-3} - \tilde{p}_{2n-5}\tilde{p}_{2n+1}).$$

Since

$$\tilde{p}_5\tilde{p}_7 - \tilde{p}_3\tilde{p}_9 = x_1^4 x_2^2(x_1^4 + (x_2+1)^2),$$

we have the desired result by induction. $\square$

We thus can verify (30) algebraically by using (20), (32), and (36). Since the proof of equations (29) through (32) was sufficient, thus recurrence (17) is proven for $n \neq 0$ or -1. $\square$

3.3 Proof of the second recurrence

We now prove the recurrence (18) via the following two observations.

Lemma 6.

$$\begin{aligned} p_{2n-1}p_{2n+3} - p_{2n+1}^2 &= x_1^{4n-4}x_2^{2n-3}(x_1^4 + (x_2+1)^2) \qquad \text{for } n \geq 2 \qquad (37)\\ &= x_1^{-4n-4}x_2^{-2n-1}(x_1^4 + (x_2+1)^2) \qquad \text{for } n \leq -1. \qquad (38)\end{aligned}$$

Lemma 7.

$$\begin{aligned} p_{2n+3} &= (x_1^4 + (x_2+1)^2)p_{2n+1} - x_1^4x_2^2 p_{2n-1} \qquad \text{for } n \geq 2, \qquad (39)\\ p_{2n-1} &= (x_1^4 + (x_2+1)^2)p_{2n+1} - x_1^4x_2^2 p_{2n+3} \qquad \text{for } n \leq -1. \qquad (40)\end{aligned}$$

Proof. (Lemma 6) We easily derive (37) from Lemmas 3, 4 and 5 by the following derivation,

$$\begin{aligned} &p_{2n-1}p_{2n+3} - p_{2n+1}^2\\ &\quad = ((x_2+1)\tilde{p}_{2n-1} - x_1^4x_2\tilde{p}_{2n-3})((x_2+1)\tilde{p}_{2n+3} - x_1^4x_2\tilde{p}_{2n+1}) - ((x_2+1)\tilde{p}_{2n+1} - x_1^4x_2\tilde{p}_{2n-1})^2\\ &\quad = (x_2+1)^2(\tilde{p}_{2n-1}\tilde{p}_{2n+3} - \tilde{p}_{2n+1}^2) \ + x_1^8x_2^2(\tilde{p}_{2n-3}\tilde{p}_{2n+1} - \tilde{p}_{2n-1}^2) \ - x_1^4x_2(x_2+1)(\tilde{p}_{2n-3}\tilde{p}_{2n+3} - \tilde{p}_{2n+1}\tilde{p}_{2n-1})\\ &\quad = -(x_2+1)^2(x_1^{4n-4}x_2^{2n-2}) - x_1^8x_2^2(x_1^{4n-8}x_2^{2n-4} + x_1^4x_2(x_2+1)(x_1^4+(x_2+1)^2)x_1^{4n-8}x_2^{2n-4})\\ &\quad = x_1^{4n-4}x_2^{2n-3}(x_1^4 + (x_2+1)^2).\end{aligned}$$

The proof of (38) is similar and thus Lemma 6 is proved. $\square$

Proof. (Lemma 7) Like Lemma 3, Lemma 7 can also be proven using an inclusion-exclusion argument. This one relies on the fact that G_{2n+3} is inductively built from G_{2n+1} by adjoining an octagon, an arc, and two squares. Graph $\tilde{G}_5 \sqcup G_{2n+1}$ has $(x_1^4 + (x_2+1)^2)p_{2n+1}$ as the sum of the weight of all its perfect matchings. Most perfect matchings of $\tilde{G}_5 \sqcup G_{2n+1}$ map to a perfect matching of G_{2n+3} (resp. G_{2n-1}) with the same weight. The only perfect matchings that do not participate in the bijection are those that contain a pair of diagonals meeting a pair of horizontals. The sum of the weights of all such perfect matchings is $x_1^4x_2(x_2+1)p_{2n-1}$ (resp. $x_1^4x_2(x_2+1)p_{2n+3}$). However, we have neglected the perfect matchings of G_{2n+3} (resp. G_{2n-1}) that use the one arc not appearing in $\tilde{G}_5 \sqcup G_{2n+1}$. Correcting for this we add back $x_1^4x_2p_{2n-1}$ (resp. $x_1^4x_2p_{2n+3}$). After these subtractions and additions we do indeed obtain equation (39) for $n \geq 2$ (resp. (40) for $n \leq -1$). $\square$

We now are ready to prove (18). Using the first recurrence, (17), we can rewrite the lefthand side of (18) as

$$(p_{2n-1}p_{2n+1} - x_1^{|4n-2|-2}x_2^{|2n-2|-1})(p_{2n+1}p_{2n+3} - x_1^{|4n+2|-2}x_2^{|2n|-1})$$

which reduces to

$$p_{2n+1}^2(p_{2n-1}p_{2n+3}) - p_{2n+1}x_1^{|4n-2|-2}x_2^{|2n-2|-1}(p_{2n+3} + x_1^4x_2^2p_{2n-1}) + x_1^{|8n|-4}x_2^{|4n-2|-2}.$$

Using Lemma 6 and Lemma 7, this equation simplifies to $p_{2n+1}^4 + x_1^{|8n|-4}x_2^{|4n-2|-2}$. Thus the recurrence (18) is proven for $n \neq 0$ or 1. We have thus proven Theorem 4 .

3.4 A combinatorial interpretation for the semicanonical basis

It was shown in [12] that a canonical basis for the positive cone consists of cluster monomials, that is monomials of the form $x_n^p x_{n+1}^q$, as well as one additional sequence of elements, in the affine ((2, 2) or (1, 4)) case. One can think of this extraneous sequence as corresponding to the imaginary roots of the Kac Moody algebra, which are of the form $n\delta$ where $\delta = \alpha_1 + \alpha_2$ in the (2, 2) case and $2\alpha_1 + \alpha_2$ in the (1, 4) case.

As described in [12], this sequence completes the canonical basis, and is closely related to the sequence of s_n's which completes the *semicanonical* basis. The s_n's are defined as the normalized Chebyshev polynomials of the second kind in variable $z_1 = s_1 = (x_1^4 + (x_2+1)^2)/x_1^2x_2$ just as they were in the (2, 2)-case [12]. Using this definition, we obtain a combinatorial interpretation for the s_n's in the (1, 4)-case just as we did in the (2, 2)-case. In both cases, the non-cluster monomial elements of the semicanonical basis were discovered as an auxiliary sequence in the proof of x_n's combinatorial interpretation.

Theorem 5. *In the* $(1,4)$*-case, the Laurent polynomials* $s_n(x_1, x_2)$*, defined as* $\overline{S}_n(z_1)$*, are precisely* $\tilde{p}_{2n+3}(x_1, x_2)/\tilde{m}_{2n+3}(x_1, x_2)$ *where the* $\tilde{p}_{2n+3}$*'s are defined in section 3.2 (between Proposition 2 and Lemma 3) and* $\tilde{m}_{2n+3}(x_1, x_2) = x_1^{2n}x_2^n$.

Therefore the s_n's have a combinatorial interpretation in terms of the graphs $\tilde{G}_{2n+1}$ of section 3.2.

Proof. Our method of proof is analogous to our proof of Lemmas 1 and 3. We note that a perfect matching of $\tilde{G}_{2n+3}$ can be decomposed into a perfect matching of $\tilde{G}_{2n+1}$ and $\tilde{G}_5$ (with the graph $\tilde{G}_5$ on the righthand side) or will utilize the rightmost arc. However, there is not a bijection between perfect matchings avoiding the rightmost arc and perfect matchings of $\tilde{G}_{2n+1} \sqcup \tilde{G}_5$ since we must exclude those matchings of $\tilde{G}_{2n+1} \sqcup \tilde{G}_5$ that use the two rightmost horizontal edges of $\tilde{G}_{2n+1}$ and the two diagonal edges of $\tilde{G}_5$. In conclusion, we obtain

$$\begin{aligned} \tilde{p}_{2n+3} &= \tilde{p}_{2n+1}\tilde{p}_5 + x_1^4 x_2 \tilde{p}_{2n-1} - x_1^4 x_2(x_2+1)\tilde{p}_{2n-1} & (41) \\ &= \tilde{p}_{2n+1}\tilde{p}_5 - x_1^4 x_2^2 \tilde{p}_{2n-1}. & (42) \end{aligned}$$

After dividing by $\tilde{m}_{2n+3}$, $\tilde{m}_{2n+1}$, $\tilde{m}_{2n-1}$, and $\tilde{m}_5$ accordingly, equation (42) reduces to

$$\frac{\tilde{p}_{2n+3}}{\tilde{m}_{2n+3}} = \frac{\tilde{p}_5}{\tilde{m}_5}\frac{\tilde{p}_{2n+1}}{\tilde{m}_{2n+1}} - \frac{\tilde{p}_{2n-1}}{\tilde{m}_{2n-1}} \qquad (43)$$

and thus the $\tilde{p}_{2n+3}/\tilde{m}_{2n+3}$'s satisfy the same recurrence as the normalized Chebyshev polynomials of the second kind [4, Equation (3.2)] or [14, Equation (2)]; and thus the same recurrence as the s_n's. $\square$

4 Comments and open problems

In both the (2,2)- and (1,4)- cases, we have now shown that the sequence of Laurent polynomials $\{x_n\}$ defined by the appropriate recurrences have numerators with positive coefficients. A deeper combinatorial understanding of the $(1,4)$ case might, in combination with what is already known about the other cases of (b,c) with $bc \leq 4$, give us important clues into how one might construct suitable graphs G_n for the cases $bc > 4$.

Another direction follows from the combinatorial interpretation of semicanonical basis elements in the $(1,4)$-case. This analysis motivates the question of whether or not one can find a family of graphs such that the sum of the weights of their perfect matchings are precisely the numerators of the z_n's. This would give combinatorial interpretation to not only the semicanonical basis, but also the canonical basis.

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