Truncations of Random Unitary Matrices and Young Tableaux

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Abstract

Let U be a matrix chosen randomly, with respect to Haar measure, from the unitary group $U(d)$. For any $k \leq d$, and any $k \times k$ submatrix U_k of U, we express the average value of $|\operatorname{Tr}(U_k)|^{2n}$ as a sum over partitions of n with at most k rows whose terms count certain standard and semistandard Young tableaux. We combine our formula with a variant of the Colour-Flavour Transformation of lattice gauge theory to give a combinatorial expansion of an interesting family of unitary matrix integrals. In addition, we give a simple combinatorial derivation of the moments of a single entry of a random unitary matrix, and hence deduce that the rescaled entries converge in moments to standard complex Gaussians. Our main tool is the Weingarten function for the unitary group.

1 Introduction and Statement of the Main Theorem

Consider the unitary group $U(d)$ as a probability space under normalized Haar measure dU. Given a random variable $X : U(d) \to \mathbb{C}$, its expected value is defined to be

$$\mathbb{E}_{U(d)}(X) = \int_{U(d)} X\,dU.$$

When studying a random variable X, one often wishes to know its moments

$$\mathbb{E}_{U(d)}(X^m \overline{X}^n),$$

since in many situations the moments of X uniquely determine its distribution. X is called a polynomial random variable if it is polynomial in the entries of U, i.e. if there is a polynomial $f \in \mathbb{C}[x_{11}, \ldots, x_{dd}]$ such that

$$X(U) = f(u_{11}, \ldots, u_{dd})$$

for all $U \in U(d)$. If X is a polynomial random variable, then

$$\mathbb{E}_{U(d)}(X^m \overline{X}^n) = 0 \text{ if } m \neq n,$$

(see [1]), so knowledge of the moments of X reduces to knowledge of the quantities

$$\mathbb{E}_{U(d)}(|X|^{2n}).$$

It has recently been shown that certain polynomial random variables on the unitary group encode interesting combinatorial information via their moments. For a random matrix $U \in U(d)$, write its characteristic polynomial as

$$\det(U - zI) = (-1)^d \sum_{j=0}^{d} (-1)^j \operatorname{Sc}_j(U) z^{d-j}.$$

$\operatorname{Sc}_j(U)$ is called the j-th *secular coefficient* of U. In particular,

$$\operatorname{Sc}_1(U) = \operatorname{Tr}(U)$$
$$\operatorname{Sc}_d(U) = \det(U).$$

The combinatorial significance of the secular coefficients of a random Haar-distributed unitary matrix is that their moments enumerate magic squares. It is shown in [2] that for $jn \leq d$,

$$\mathbb{E}_{U(d)}(|\operatorname{Sc}_j(U)|^{2n}) = H_n(j),$$

where $H_n(j)$ is the number of $n \times n$ matrices whose entries are nonnegative integers and whose rows and columns all sum to j.

For the $Sc_1(U) = \operatorname{Tr}(U)$, this reduces to the fact that for $n \leq d$, $\mathbb{E}_{U(d)}(|\operatorname{Tr}(U)|^{2n}) = n!$, the number of $n \times n$ permutation matrices. The natural question of what happens when the condition $n \leq d$ is removed was answered by Rains in [6], where it was proved that for any d and n

$$\mathbb{E}_{U(d)}(|\operatorname{Tr}(U)|^{2n}) = \sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f^\lambda)^2,$$

where the sum is over all partitions λ of n and f^λ denotes the number of Standard Young Tableaux of shape λ. The significance of this result is that, by the RSK correspondence, $\sum_{\lambda \vdash n, \ell(\lambda) \leq d} (f^\lambda)^2$ is the number $f(n,d)$ of permutations in the symmetric group S_n having no increasing subsequence of length greater than d. Thus Rains's result is equivalent to the remarkable fact that the exponential generating series for $f(n,d)$ has a "closed form" as an integral over the unitary group $U(d)$:

$$\sum_{n=0}^{\infty} \frac{f(n,d)}{n!} \frac{z^n}{n!} = \int_{U(d)} e^{\sqrt{z} \operatorname{Tr}(U+U^*)} dU.$$

The elegant method of proof employed in [2], [6] consists of two main ingredients:

- The secular coefficients of U are the elementary symmetric functions applied to the eigenvalues of U.
- The elementary symmetric functions can be written as linear combinations of Schur functions, which are the irreducible characters of the unitary group and thus satisfy orthogonality relations.

In this paper we will study polynomial random variables on $U(d)$ that are not symmetric functions of the eigenvalues. For example, suppose that we are given a random matrix $U \in U(d)$, and we want to calculate the moments of a single entry of U

$$\mathbb{E}_{U(d)}(|u_{ij}|^{2n})$$

(note that since the permutation matrices are in $U(d)$ and Haar measure is unitary invariant, all entries of U are equidistributed). More generally, for a positive integer k with $1 \leq k \leq d$, let U_k denote the $k \times k$ upper left corner of U. We could ask about the moments of the matrix U_k

$$\mathbb{E}_{U(d)}(|\operatorname{Tr}(U_k)|^{2n})$$

(again, since the permutation matrices are in $U(d)$, the traces of any two $k \times k$ submatrices of U are equidistributed). The methods used in [6], [2] will not work in this situation, since we are no longer dealing with the eigenvalues of a unitary matrix, but rather the eigenvalues of one of its submatrices, so that Schur orthogonality is not directly applicable. However, quite surprisingly, there is a combinatorial formula for these moments that maintains the structure of that for the full matrix case.

Theorem 1. *Let U be a matrix chosen randomly with respect to Haar measure from the unitary group $U(d)$, and let U_k be its $k \times k$ upper left corner. We have*

$$\mathbb{E}_{U(d)}(|\operatorname{Tr}(U_k)|^{2n}) = \sum_{\lambda \vdash n, \ell(\lambda) \leq k} (f^\lambda)^2 \frac{s_\lambda(1^k)}{s_\lambda(1^d)}.$$

Equivalently, for any $z \in \mathbb{C}$ we have

$$\int_{U(d)} e^{\sqrt{z}\operatorname{Tr}(U_k + U_k^*)} dU = \sum_{n=0}^{\infty} \left(\sum_{\lambda \vdash n, \ell(\lambda) \leq k} \frac{(f^\lambda)^2}{n!} \frac{s_\lambda(1^k)}{s_\lambda(1^d)} \right) \frac{z^n}{n!}.$$

Here, for a positive integer r, $s_\lambda(1^r)$ denotes the number of Semistandard Young Tableaux on the shape λ with entries from the set $[r] = \{1, \ldots, r\}$. The notation $s_\lambda(1^r)$ is shorthand for the value of the Schur function s_λ obtained by setting the first r variables equal to 1 and making the remaining variables 0.

Note that we recover Rains's result by setting $k = d$ in Theorem 1.

2 Applications of the Main Theorem

Before proving Theorem 1, we will give two applications. First we use the Colour-Flavour Transformation of lattice gauge theory to give another formulation of Theorem 1, which gives a combinatorial expansion for an interesting type of matrix integral. Then we use Theorem 1 to give simple combinatorial proofs of known probabalistic results regarding the moments and limiting distribution of the individual entries of a random unitary matrix.

The matrix U_k is called the kth *truncation* of U. Truncations of random unitary matrices have been the subject of several recent articles (see [3], [5], [8]). Interest in truncations of random unitary matrices stems from their relation to the physics of quantum chaotic scattering [8].

Truncations of random unitary matrices are closely related to the so-called *Colour-Flavour Transformation* (CFT) from physics introduced in [11]. As explained in [10], the CFT trades an integration over a "colour" gauge group for an integration over a certain supersymmetric coset space, or "flavour" space. The physical terminology comes from lattice gauge theory. There are several versions of the CFT, see [3] for details. The one that we will use can be formulated mathematically as follows (see [3], [10]). Let $k \leq d$, and let $X, Y \in \mathbb{C}^{d\times k}$ be constant matrices. Then

$$\int_{U(d)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)}dU = \int_{U(k)} e^{\mathrm{Tr}(X^*XV^*+VY^*Y)}\det(VY^*X)^{k-d}dV.$$

This variant of the CFT shows a certain recursive property of integration on the unitary group. In [10], the interesting observation was made that the integral on the left does not change if we replace the unitary group $U(d)$ with the special unitary group $SU(d)$ provided $k < d$:

$$\int_{U(d)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)}dU = \int_{SU(d)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)}dU, \ k < d.$$

Now let x be a nonnegative real number and take $X(x) = Y(x) \in \mathbb{C}^{d\times k}$ to be the matrix with $x^{\frac{1}{4}}$ along the main diagonal and 0's elsewhere. Then

$$\begin{aligned} Y(x)^*UX(x) &= \sqrt{x}U_k \\ X(x)^*U^*Y(x) &= \sqrt{x}U_k^* \\ VY(x)^*X(x) &= \sqrt{x}V. \end{aligned}$$

Thus applying the CFT yields

Corollary 2.1. *Let k, d be positive integers with $k \leq d$, and let x be a nonnegative real number. Define*

$$f(n,d,k) := \sum_{\lambda\vdash n, \ell(\lambda)\leq k} \frac{(f^\lambda)^2}{n!}\frac{s_\lambda(1^k)}{s_\lambda(1^d)}.$$

The exponential generating series for $f(n,d,k)$ has two different closed forms, one as an integral over $U(d)$ and one as an integral over $U(k)$:

$$\sum_{n=0}^{\infty} f(n,d,k)\frac{x^n}{n!} = \int_{U(d)} e^{\sqrt{x}\operatorname{Tr}(U_k+U_k^*)} = x^{\frac{k(k-d)}{2}} \int_{U(k)} e^{\sqrt{x}\operatorname{Tr}(V+V^*)} \det(V)^{k-d} dV.$$

Also, in the range $k < d$, we have a third closed form as an integral over the group $SU(d)$:

$$\sum_{n=0}^{\infty} f(n,d,k)\frac{x^n}{n!} = \int_{SU(d)} e^{\sqrt{x}\operatorname{Tr}(U_k+U_k^*)} dU.$$

We remarked in the Introduction that the extreme case $k=d$ of Theorem 1 is Rains's result, and is of relevance to the longest increasing subsequence problem. The other extreme $k=1$ is also of interest, since this corresponds to the computation of the moments of a single entry of a random unitary matrix.

Corollary 2.2. *In the special case $k=1$, we have*

$$\mathbb{E}_{U(d)}(|u_{ij}|^{2n}) = \frac{n!}{d(d+1)\dots(d+n-1)} = \binom{d+n-1}{n}^{-1}.$$

Proof. For $k=1$ the only contribution to the sum is made by the single partition whose diagram is a row of n boxes. Thus we have

$$\int_{U(d)} |u_{ij}|^{2n} dU = \frac{1}{s_n(1^d)}.$$

The generalized hook length formula asserts that

$$s_\lambda(1^d) = \prod_{\Box \in \lambda} \frac{d + c(\Box)}{h(\Box)},$$

where $c(\Box)$ is the content of the box, and $h(\Box)$ is its hook length (see [9]). For the single row partition of n, this gives

$$s_n(1^d) = \frac{d(d+1)\dots(d+n-1)}{n!},$$

and the result follows. ☐

Explicitly knowing the moments of u_{ij} makes it easy to determine its limiting distribution. Recall that if x, y are Gaussian random variables with mean 0 and variance $1/2$, then the random variable $z = x + iy$ is called a standard complex Gaussian.

Corollary 2.3. *As $d \to \infty$, the random variable $\sqrt{d}u_{ij}$ converges in moments to a standard complex Gaussian random variable.*

Proof. It is well-known (see for instance [5]) that the moments of a standard complex Gaussian z are given by

$$\mathbb{E}(z^m \overline{z}^n) = \delta_{mn} n!.$$

Corollary 1.2 shows that

$$\begin{aligned}
\mathbb{E}_{U(d)}((\sqrt{d}u_{ij})^m(\sqrt{d}\overline{u}_{ij})^n) &= \delta_{mn}\frac{d^n n!}{d(d+1)\dots(d+n-1)} \\
&= \delta_{mn}\frac{n!}{(1)(1+\frac{1}{d})\dots(1+\frac{n-1}{d})} \\
&\to \delta_{mn} n!.
\end{aligned}$$

□

3 The Weingarten Function

In order to prove Theorem 1, one needs to connect unitary expectations to symmetric function theory by some method other than applying symmetric functions to eigenvalues. This can be done using the Weingarten function introduced in [1], which is a powerful tool for computing the moments of polynomial random variables on the unitary group. The Weingarten function has already been used in free probability theory to prove asymptotic freeness results for random unitary matrices (see the recent book [4] for a clear account of this).

For any positive integers d and n, define a function $\mathrm{Wg}(d, n, \cdot) : S_n \to \mathbb{Q}$ by

$$\mathrm{Wg}(d, n, \sigma) := \frac{1}{n!^2} \sum_{\lambda \vdash n,\ \ell(\lambda) \leq d} \frac{(f^\lambda)^2}{s_\lambda(1^d)} \chi^\lambda(\sigma),$$

where χ^λ is the irreducible character of S_n labelled by λ.

The following integration formula was proved in [1].

Theorem 2. *Let $i, j, i', j' : [n] \to [d]$ be any functions. Then*

$$\begin{aligned}
&\int_{U(d)} u_{i(1)j(1)} \dots u_{i(n)j(n)} \overline{u}_{i'(1)j'(1)} \dots \overline{u}_{i'(n)j'(n)} dU \\
&= \sum_{\sigma,\tau \in S_n} \delta_{i(1)i'(\sigma(1))} \dots \delta_{i(n)i'(\sigma(n))} \delta_{j(1)j'(\tau(1))} \dots \delta_{j(n)j'(\tau(n))} \,\mathrm{Wg}(d, n, \tau\sigma^{-1}),
\end{aligned}$$

where δ is the Kronecker delta.

We can succinctly express the moments of $\mathrm{Tr}(U_k)$ in terms of the Weingarten function as follows:

Lemma 1. *For any positive integers n and k, where $1 \leq k \leq d$, we have*

$$\mathbb{E}_{U(d)}(|\operatorname{Tr}(U_k)|^{2n}) = n! \sum_{\alpha} \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \operatorname{Wg}(d, n, \sigma),$$

where the outer sum runs over all weak k-part compositions α of n, and the inner sum runs over all permutations in the Young subgroup S_α of S_n.

Proof. This is really just a calculation. We expand

$$\begin{aligned}
|\operatorname{Tr}(U_k)|^{2n} &= |u_{11} + \cdots + u_{kk}|^{2n} \\
&= (u_{11} + \cdots + u_{kk})^n (\overline{u}_{11} + \cdots + \overline{u}_{kk})^n \\
&= \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} u^\alpha \overline{u}^\beta,
\end{aligned}$$

where we are summing over all pairs of weak k-part compositions of n, $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$. We are using multi-index notation,

$$\begin{aligned}
\binom{n}{\alpha} &= \binom{n}{a_1, \ldots, a_k} \\
u^\alpha &= u_{11}^{a_1} \ldots u_{kk}^{a_k} \\
\binom{n}{\beta} &= \binom{n}{b_1, \ldots, b_k} \\
\overline{u}^\beta &= \overline{u}_{11}^{b_1} \ldots \overline{u}_{kk}^{b_k}
\end{aligned}$$

Hence

$$\mathbb{E}_{U(d)}(|\operatorname{Tr}(U_k)|^{2n}) = \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} \mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta).$$

We will use the Weingarten integration formula to evaluate the expectation $\mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta)$ for a fixed pair of compositions α, β. Implicitly define coordinate functions $i_\alpha, j_\alpha, i_\beta, j_\beta : [n] \to [d]$ by setting

$$\begin{aligned}
&\int_{U(d)} u_{i_\alpha(1) j_\alpha(1)} \ldots u_{i_\alpha(n) j_\alpha(n)} \overline{u}_{i_\beta(1) j_\beta(1)} \ldots \overline{u}_{i_\beta(n) j_\beta(n)} dU \\
&:= \int_{U(d)} u_{11}^{a_1} \ldots u_{kk}^{a_k} \overline{u}_{11}^{b_1} \ldots \overline{u}_{kk}^{b_k} dU \\
&= \mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta).
\end{aligned}$$

Applying the Weingarten integration formula, we have

$$\begin{aligned}
&\mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta) \\
&= \sum_{\sigma, \tau \in S_n} \delta_{i_\alpha(1) i_\beta(\sigma(1))} \ldots \delta_{i_\alpha(n) i_\beta(\sigma(n))} \delta_{j_\alpha(1) j_\beta(\tau(1))} \ldots \delta_{j_\alpha(n) j_\beta(\tau(n))} \operatorname{Wg}(d, n, \tau\sigma^{-1}).
\end{aligned}$$

Since we are only taking entries from the diagonal, we have $i_\alpha = j_\alpha$ and $i_\beta = j_\beta$. Moreover, the level sets of these functions are easy to read off:

$$\begin{aligned} i_\alpha^{-1}(1) &= [1, a_1] \\ i_\alpha^{-1}(2) &= [a_1+1, a_1+a_2] \\ &\vdots \\ i_\alpha^{-1}(k) &= [n-a_k+1, n] \end{aligned}$$

and

$$\begin{aligned} i_\beta^{-1}(1) &= [1, b_1] \\ i_\beta^{-1}(2) &= [b_1+1, b_1+b_2] \\ &\vdots \\ i_\beta^{-1}(k) &= [n-b_k+1, n] \end{aligned}$$

Hence

$$\begin{aligned} \prod_{r=1}^{a_1} \delta_{i_\alpha(r) i_\beta(\sigma(r))} &= \prod_{r=1}^{a_1} \delta_{1 i_\beta(\sigma(r))} \\ \prod_{r=a_1+1}^{a_1+a_2} \delta_{i_\alpha(r) i_\beta(\sigma(r))} &= \prod_{r=a_1+1}^{a_1+a_2} \delta_{2 i_\beta(\sigma(r))} \\ &\vdots \\ \prod_{r=n-a_k+1}^{n} \delta_{i_\alpha(r) i_\beta(\sigma(r))} &= \prod_{r=n-a_k+1}^{n} \delta_{k i_\beta(\sigma(r))}, \end{aligned}$$

Thus in order for the product

$$\delta_{i_\alpha(1) i_\beta(\sigma(1))} \dots \delta_{i_\alpha(n) i_\beta(\sigma(n))}$$

to be nonzero, we see that σ must bijectively map the interval $[1, a_1]$ onto the interval $[1, b_1]$, and σ must also bijectively map the interval $[a_1+1, a_1+a_2]$ onto the interval $[b_1+1, b_1+b_2]$, etc. Similarly, in order for the product

$$\delta_{j_\alpha(1) j_\beta(\tau(1))} \dots \delta_{j_\alpha(n) j_\beta(\tau(n))}$$

to be nonzero, we see that τ must bijectively map the interval $[1, a_1]$ onto the interval $[1, b_1]$, and τ must also bijectively map the interval $[a_1+1, a_1+a_2]$ onto the interval $[b_1+1, b_1+b_2]$, etc. Thus we see that the expectation $\mathbb{E}_{U(d)}(u^\alpha \overline{u}^\beta)$ is zero unless:

- $\alpha = \beta$, i.e. these two are the same weak k-part composition of n,

- σ, τ are both in the Young subgroup S_α, i.e. the subgroup of permutations in S_n that permute the first a_1 symbols amongst themselves, the next a_2 symbols amongst themselves, etc.

Thus

$$\begin{aligned}
\mathbb{E}_{U(d)}(|\operatorname{Tr}(U_k)|^{2n}) &= \sum_\alpha \binom{n}{\alpha}^2 \mathbb{E}_{U(d)}(u^\alpha \overline{u}^\alpha) \\
&= \sum_\alpha \binom{n}{\alpha}^2 \sum_{\sigma,\tau \in S_\alpha} \mathrm{Wg}(d, n, \tau\sigma^{-1}) \\
&= \sum_\alpha \binom{n}{\alpha}^2 \alpha! \sum_{\sigma \in S_\alpha} \mathrm{Wg}(d, n, \sigma) \\
&= n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \mathrm{Wg}(d, n, \sigma)
\end{aligned}$$

□

4 Proof of the Main Theorem

We are now in a position to prove Theorem 1.

Proof. We work with the sum on the right hand side of Lemma 1. Plugging in the definition of the Weingarten function, we have

$$n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \mathrm{Wg}(d, n, \sigma) = n! \sum_\alpha \binom{n}{\alpha} \sum_{\sigma \in S_\alpha} \frac{1}{n!^2} \sum_{\lambda \vdash n,\ \ell(\lambda) \le d} \frac{(f^\lambda)^2}{s_\lambda(1^d)} \chi^\lambda(\sigma).$$

Changing order of summation, this becomes

$$\sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{1}{s_\lambda(1^d)} \sum_\alpha \frac{1}{\alpha!} \sum_{\sigma \in S_\alpha} \chi^\lambda(\sigma) = \sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{1}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \rangle_{S_\alpha},$$

where the inner product $\langle \cdot, \cdot \rangle_{S_\alpha}$ is the averaged dot product on the space $CF(S_\alpha)$ of complex-valued class functions on the group S_α. Note that this sum may be written as

$$\sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{1}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \downarrow_{S_\alpha}^{S_n} \rangle_{S_\alpha},$$

where $\chi^\lambda \downarrow_{S_\alpha}^{S_n}$ is the restriction of the irreducible character χ^λ of S_n to the subgroup S_α. Now, the function which is identically 1 is the character of the trivial representation of S_α. Thus we may apply Frobenius reciprocity (see for instance [7]);

$$\langle 1, \chi^\lambda \downarrow_{S_\alpha}^{S_n} \rangle_{S_\alpha} = \langle 1 \uparrow_{S_\alpha}^{S_n}, \chi^\lambda \rangle_{S_n},$$

where $1 \uparrow_{S_\alpha}^{S_n}$ is the induction of the trivial character of S_α to S_n.

The final step in the proof relies on the characteristic map $\mathrm{ch}_n : CF(S_n) \to \Lambda^n$, where Λ^n is the inner product space of degree n symmetric functions equipped with the Hall inner product $\langle \cdot, \cdot \rangle_{\Lambda^n}$ (see [7] or [9]). The characteristic function is an isometry, and has the following important properties:

$$\begin{aligned} \mathrm{ch}_n(1 \uparrow_{S_\alpha}^{S_n}) &= h_\alpha \\ \mathrm{ch}_n(\chi^\lambda) &= s_\lambda, \end{aligned}$$

where h_α is the complete homogeneous symmetric function indexed by α, and s_λ is the Schur function indexed by λ. Thus we have

$$\langle 1 \uparrow_{S_\alpha}^{S_n}, \chi^\lambda \rangle_{S_n} = \langle \mathrm{ch}_n(1 \uparrow_{S_\alpha}^{S_n}), \mathrm{ch}_n(\chi^\lambda) \rangle_{S_n} = \langle h_\alpha, s_\lambda \rangle_{\Lambda^n}.$$

It is well-known that the Schur functions constitute an orthonormal basis for Λ^n, and that the coordinates of the complete homogeneous symmetric functions with respect to the basis of Schur functions are the Kostka numbers (see [9]). That is,

$$h_\alpha = \sum_{\mu \vdash n} K_{\mu\alpha} s_\mu,$$

where the Kostka number $K_{\mu\alpha}$ is by definition the number of semistandard Young tableaux on the diagram of μ with content vector α. Thus,

$$\langle h_\alpha, s_\lambda \rangle_{\Lambda^n} = \sum_{\mu \vdash n} K_{\mu\alpha} \langle s_\mu, s_\lambda \rangle_{\Lambda^n} = K_{\lambda,\alpha}.$$

Thus we have

$$\begin{aligned} \sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{1}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \rangle_{S_\alpha} &= \sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{1}{s_\lambda(1^d)} \sum_\alpha K_{\lambda\alpha} \\ &= \sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{s_\lambda(1^k)}{s_\lambda(1^d)}, \end{aligned}$$

where the last equality follows from the fact that, by definition,

$$\sum_\alpha K_{\lambda\alpha} = s_\lambda(1^k),$$

since the sum runs over all weak k-part compositions of n.

Finally, we remark that if λ is a partition of n with $\ell(\lambda) > k$, then $s_\lambda(1^k) = 0$. Hence,

$$\sum_{\lambda \vdash n,\ \ell(\lambda) \le d} (f^\lambda)^2 \frac{s_\lambda(1^k)}{s_\lambda(1^d)} = \sum_{\lambda \vdash n,\ \ell(\lambda) \le k} (f^\lambda)^2 \frac{s_\lambda(1^k)}{s_\lambda(1^d)},$$

which proves our theorem. $\square$

5 Conclusion

The results in this paper arose from the author's attempts to evaluate known unitary integrals via the Weingarten function. It was observed that with the Weingarten method one gets information about related integrals where the matrices in the integrand are truncated with no extra work.

The notion of Weingarten function has been defined for the orthogonal and symplectic groups $O(d)$ and $Sp(d)$, and similar expansion formulas have been proved for expectations of polynomial random variables on these groups [1]. There are also versions of the Colour-Flavour transformation for the orthogonal and symplectic groups (indeed, the Colour-Flavour transformations are all intimately related to Howe duality, see [12]). Thus there are likely analogues of Theorem 1 for truncations of random orthogonal and symplectic matrices.

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