A Two Parameter Chromatic Symmetric Function

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Abstract

We introduce and develop a two-parameter chromatic symmetric function for a simple graph G over the field of rational functions in q and t , $\mathbb{Q}(q,t)$. We derive its expansion in terms of the monomial symmetric functions, m_λ, and present various correlation properties which exist between the two-parameter chromatic symmetric function and its corresponding graph.

Additionally, for the complete graph G of order n, its corresponding two parameter chromatic symmetric function is the Macdonald polynomial $Q_{(n)}$. Using this, we develop graphical analogues for the expansion formulas of the two-row Macdonald polynomials and the two-row Jack symmetric functions.

Finally, we introduce the "complement" of this new function and explore some of its properties.

1. Preliminaries.

We briefly define some of the basic concepts used in the development of our two parameter chromatic symmetric function. In general, our notation will be consistent with that of [1].

Let G be a finite, simple graph; G has no multiple edges or loops. Denote the edge set of G by $E(G)$ and the vertex set of G by $V(G)$. The *order* of the graph G, denoted $o(G)$, is the size of its vertex set $V(G)$ and the *size* of the graph G, denoted $s(G)$, is equal to the number of edges in $E(G)$. A *subgraph* of G , G' , is a graph whose vertex set and edge set are contained in those of G. For a subset $V'(G) \subseteq V(G)$, the *subgraph induced by* $V'(G)$, G_I , is the subgraph of G which contains all edges in $E(G)$ which connect any two vertices in $V'(G)$.

For the graph G, denote the edge of $E(G)$ which joins the vertices $v_i, v_j \in V(G)$ by v_iv_j ; we say that v_i and v_j are the *endvertices* of the edge v_iv_j. A *walk* in G is a sequence of vertices and edges, $v_1, v_1v_2, \ldots, v_{l-1}v_l, v_l$, denoted $v_1 \ldots v_l$; the *length* of this walk is l. A *path* is a walk with distinct vertices and a *trail* is a walk with distinct edges. A trail whose endvertices are equal, $v_1 = v_l$, is called a *circuit.* A walk of length ≥ 3 whose vertices are all distinct, except for coinciding endvertices, is called a *cycle.* The graph G

is said to be *connected* if for every pair of vertices $\{v_i, v_j\} \in V(G)$, there is a path from v_i to v_j. A *tree* is a connected, acyclic graph.

Let $V(G) = \{v_1, \ldots, v_n\}$. Denote the number of edges emminating from the vertex $v_i \in V(G)$ by $d(v_i)$, the *degree* of the vertex v_i. The *degree sequence* of G, denoted by $deg(G)$, is a weakly decreasing sequence (or partition) of nonnegative integers, $deg(G) = (d_1, \ldots, d_n)$, such that the length of $deg(G)$ is equal to $|V(G)|$ and $(d_1, \ldots, d_n)$ represents the degrees of the vertices of $V(G)$, arranged in decreasing order. Since each edge of G has two endvertices, it follows that $\sum_{i=1}^{n} d_i = 2s(G)$; thus, $deg(G) \vdash 2s(G)$.

A *coloring* of the graph G is a function $k : V(G) \to \mathbb{N}$. The coloring k is said to be *proper* if $k(v_i) \neq k(v_j)$ whenever $v_iv_j \in E(G)$.

Additionally, we will use the following consistent with [2].

$$\begin{aligned}
(a;q)_0 &= 1 \\
(a;q)_n &= \prod_{i=0}^{n-1}(1 - aq^i) \\
(a;q)_n &= \frac{(a;q)_\infty}{(aq^n;q)_\infty} \\
(a_1, \ldots, a_m; q)_n &= (a_1;q)_n \cdots (a_m;q)_n \\
(a;q) &= (a;q)_1
\end{aligned}$$

2. A Two-Parameter Chromatic Symmetric Function.

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and let $k : V(G) \to \mathbb{N}$ be a coloring from the set of vertices of the graph G into $\mathbb{N} = \{1, 2, \ldots\}$. An edge $v_iv_j \in E(G)$ is *colored c by k* if $k(v_i) = k(v_j) = c$. Denote $m_i(k)$ to be the number of monochromatic edges of G which are colored $i \in \mathbb{N}$ with respect to the coloring k. Denote $R(k)$ to be the range of the coloring k.

For $i \in \mathbb{N}$, as in [6], set

$$V_i \;=\; |\{v_j \in V(G) :\; k(v_j) = i\}| \tag{1}$$

i.e. the number of vertices of $V(G)$ colored i by k. For $i \in R(k)$, define

$$m_i = \begin{cases} (m_i(k) + 1) & \text{if } \; (m_i(k) + 1) \leq V_i \\ V_i & \text{otherwise.} \end{cases} \tag{2}$$

Let $x = \{x_1, x_2, \ldots\}$ be a set of commuting indeterminates. For the coloring $k : V(G) \to \mathbb{N}$, set

$$x^k \;=\; \prod_{i=1}^{n} x_{k(v_i)} \tag{3}$$

for $v_i \in V(G)$.

Definition 2.1. For a simple graph G , $o(G) = n$,

$$Y_G(x;q,t) = \sum_k \binom{n}{V_1, V_2, \ldots}^{-1} \left(\prod_{i \in R(k)} \frac{(t;q)_{m_i}}{(q;q)_{m_i}} \right) x^k$$

where k ranges over all colorings of G.

It follows from Definition 2.1 and (3) that $Y_G(x;q,t)$ is a symmetric function of degree n.

Remark 2.1. The papers [6] and [7], by Richard Stanley, served as inspiration for this work. Note however, that his chromatic symmetric function described is [6] and [7] and the present two-parameter chromatic symmetric function are entirely different. Some of the prominent differences include, for example, that the function in this paper is a two-parameter symmetric function in q and t and that the colorings considered here are not necessarily proper. Even if we set $q = \frac{1}{t}$ to kill the terms corresponding to colorings that are not proper, the remaining coefficients are different from Stanley's. See [6] and [7] for further details.

Definition 2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and G be a simple graph.

A *general distinct coloring* is a coloring of G , $k_\lambda^m : V(G) \to \mathbb{N}$, which sends λ_i-many vertices to one color and λ_j-many vertices to another color, for all $i \neq j$.

The *basic coloring of* G *of type* λ , $k_\lambda : V(G) \to \mathbb{N}$, is the set of all general distinct colorings $\{k_\lambda^m\}$ of the graph G.

Remark 2.2. Note that for $k_\lambda = \{k_\lambda^m\}$, each general distinct coloring $k_\lambda^m : V(G) \to \mathbb{N}$ corresponds to a unique, ordered grouping of the vertices of $V(G)$ into disjoint subsets of size λ_i , $1 \leq i \leq n$.

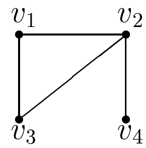
In other words, the map k_λ^m is a general distinct coloring if it partitions $V(G)$ into disjoint subsets of size $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that the vertices in each subset are all mapped to the same color and such that the vertices in distinct subsets are mapped to distinct colors by k_λ^m.

Additionally, for $o(G) = d$, there are $\binom{d}{\lambda_1,\ldots,\lambda_n}$-many general distinct colorings, k_λ^m, within k_λ ; $|k_\lambda| = \binom{d}{\lambda_1,\ldots,\lambda_n}$.

Example 2.1. Let $\lambda = (3,2,1,1)$ and $V(G) = \{v_1, \ldots, v_7\}$. Let $\{j\}$ denote a subset of vertices of $V(G)$ of size j. The basic coloring of G of type $\lambda = (3,2,1,1)$ includes all general distinct colorings $k_\lambda^m : V(G) \to \mathbb{N}$ such that $k_\lambda^m(\{3\}) \neq k_\lambda^m(\{2\}) \neq k_\lambda^m(\{1\}) \neq k_\lambda^m(\{1\})$; each m corresponds to a specific ordered grouping, $(\{3\}, \{2\}, \{1\}, \{1\})$, of disjoint j-element subsets of $V(G)$, $j \in \{1,2,3,3\}$. Note that $|k_\lambda| = \binom{7}{3,2,1,1} = 420$, the number of general distinct colorings, k_λ^m, included in the basic coloring k_λ:

$$k_\lambda = \{(\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_6\}, \{v_7\}), (\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_7\}, \{v_6\}), \ldots\}. \quad \square$$

Example 2.2. Consider the simple graph G such that $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4\}$.



There are five possible basic colorings $k : V(G) \to \mathbb{N}$: (1.) the coloring of type $\lambda = (1^4)$ sending each vertex to a different color, (2.) the coloring of type $\lambda = (4)$ sending all vertices to the same color, (3.) the coloring of type $\lambda = (3,1)$ which sends three vertices to the same color and the remaining one to a different color, (4.) the coloring of type $\lambda = (2,1,1)$ sending two vertices to the same color and sending the remaining two vertices to two other distinct colors, and (5.) the coloring of type $\lambda = (2,2)$ which sends two vertices to the same color and the remaining two vertices to the same color (distinct from the first).

Restrict the number of variables to four such that $x = \{x_1, x_2, x_3, x_4\}$. Therefore, the range of k becomes $\{1,2,3,4\}$, $k : V(G) \to \{1,2,3,4\}$. We will compute $Y_G(x; q, t)$ via computing the function of each of the five basic colorings.

Within the first basic coloring, there are $\binom{4}{1,1,1,1} = 4!$ general distinct colorings, each with $m_i = 1$ for all $i \in \{1,2,3,4\}$:

$$\frac{(t;q)^4}{(q;q)^4}\, x_1x_2x_3x_4.$$

For the second basic coloring, there is $\binom{4}{4} = 1$ general distinct coloring and four specific colorings. Since the range of the coloring is restricted to $\{1,2,3,4\}$, each of these gives $m_i = 4$ for all $i \in \{1,2,3,4\}$:

$$\frac{(t;q)_4}{(q;q)_4}(x_1^4 + x_2^4 + x_3^4 + x_4^4).$$

There are $\binom{4}{3,1} = 4$ general distinct colorings within the third basic coloring. For $\{3\}$, three of these give $m_i = 3$ and one gives $m_i = 2$, $k_\lambda^m = (\{v_1, v_3, v_4\}, \{v_2\})$ for all $i \in \{1,2,3,4\}$:

$$\frac{3}{4}\,\frac{(t;q)_3(t;q)}{(q;q)_3(q;q)}(x_1^3x_2 + x_2^3x_3 + \ldots) + \frac{1}{4}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)}(x_1^3x_2 + x_2^3x_3 + \ldots).$$

Within the fourth basic coloring, there are $\binom{4}{2,1,1} = 12$ general distinct colorings. For the subset $\{2\}$, eight of these give $m_i = 2$ and four give $m_i = 1$ for all $i \in \{1,2,3,4\}$. Thus, we have:

$$\frac{2}{3}\,\frac{(t;q)_2(t;q)(t;q)}{(q;q)_2(q;q)(q;q)}(x_1^2x_2x_3 + x_1x_2^2x_3 + \ldots) + \frac{1}{3}\,\frac{(t;q)^3}{(q;q)^3}(x_1^2x_2x_3 + x_2^2x_1x_3 + \ldots).$$

Lastly, within the fifth basic coloring, there are $\binom{4}{2,2} = 6$ general distinct colorings, yielding:

$$\frac{2}{3}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)}(x_1^2x_2^2 + x_2^2x_3^2 + \ldots) + \frac{1}{3}\,\frac{(t;q)_2(t;q)_2}{(q;q)_2(q;q)_2}(x_1^2x_2^2 + x_2^2x_3^2 + \ldots).$$

Thus,

$$\begin{aligned}
Y_G(x;q,t) &= \frac{(t;q)^4}{(q;q)^4}\,x_1x_2x_3x_4 + \frac{(t;q)_4}{(q;q)_4}(x_1^4 + x_2^4 + x_3^4 + x_4^4) \\
&+ \left(\frac{3}{4}\,\frac{(t;q)_3(t;q)}{(q;q)_3(q;q)} + \frac{1}{4}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)}\right)(x_1^3x_2 + x_2^3x_3 + \ldots) \\
&\quad + \left(\frac{2}{3}\,\frac{(t;q)_2(t;q)(t;q)}{(q;q)_2(q;q)(q;q)} + \frac{1}{3}\,\frac{(t;q)^3}{(q;q)^3}\right)(x_1^2x_2x_3 + x_1x_2^2x_3 + \ldots) \\
&+ \left(\frac{2}{3}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)} + \frac{1}{3}\,\frac{(t;q)_2(t;q)_2}{(q;q)_2(q;q)_2}\right)(x_1^2x_2^2 + x_2^2x_3^2 + \ldots). \quad \square
\end{aligned}$$

As in [5], a *set partition* P of the set S is a collection of disjoint subsets $\{S_1, \ldots, S_r\}$ whose union is S. The set partition P has *type* μ if $\mu = (\,|S_1|, \ldots, |S_r|\,)$ where $|S_1| \geq \ldots \geq |S_r|$.

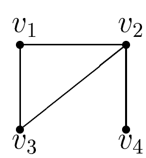
Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n. Denote

$$W'_\lambda = W'_{\lambda_1} \uplus \ldots \uplus W'_{\lambda_r} \tag{4}$$

to be the disjoint union of subsets of $V(G)$ such that for $1 \leq i \leq r$, W'_{λ_i} is a subset of $V(G)$ of size λ_i and $W'_{\lambda_i} \cap W'_{\lambda_j} = \emptyset$ for all $i \neq j$. Thus, W'_λ is a set partition of $V(G)$ of type λ.

Now, for $\lambda \vdash n$ and the graph G, restrict the set partition W'_λ of $V(G)$ to all of the possible distinct *ordered* subset compositions of $V(G)$ where each distinct ordered subset composition is a unique, ordered grouping $V(G)$ as dicated by the partition λ. Denote this new "restricted set" of W'_λ as W_λ.

Example 2.3. Consider the graph



and the partition $\lambda = (2, 2)$. Then,

$$\begin{aligned}
W_\lambda = \\
&\{\,\{v_1, v_2\} \cup \{v_3, v_4\}\,, \\
&\{v_3, v_4\} \cup \{v_1, v_2\} \\
&\{v_1, v_3\} \cup \{v_2, v_4\} \\
&\{v_2, v_4\} \cup \{v_1, v_3\} \\
&\{v_1, v_4\} \cup \{v_2, v_3\} \\
&\{v_2, v_3\} \cup \{v_1, v_4\}\ \}. \quad \square
\end{aligned}$$

Moreover, let $W^*_{\lambda_i}$ be the set of all distinct two-element subsets $\{v_i, v_j\}$, $i \neq j$, of W_{λ_i}.

Viewing each two element subset $\{v_i, v_j\} \in W^*_{\lambda_i}$ as the possible edge $v_iv_j \in E(G)$, define:

$$P_{\lambda_i} = \begin{cases} (|W^*_{\lambda_i} \cap E(G)| + 1) & \text{if } (|W^*_{\lambda_i} \cap E(G)| + 1) \leq |W_{\lambda_i}| \\ |W_{\lambda_i}| & \text{otherwise,} \end{cases} \tag{5}$$

where $P_{\lambda_i} = 1$ if $(W^*_{\lambda_i} \cap E(G)) = \emptyset$.

For a partition $\mu = (\mu_1, \ldots, \mu_l)$, the *monomial symmetric function*, m_μ, is given by: $m_\mu = \sum_{i_1 < \ldots < i_l} x^{\mu_1}_{i_1} x^{\mu_2}_{i_2} \cdots x^{\mu_l}_{i_l}$.

Proposition 2.1. For the simple graph G of order n,

$$Y_G(x; q, t) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_r}^{-1} \left(\sum_{W_\lambda \subseteq V(G)} \left(\prod_{i=1}^{r} \frac{(t; q)_{P_{\lambda_i}}}{(q; q)_{P_{\lambda_i}}} \right) \right) m_\lambda$$

where $W_\lambda \subseteq V(G)$ runs over all possible distinct ordered subset compositions for the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$; W_λ and P_{λ_i} as defined above.

Proof. Since $Y_G(x; q, t)$ is a symmetric function of degree n, it can be expressed in terms of monomial symmetric functions, m_λ, such that $\lambda \vdash n$. Since $k : V(G) \to \mathbb{N}$ ranges over all possible colorings of G, we obtain the functions m_λ such that $\lambda = (\lambda_1, \ldots, \lambda_r)$ runs through all partitions of n, where $\lambda_i \equiv V_j$, j ranging throughout $R(k)$ such that $|V_j| = \lambda_i$.

For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, there are $\binom{n}{\lambda_1, \ldots, \lambda_r}$ possible distinct general colorings within the basic coloring k_λ; sending λ_i-many vertices to the same color $j \in R(k)$, $1 \leq i \leq r$, and where the vertices of λ_i are sent to a distinct color from those of λ_m , $\forall i \neq m$.

Since $W_\lambda = W_{\lambda_1} \uplus \ldots \uplus W_{\lambda_r}$ partitions the vertices of $V(G)$ into all possible disjoint subsets such that $|W_{\lambda_i}| = \lambda_i$, and since $W_\lambda \subseteq V(G)$ runs over all *distinct ordered* W_λ (with respect to the composition of W_{λ_i} , $\forall i$), we obtain all distinct general colorings k^i_λ of G within k_λ. Since the "specific" colorings within each k^m_λ have the same coefficient (ref. Example 2.2), we may consider the coefficient of m_λ , $\lambda \vdash n$, via the general coefficients for a coloring of type λ , k_λ , with respect to the individual coefficients for each k^m_λ. Since $l(\lambda) = |R(k)|$ for the coloring k, one can see by comparing the m_i to the P_{λ_i} that these terms coincide. Thus, the coefficent of the monomial m_λ in $Y_G(x; q, t)$ is equal to

$$\binom{n}{\lambda_1, \ldots, \lambda_r}^{-1} \left(\sum_{W_\lambda \subseteq V(G)} \left(\prod_{i=1}^{r} \frac{(t; q)_{P_{\lambda_i}}}{(q; q)_{P_{\lambda_i}}} \right) \right). \quad \blacksquare$$

Example 2.4. For the graph G of Example 2.2, it is easily seen that

$$\begin{aligned}
Y_G(x;q,t) &= \frac{(t;q)^4}{(q;q)^4}\, m_{(1,1,1,1)} + \frac{(t;q)_4}{(q;q)_4}\, m_{(4)} \\
&+ \left(\frac{3}{4}\,\frac{(t;q)_3(t;q)}{(q;q)_3(q;q)} + \frac{1}{4}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)}\right) m_{(3,1)} \\
&+ \left(\frac{2}{3}\,\frac{(t;q)_2(t;q)(t;q)}{(q;q)_2(q;q)(q;q)} + \frac{1}{3}\,\frac{(t;q)^3}{(q;q)^3}\right) m_{(2,1,1)} \\
&+ \left(\frac{2}{3}\,\frac{(t;q)_2(t;q)}{(q;q)_2(q;q)} + \frac{1}{3}\,\frac{(t;q)_2(t;q)_2}{(q;q)_2(q;q)_2}\right) m_{(2,2)}. \quad \square
\end{aligned}$$

3. Some Properties of $\mathbf{Y_G(x;q,t)}$.

In this section, we will explore some of the basic properties and correlations between a finite, simple graph G and the symmetric function $Y_G(x;q,t)$.

Proposition 3.1. Let G be a simple graph. G has order d and size s if the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(d-2)}}{(q;q)_2(q;q)^{(d-2)}}\, m_{(2,1^{(d-2)})} \tag{6}$$

in $Y_G(x;q,t)$ is $\frac{2s}{d(d-1)}$.

Proof. Let G be a graph of order d and size s. The multiplicity of the term (6) in $Y_G(x;q,t)$ corresponds to $\binom{d}{2,1^{(d-2)}}^{-1}$ multiplied by the number of pairs of vertices $\{v_i,v_j\} \in V(G)$ such that $v_iv_j \in E(G)$ (where $P_{(2)} = 2$) multiplied by $(d-2)!$:

For $\{v_i,v_j\} \in V(G)$ such that $v_iv_j \in E(G)$, consider the number of possible general distinct colorings $k^m_{(2,1^{(d-2)})} : V(G) \to \mathbb{N}$ of type $(2,1^{(d-2)})$ such that $k^m_{(2,1^{(d-2)})}(v_i) = k^m_{(2,1^{(d-2)})}(v_j)$ and such that $k^m_{(2,1^{(d-2)})}$ distinguishes all remaining vertices in $V(G)$ from each other and from v_i and v_j. Since $s(G) = s$, there are s possible such two element subsets $\{v_i,v_j\}$ of $V(G)$. For each of these subsets, since $o(G) = d$, there are $(d-2)$ remaining vertices in $V(G) \setminus \{v_i,v_j\}$. Thus, there are $(d-2)!$ different general distinct colorings $k^m_{(2,1^{(d-2)})}$ distinguishing among $V(G) \setminus \{v_i,v_j\}$ and $\{v_i,v_j\}$. Hence, the multiplicity of the desired term is

$$\binom{d}{2,1^{(d-2)}}^{-1} s\,(d-2)! \;=\; \frac{2s}{d(d-1)}. \quad \blacksquare$$

Remark 3.1. Conversely to Proposition 3.1, consider $Y_G(x;q,t)$ in which the term

$$\frac{(t;q)_2(t;q)^r}{(q;q)_2(q;q)^{(r}}\, m_{(2,1^r)}$$

appears.

Note that the monomial symmetric function $m_{(2,1^r)}$, for some $r \geq 0$, appears in $Y_G(x;q,t)$ if and only if $o(G) = 2+r$ since $(2,1^r) \vdash (2+r)$. Furthermore, by Proposition 2.1, the coefficient of the monomial $m_{(2,1^r)}$ is equal to

$$\binom{2+r}{2,1^r}^{-1}\Bigg(\sum_{W_\lambda \subseteq V(G)}\Bigg(\prod_{i=1}^{2+r}\frac{(t;q)_{P_{\lambda_i}}}{(q;q)_{P_{\lambda_i}}}\Bigg)\Bigg).$$

For

$$\sum_{W_\lambda \subseteq V(G)}\Bigg(\prod_{i=1}^{2+r}\frac{(t;q)_{P_{\lambda_i}}}{(q;q)_{P_{\lambda_i}}}\Bigg),$$

we have $P_{\lambda_1} = 2$ and $P_{\lambda_2} = \ldots = P_{\lambda_r} = 1$. Hence, by definition of P_{λ_i}, and since $|W_{\lambda_i}| = 2$, it follows that the multiplicity of (6) is:

$$\begin{aligned}&\binom{2+r}{2,1^r}^{-1} \cdot |E(G)| \cdot r!\\ &= \frac{2}{(2+r)!} \cdot |E(G)| \cdot r!\\ &= \frac{2|E(G)|}{(2+r)(1+r)}.\end{aligned}$$

Therefore, given the multiplicity of (6), we may recover $|E(G)|$.

Proposition 3.2. Let G and H be graphs with degree sequences $deg(G)$ and $deg(H)$, respectively. Then $o(G) = o(H) = d$, $s(G) = s(H) \leq d$, and $deg(G) = deg(H)$ if and only if the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(d-2)}}{(q;q)_2(q;q)^{(d-2)}}m_{(2,1^{(d-2)})}$$

is $\leq \frac{2}{(d-1)}$ and is equal in both $Y_G(x;q,t)$ and $Y_H(x;q,t)$ and if the coefficients of $m_{((d-1),1)}$ in $Y_G(x;q,t)$ and $Y_H(x;q,t)$ are equal.

Proof. $(\Rightarrow)$ Suppose that $o(G) = o(H) = d$, $s(G) = s(H) \leq d$, and $deg(G) = deg(H)$. Let $deg(G) = (\beta_1,\ldots,\beta_n) = deg(H)$; $\beta_1 \geq \ldots \geq \beta_n$, $n \leq d$, and $\sum_{i=1}^n \beta_i = 2s(G)$. By Proposition 3.1, we know that the multiplicity of the term (6) is equal in both Y_G and Y_H and is $\leq \frac{2}{(d-1)}$. By the definitions of $Y_G(x;q,t)$ and $Y_H(x;q,t)$, the coefficients of $m_{((d-1),1)}$ are given by

$$\begin{aligned}&\binom{d}{(d-1),1}^{-1}\Bigg(\sum_{W_\lambda \subseteq V(G)}\Bigg(\prod_{i=1}^{2}\frac{(t;q)_{P_{\lambda_i}}}{(q;q)_{P_{\lambda_i}}}\Bigg)\Bigg)\\ &= \frac{1}{d}\Bigg(\sum_{W_{((d-1),1)} \subset V(G)}\frac{(t;q)_{P_{(d-1)}}(t;q)_{P_{(1)}}}{(q;q)_{P_{(d-1)}}(q;q)_{P_{(1)}}}\Bigg).\end{aligned}$$

Note that there are d-many distinct subsets $W_{((d-1),1)} = W_{(d-1)} \uplus W_{(1)}$ of $V(G)$ (resp., $V(H)$). Moreover, note that each $\beta_i \in deg(G)$ (resp., $deg(H)$) directly corresponds to one vertex $v_j \in V(G)$, where β_i indicates the degree of the vertex v_j , $d(v_j) = \beta_i$. Thus, sending the vertex v_j to $W_{(1)}$ amounts to removing all edges from $E(G)$ (resp., $E(H)$) which are incident with the vertex v_j in the computation of $|W^*_{(d-1)} \cap E(G)| + 1 = P_{(d-1)}$, $W_{(d-1)} = V(G) \setminus \{v_j\}$. This implies that $|W^*_{(d-1)} \cap E(G)| = |E(G)| - \beta_i$ (resp. for H). Repeating this for each $\beta_i \in deg(G) = deg(H)$ and the corresponding two vertices (one for $deg(G)$ and possibly a different one for $deg(H)$) gives the coefficients of $m_{((d-1),1)}$ in $Y_G(x; q, t)$ and $Y_H(x; q, t)$ to be equal.

($\Leftarrow$) From Proposition 3.1, the multiplicity of the term (6) being equal and $\leq \frac{2}{(d-1)}$ in Y_G and Y_H tells us that $o(G) = o(H) = d$ and that $s(G) = s(H) \leq d$.

Suppose that the coefficients of the term $m_{((d-1),1)}$ in both $Y_G(x; q, t)$ and $Y_H(x; q, t)$ are equal. We must show that $deg(G) = deg(H)$. For $1 \leq l \leq (d-1)$, consider the multiplicity K_l of the term

$$\frac{(t;q)_l(t;q)}{(q;q)_l(q;q)} m_{((d-1),1)}$$

in $Y_G(x; q, t)$ and $Y_H(x; q, t)$.

Suppose that $l = (d-1)$. Then there exists $K_{(d-1)}$ vertices in $V(G)$ such that $|W^*_{(d-1)} \cap E(G)| = (d-1)$ or $(d-2)$, and similarly for $V(H)$. We need to show that the number of vertices in $V(H)$ such that $|W^*_{(d-1)} \cap E(G)| = (d-1)$ (resp. $(d-2)$) is equal to the number of vertices in $V(H)$ such that $|W^*_{(d-1)} \cap E(H)| = (d-1)$ (resp. $(d-2)$).

Note that the multiplicity of

$$\frac{(t;q)_{(d-1)}(t;q)}{(q;q)_{(d-1)}(q;q)} m_{((d-1),1)} \tag{7}$$

corresponds to the number of vertices in $V(G)$ and $V(H)$ such that $d(v_i) = 1$ or $d(v_i) = 0$. Consider the vertices $v_i \in V(G)$ and $v_j \in V(H)$, for which $W_{(1)} = \{v_i\}$ and $W_{(1)} = \{v_j\}$ in $W_{((d-1),1)}$, such that $P_{(d-1)} \leq (d-2)$. For each $W_{((d-1),1)} \subset V(G)$ and $W_{((d-1),1)} \subset V(H)$ such that $P_{(d-1)}$ is equal for both $V(G)$ and $V(H)$ and $P_{(d-1)} \leq (d-2)$, we have that $|W^*_{(d-1)} \cap E(G)| = (P_{(d-1)} - 1) = |W^*_{(d-1)} \cap E(H)|$, by definition of $P_{(d-1)}$. Since the multiplicity of the coefficient of

$$\frac{(t;q)_{P_{(d-1)}}(t;q)}{(q;q)_{P_{(d-1)}}(q;q)} m_{((d-1),1)}$$

in Y_G and Y_H is equal, the number of vertices $v_i \in V(G)$ and $v_j \in V(H)$ such that $d(v_i) = d(v_j) = s - P_{(d-1)}$ must be equal. (Note: $P_{(d-1)} + 1 = s - d(v_i) + 1$.) Thus, since $o(G) = o(H)$, $s(G) = s(H)$, and $\sum deg(G) = \sum deg(H)$, the number of vertices with degree 0 in G equals the number of vertices with degree 0 in H and, similarly, the number of vertices with degree 1 in G equals the number of vertices with degree 1 in H. Therefore, $deg(G) = deg(H)$. ■

Proposition 3.3. Let G be a simple graph of order d. Any induced subgraph of G , G_I , of order $(d-1)$ is connected if and only if the multiplicity of the term

$$\frac{(t;q)_{(d-1)}(t;q)}{(q;q)_{(d-1)}(q;q)}m_{((d-1),1)}$$

in $Y_G(x;q,t)$ is one.

Proof. ($\Rightarrow$) Suppose that any induced subgraph of G , G_I , of order $(d-1)$ is connected. Then, $|E(G_I)| \geq (d-2)$. Hence, for all possible subsets $W_{(d-1)} \subset V(G)$, $W_{(d-1)} \subset W_{((d-1),1)}$, it follows that $P_{(d-1)} = (d-1)$. Hence, the multiplicity term (7) in $Y_G(x;q,t)$ is one.

($\Leftarrow$) Suppose that the multiplicity of term (7) in $Y_G(x;q,t)$ is one. Then, for all possible $(d-1)$-element subsets $W_{(d-1)} \subset V(G)$, $|W^*_{(d-1)} \cap E(G)| \geq (d-2)$. Therefore, every induced subgraph G_I of order $(d-1)$ must be connected. ■

Remark 3.2. By Proposition 3.3, for a graph G of order d, if the multiplicity of (7) is one in $Y_G(x;q,t)$, then G is not a tree.

Proposition 3.4. Let G be a simple graph. G has order d and is a cycle of size d if and only if the multiplicity of the term

$$\frac{(t;q)_{(d-1)}(t;q)}{(q;q)_{(d-1)}(q;q)}m_{((d-1),1)}$$

in $Y_G(x;q,t)$ is one and the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(d-2)}}{(q;q)_2(q;q)^{(d-2)}}m_{(2,1^{(d-2)})}$$

is $\frac{2}{(d-1)}$.

Proof. ($\Rightarrow$) If $o(G) = s(G) = d$, we know from Proposition 3.1 that the multiplicity of the term (6) is $\frac{2}{(d-1)}$. Consider the multiplicity of the term (7). Since G is a cycle of length d and $o(G) = s(G) = d$, we know that $d(v_i) = 2$ for all $v_i \in V(G)$. Thus, the number of subsets $W_\lambda \subseteq V(G)\,, W_\lambda = W_{(d-1)} \uplus W_{(1)}$, such that $P_{(d-1)} = (d-1)$ and $P_{(1)} = 1$ is exactly d many, since any choice of $(d-1)$ vertices is connected by $(d-2)$ edges. This implies that the multiplicity of the desired term is

$$d\binom{d}{(d-1),1}^{-1} = 1.$$

($\Leftarrow$) From Proposition 3.1 and Remark 3.1, if the multiplicity of the term (6) is $\frac{2}{(d-1)}$ for some d, we know that G has order and size d. By Proposition 3.3, the multiplicity of term (7) being one implies that any $(d-1)$ element subset of $V(G)$ is connected. Since $o(G) = s(G) = d$, the only connected graph fitting this description is a cycle of length d. ■

4. $\mathbf{Y_G(x;q,t)}$ and Macdonald Polynomials.

Denote the ring of symmetric functions over the field $\mathbb{F}$ as $\Lambda_{\mathbb{F}}$ and let $\Lambda_{\mathbb{F}}^n$ denote its n^{th} graded space. The space $\Lambda_{\mathbb{F}}^n$ consists of all symmetric functions of total degree $n \in \mathbb{Z}$, indexed by the partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ for which $\sum_i \lambda_i = n$. Five important bases of $\Lambda_{\mathbb{F}}^n$ are: the monomial symmetric functions m_λ, the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r}$, the complete symmetric functions h_λ, the Schur functions s_λ, and the power sum symmetric functions $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_r}$. Of these five bases, all except the power sum symmetric functions are $\mathbb{Z}$-bases; the power sum symmetric functions are a $\mathbb{Q}$-basis.

Let $H = \mathbb{Q}(q,t)$ be the field of rational functions in q and t. In 1988, Macdonald introduced a new class of two-parameter symmetric functions $P_\lambda(q,t)$, over the ring Λ_H, which generalize several classes of symmetric functions. In particular, taking $q = t$ we obtain the Schur functions, setting $t = 1$ we have the monomial symmetric functions, and letting $q = 0$ gives the Hall-Littlewood functions.

We know from [4] that the (P_λ) are a basis of Λ_H^n. Further, with respect to the scalar product:

$$< p_\lambda, p_\mu >= \delta_{\lambda,\mu} \prod_i i^{m_i} m_i! \prod_{j=1}^{l(\lambda)} \frac{1-q^{\lambda_j}}{1-t^{\lambda_j}}$$

we have that

$$< P_\lambda, P_\mu >= 0 \quad \text{if} \quad \lambda \neq \mu,$$

where m_i denotes the multiplicity of i in λ and $l(\lambda)$ denotes the length of λ. We also know that for each λ, there exists a unique $P_\lambda(q,t)$ such that:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu \quad \text{where} \quad c_{\lambda\mu} \in \mathbb{Q}(q,t).$$

Define:

$$Q_\lambda = \frac{P_\lambda}{< P_\lambda, P_\lambda >}.$$

Then, the bases (P_λ) and (Q_λ) of Λ_H^n are dual to each other, $< Q_\lambda, P_\mu >= \delta_{\lambda,\mu}$, and from [4], for $\gamma = (n)$:

$$Q_{(n)} = \sum_{|\lambda|=n} \prod_i \frac{1}{i^{m_i} m_i!} \prod_{j=1}^{l(\lambda)} \frac{1-t^{\lambda_j}}{1-q^{\lambda_j}} p_\lambda$$

where we set $Q_0 = 1$ and $Q_{-m} = 0$ for $m \in \mathbb{Z}^+$.

There turns out to be an interesting connection between our two parameter chromatic symmetric function $Y_G(x;q,t)$ and the Macdonald polynomials Q_λ. We motivate this connection via the following definitions and proposition.

The *complete graph* of order n, denoted K^n, is the graph G which has size $\binom{n}{2}$; every two vertices in $V(G)$ are adjacent. We know from [4] that for $n \in \mathbb{Z}^+$, the Macdonald polynomial

$$Q_{(n)} = \sum_{\lambda \vdash n} \frac{(t;q)_\lambda}{(q;q)_\lambda} m_\lambda$$

where we define

$$\frac{(t;q)_\lambda}{(q;q)_\lambda} = \prod_{i=1}^{r} \frac{(t;q)_{\lambda_i}}{(q;q)_{\lambda_i}}$$

for $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$.

The following proposition is immediate.

Proposition 4.1. Let G be the complete graph of order n, $G = K^n$, for $n \in \mathbb{Z}^+$. Then

$$Y_G(x;q,t) = Q_{(n)}(x;q,t).$$

From [3], we have the following combinatoral formula for a two-row Macdonald polynomial Q_λ, $\lambda = (\lambda_1, \lambda_2)$:

$$Q_{(\lambda_1,\lambda_2)} = \sum_{i=0}^{\lambda_2} a_i^{\lambda_1-\lambda_2}\, Q_{(\lambda_1+i)}\, Q_{(\lambda_2-i)} \tag{8}$$

where

$$a_i^{\lambda_1-\lambda_2} = \left(\frac{(t^{-1};q)_i(q^{\lambda_1-\lambda_2};q)_i(1-q^{\lambda_1-\lambda_2+2i})}{(q;q)_i(q^{\lambda_1-\lambda_2+1}t;q)_i(1-q^{\lambda_1-\lambda_2})}\right)t^i$$

and $a_0^{\lambda_1-\lambda_2} = 1$.

Using the symmetric function $Y_G(x;q,t)$, we give a graphical analogue of this two-row formula for any partition $\lambda = (\lambda_1, \lambda_2)$.

Let G be the complete graph of order $(\lambda_1 + \lambda_2)$, $G = K^{(\lambda_1+\lambda_2)}$. Then, $V(G) = \{v_1, \ldots, v_{\lambda_1+\lambda_2}\}$. Denote W_i to be the subset of $V(G)$ containing vertices $\{v_j\}$ for $1 \leq j \leq i$:

$$W_i \;=\; \{v_1, \ldots, v_i\}. \tag{9}$$

Denote W_i^c to be the subset of $V(G)$ containing the vertices $\{v_m\}$ such that $(i+1) \leq m \leq (\lambda_1 + \lambda_2)$,

$$W_i^c \;=\; \{v_{(i+1)}, \ldots, v_{(\lambda_1+\lambda_2)}\}, \tag{10}$$

and set $W_0 = \emptyset$.

Let $G[V \setminus W_i]$ denote the subgraph of $G = K^{(\lambda_1+\lambda_2)}$ obtained by deleting the vertices in $W_i \subseteq V(G)$ and all edges in $E(G)$ which are incident with them.

Theorem 4.1. Let $G = K^{(\lambda_1+\lambda_2)}$. For the partition $\lambda = (\lambda_1, \lambda_2)$,

$$Q_\lambda \;=\; Q_{(\lambda_1,\lambda_2)} \;=\; \sum_{i=0}^{\lambda_2} a_{(\lambda_2-i)}^{\lambda_1-\lambda_2}\; Y_{G[V\setminus W_i]}(x;q,t)\; Y_{G[V\setminus W_i^c]}(x;q,t)$$

where

$$Y_{G[V\setminus W_i^c]}(x;q,t) \;=\; 1 \qquad \text{if} \quad V \setminus W_i^c = \emptyset$$

and where $a_{(\lambda_2-i)}^{\lambda_1-\lambda_2}$ is defined above.

Proof. Note that the complete graph $G = K^{(\lambda_1+\lambda_2)}$ contains all of the complete graphs K^l for $0 < l < (\lambda_1 + \lambda_2)$. Since $G[V \setminus W_i]$ is the complete graph on $(\lambda_1 + \lambda_2 - i)$-many vertices, $G[V \setminus W_i] = K^{(\lambda_1+\lambda_2-i)}$, it follows that $Y_{G[V\setminus W_i]}(x; q, t) = Q_{(\lambda_1+\lambda_2-i)}$. Similarly, $G[V \setminus W_i^c] = K^i$ which in turn implies that $Y_{G[V\setminus W_i^c]}(x; q, t) = Q_{(i)}$. Expressing (8) as

$$Q_{(\lambda_1+\lambda_2)} \;=\; \sum_{i=0}^{\lambda_2} a^{\lambda_1-\lambda_2}_{(\lambda_2-i)}\, Q_{(\lambda_1+\lambda_2-i)}\, Q_{(i)}$$

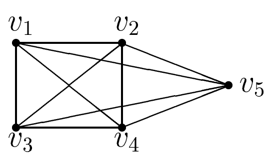
the result follows. ■

Example 4.1. Consider the expression of the two-row Macdonald polynomial $Q_{(3,2)}$. By Theorem 4.1, we have

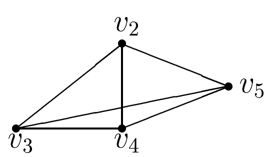
$$Q_{(3,2)} \;=\; \sum_{i=0}^{2} a_{(2-i)}\, Y_{G[V\setminus W_i]}(x; q, t)\, Y_{G[V\setminus W_i^c]}(x; q, t)$$

where $G = K^5$. Thus,

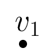
$$\begin{aligned} Q_{(3,2)} \;=\; & a_{(2)}\, Y_{G[V\setminus W_0]}(x; q, t)\, Y_{G[V\setminus W_0^c]}(x; q, t) \\ & + a_{(1)}\, Y_{G[V\setminus W_1]}(x; q, t)\, Y_{G[V\setminus W_1^c]}(x; q, t) \\ & + a_{(0)}\, Y_{G[V\setminus W_2]}(x; q, t)\, Y_{G[V\setminus W_2^c]}(x; q, t). \end{aligned}$$



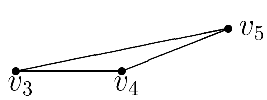
$G[V\backslash W_0]$



$G[V\backslash W_1]$



$G[V\backslash W_1^c]$



$G[V\backslash W_2]$

$G[V\backslash W_2^c]$

Computing the respective $Y_{G[V\setminus W_i]}(x; q, t)$ and $Y_{G[V\setminus W_i^c]}(x; q, t)$ for $0 \le i \le 2$ yields:

$$Q_{(3,2)} \;=\; \left(\frac{(t^{-1}; q)_2}{(q^2t; q)_2}\frac{(1-q^5)}{(1-q)}t^2\right)Q_{(5)} + \left(\frac{(t^{-1}; q)}{(q^2t; q)}\frac{(1-q^3)}{(1-q)}t\right)Q_{(4)}Q_{(1)} + Q_{(3)}Q_{(2)}. \quad \square$$

For the parameter α, the *Jack symmetric functions*, J_λ, are defined by:

$$J_\lambda \;=\; Q_\lambda(\alpha) \;=\; \lim_{t\to 1} Q_\lambda(t^\alpha, t)$$

where we set $q = t^\alpha$ in $Q_\lambda(x; q, t)$.

In [3], we have a formula for the Jack functions J_λ, $\lambda = (\lambda_1, \lambda_2)$:

$$J_{(\lambda_1,\lambda_2)} \;=\; Q_{(\lambda_1,\lambda_2)}(\alpha) \;=\; \sum_{i=0}^{\lambda_2} a_i^{\lambda_1-\lambda_2}(\alpha)\, Q_{(\lambda_1+i)}(\alpha)\, Q_{(\lambda_2-i)}(\alpha)$$

where

$$\begin{aligned} a_i^{\lambda_1-\lambda_2} &= (-1)^i \left(\frac{(1-\alpha)\cdots(1-(i-1)\alpha)}{i!} \right) \\ &\quad \cdot \left(\frac{(\lambda_1-\lambda_2+1)\cdots(\lambda_1-\lambda_2+i-1)(\lambda_1-\lambda_2+2i)}{(1+(\lambda_1-\lambda_2+1)\alpha)\cdots(1+(\lambda_1-\lambda_2+i)\alpha)} \right). \end{aligned}$$

Set $q = t^\alpha$ in the two-parameter symmetric function $Y_G(x; q, t)$. Define

$$Y_G(\alpha) \;=\; \lim_{t\to 1} Y_G(x; t^\alpha, t).$$

Similar to Theorem 4.1, we obtain a graphical analogue for the expansion of the two-row Jack symmetric functions $J_{(\lambda_1,\lambda_2)}$ using $Y_G(\alpha)$.

Corollary 4.1. Let $G = K^{(\lambda_1+\lambda_2)}$. For the partition $\lambda = (\lambda_1, \lambda_2)$,

$$J_{(\lambda_1,\lambda_2)} \;=\; Q_{(\lambda_1,\lambda_2)}(\alpha) \;=\; \sum_{i=0}^{\lambda_2} a_{(\lambda_2-i)}^{\lambda_1-\lambda_2}(\alpha)\, Y_{G[V\setminus W_i]}(\alpha)\, Y_{G[V\setminus W_i^c]}(\alpha)$$

where

$$Y_{G[V\setminus W_i^c]}(\alpha) \;=\; 1 \qquad \text{if} \quad V \setminus W_i^c = \emptyset,$$

$a_{(\lambda_2-i)}^{\lambda_1-\lambda_2}$ defined above, and $a_0^{\lambda_1-\lambda_2}(\alpha) = 1$.

5. The Symmetric Function $\mathbf{Y_G^c(x; q, t)}$.

We now introduce the "complement," $Y_G^c(x; q, t)$, of the two-parameter symmetric function $Y_G(x; q, t)$.

Define

$$m_i^c = \begin{cases} (\binom{V_i}{2} - m_i(k) + 1) & \text{if } (\binom{V_i}{2} - m_i(k) + 1) \;\le\; V_i \\ V_i & \text{otherwise.} \end{cases} \tag{11}$$

Definition 5.1.

$$Y_G^c(x; q, t) = \sum_k \binom{n}{V_1, V_2, \ldots}^{-1} \left(\prod_{i \in R(k)} \frac{(t; q)_{m_i^c}}{(q; q)_{m_i^c}} \right) x^k$$

where k ranges over all colorings of G.

Now, define

$$P^c_{\lambda_i} = \begin{cases} (|W^*_{\lambda_i}| - |W^*_{\lambda_i} \cap E(G)| + 1) & \text{if } (|W^*_{\lambda_i}| - |W^*_{\lambda_i} \cap E(G)| + 1) \le |W_{\lambda_i}| \\ |W_{\lambda_i}| & \text{otherwise.} \end{cases} \tag{12}$$

Note that $|W^*_{\lambda_i}| = \binom{|W_{\lambda_i}|}{2}$.

Proposition 5.1. For the simple graph G of order n,

$$Y^c_G(x;q,t) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_r}^{-1} \left(\sum_{W_\lambda \subseteq V(G)} \left(\prod_{i=1}^{r} \frac{(t;q)_{P^c_{\lambda_i}}}{(q;q)_{P^c_{\lambda_i}}} \right) \right) m_\lambda$$

where $W_\lambda \subseteq V(G)$ runs over all possible distinct ordered subset compositions for the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$; W_λ and $P^c_{\lambda_i}$ defined above.

Proof. Similar to the proof of Proposition 2.1, comparing the m^c_i to the $P^c_{\lambda_i}$ for the basic coloring k_λ of type $\lambda = (\lambda_1, \ldots, \lambda_r)$, and noting that $|W^*_{\lambda_i}| = \binom{|W_{\lambda_i}|}{2}$, we see that the coefficient of m_λ in $Y^c_G(x;q,t)$ is equal to

$$\binom{n}{\lambda_1, \ldots, \lambda_r}^{-1} \left(\sum_{W_\lambda \subseteq V(G)} \left(\prod_{i=1}^{r} \frac{(t;q)_{P^c_{\lambda_i}}}{(q;q)_{P^c_{\lambda_i}}} \right) \right). \quad \blacksquare \tag{13}$$

Let G be a simple, finite graph of order n. Then, the *complement* of the graph G, denoted G^c, is the graph of order n such that $v_iv_j \in E(G^c)$ if and only if $v_iv_j \notin E(G)$. Thus, if G has size d, it follows that G^c has size $(\binom{n}{2} - d)$.

Theorem 5.1. $Y^c_G(x;q,t) = Y_{G^c}(x;q,t)$.

Proof. Let G be a simple graph of order n with complement G^c. We want to show that for $\lambda \vdash n$, the coefficient of the monomial symmetric function m_λ in $Y^c_G(x;q,t)$ and $Y_{G^c}(x;q,t)$ are equal.

For $\lambda \vdash n$, the coefficient of m_λ in $Y^c_G(x;q,t)$ is given by (13) and the coefficient of m_λ is $Y_{G^c}(x;q,t)$ is given by

$$\binom{n}{\lambda_1, \ldots, \lambda_r}^{-1} \left(\sum_{W_\lambda \subseteq V(G^c)} \left(\prod_{i=1}^{r} \frac{(t;q)_{P_{\lambda_i}}}{(q;q)_{P_{\lambda_i}}} \right) \right) \tag{14}$$

where

$$P_{\lambda_i} = \begin{cases} (|W^*_{\lambda_i} \cap E(G^c)| + 1) & \text{if } (|W^*_{\lambda_i} \cap E(G^c)| + 1) \le |W_{\lambda_i}| \\ |W_{\lambda_i}| & \text{otherwise.} \end{cases} \tag{15}$$

Since $o(G) = o(G^c) \Rightarrow V(G) = V(G^c)$, we have that $W_\lambda \subseteq V(G) \equiv W_\lambda \subseteq V(G^c)$. Thus, for $1 \le i \le r$, $|W_{\lambda_i}|$ is equal for both G and G^c and similarly, $|W^*_{\lambda_i}|$ is equal for both G and G^c. By definition of G^c,

$$|W^*_{\lambda_i} \cap E(G^c)| \;=\; (\binom{|W_{\lambda_i}|}{2} - |W^*_{\lambda_i} \cap E(G)|) \;=\; (|W^*_{\lambda_i}| - |W^*_{\lambda_i} \cap E(G)|\,).$$

This implies that, with respect to G^c and G , $P_{\lambda_i} = P^c_{\lambda_i}$. Therefore, the coeffients (13) and (14) of m_λ in $Y^c_G(x;q,t)$ and $Y_{G^c}(x;q,t)$ are equal. ■

Using $Y^c_G(x;q,t)$, we obtain the following analogues to Propositions 3.1 – 3.4 for G^c.

Proposition 5.1. Let G be a simple graph. G^c has order n and size p if and only if the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(n-2)}}{(q;q)_2(q;q)^{(n-2)}}\, m_{(2,1^{(n-2)})} \tag{16}$$

is $\frac{2p}{n(n-1)}$ in $Y^c_G(x;q,t)$.

Proposition 5.2. Let G and H be graphs with degree sequences $deg(G)$ and $deg(H)$, respectively. Then $o(G^c) = o(H^c) = n$, $s(G^c) = s(H^c) \leq n$, and $deg(G^c) = deg(H^c)$ if and only if the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(n-2)}}{(q;q)_2(q;q)^{(n-2)}} m_{(2,1^{(n-2)})}$$

is $\leq \frac{2}{(n-1)}$ and is equal in both $Y^c_G(x;q,t)$ and $Y^c_H(x;q,t)$ and if the coefficients of $m_{((n-1),1)}$ in $Y^c_G(x;q,t)$ and $Y^c_H(x;q,t)$ are equal.

Proposition 5.3. Let G be a simple graph of order n. Any induced subgraph, G^c_I , of order $(n-1)$ of G^c is connected if and only if the multiplicity of the term

$$\frac{(t;q)_{(n-1)}(t;q)}{(q;q)_{(n-1)}(q;q)} m_{((n-1),1)}$$

in $Y^c_G(x;q,t)$ is one.

Proposition 5.4. Let G be a simple graph. G^c has order n and is a cycle of size n if and only if the multiplicity of the term

$$\frac{(t;q)_{(n-1)}(t;q)}{(q;q)_{(n-1)}(q;q)} m_{((n-1),1)}$$

in $Y^c_G(x;q,t)$ is one and the multiplicity of the term

$$\frac{(t;q)_2(t;q)^{(n-2)}}{(q;q)_2(q;q)^{(n-2)}} m_{(2,1^{(n-2)})}$$

is $\frac{2}{(n-1)}$.

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