Balancing cyclic R-ary Gray codes

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Abstract

New cyclic n-digit Gray codes are constructed over $\{0, 1, \dots, R-1\}$ for all $R \geq 3$, $n \geq 2$. These codes have the property that the distribution of the digit changes (transition counts) is close to uniform: For each $n \geq 2$, every transition count is within $R - 1$ of the average R^n/n, and for the 2-digit codes every transition count is either $\lfloor R^2/2 \rfloor$ or $\lceil R^2/2 \rceil$.

1 Introduction

For $R, n \geq 2$, an **n-digit R-ary Gray code** is a sequence in which each n-digit string with digits from the set $\{0, 1, \dots, R-1\}$ occurs exactly once and any two consecutive strings differ in only one digit and the difference equals ± 1; that is, the Lee distance equals 1, where the **Lee distance** between two n-strings is defined to be

$$d_L(a_1 \, \dots \, a_n \, , \, b_1 \, \dots \, b_n) = \sum_{j=1}^{n} \min\{|a_j - b_j| \, , \, R - |a_j - b_j|\} \, .$$

When the Lee distance between the last and first strings also equals 1, the code is called **cyclic**. For instance,

$$10, 11, 01, 00 \quad \text{and} \quad 20, 21, 22, 12, 02, 01, 11, 10, 00 \tag{1}$$

are cyclic 2-digit R-ary Gray codes, with $R = 2$ and $R = 3$, respectively.

It is helpful to consider a cyclic Gray code as a Hamiltonian cycle in the **R-ary n-cube**, the graph whose vertices are all n-strings with digits from the set $\{0, 1, \ldots, R-1\}$ in which two vertices are adjacent if they differ in only one digit by ± 1. Our usual representation of this graph will have the vertices arranged in an $R^{n-1} \times R$ array, where the first $n-1$ digits of a vertex n-string are given by its row label and the last digit by its column label. Figure 1 has the Hamiltonian cycles for the cyclic Gray codes in (1).

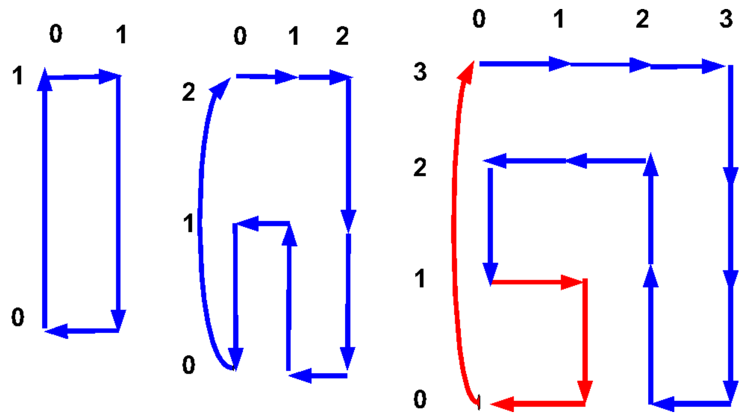

Figure 1: cyclic 2-digit R-ary Gray codes for $R = 2, 3, 4$.

The **transition sequence** of a cyclic Gray code records the successive digit changes in the code, beginning with the change from the first to the second string. For example, the transition sequences for the codes in (1) are:

$$2, 1, 2, 1 \quad \text{and} \quad 2, 2, 1, 1, 2, 1, 2, 1, 1\,. \tag{2}$$

For each $1 \leq j \leq n$, the **transition count** $TC(j)$ of the digit j is defined to be the number of times j occurs as a **transition digit** in the code. For each n, R there are many Gray codes, and one favorable characteristic for applications is a relative uniformity in the distribution of transition counts. When all transition counts are equal (this common value must necessarily equal R^n/n and n must divide R^n), the code is called **totally-balanced** or **uniform**. The binary and quaternary codes in Figure 1 are totally-balanced. In [7, 10, 11] totally-balanced cyclic n-digit binary codes are constructed for every n which is a power of 2, but there is no published construction of families of totally-balanced codes for arbitrary R. One of our results yields totally-balanced quaternary codes. (Refer to Corollary 1.)

When R^n is not divisible by n, totally-balanced cyclic codes cannot exist and it is reasonable to ask for codes in which every transition count is either $\lfloor R^n/n \rfloor$ or $\lceil R^n/n \rceil$, a phenomenon which we will call **well-balanced**. When R is even, the cyclicness forces every transition count to be even and so (unlike the cyclic ternary code given above in (1)) for even R every well-balanced cyclic R-ary Gray code must be totally-balanced. In [2], the authors asked if a well-balanced (non-cyclic) n-digit binary Gray code exists for every $n \geq 2$. Although this continues to be an open question, in [9, Chapter 3] I.

N. Suparta has obtained further evidence supporting the conjecture. In Section 2 we construct well-balanced 2-digit R-ary Gray codes for every $R \geq 2$. The codes depend on the even/odd parity of R, and are produced by a natural inductive process in which well-balanced cyclic 2-digit $(R+2)$-ary Gray codes are constructed from the R-ary ones.

A cyclic n-digit binary Gray code satisfying

$$|\, TC(i) - TC(j)\, | \leq 2 \text{ for every pair } i, j\,,$$

is referred to as **balanced**. In [2], G. S. Bhat and C. Savage constructed balanced cyclic n-digit (binary) Gray codes for every $n \geq 2$ when they completed some details which remained from the construction of J. P. Robinson and M. Cohn [7]. In a comment added to [2], F. Ruskey noted that T. Bakos [1, pp. 28–37] had much earlier constructed balanced cyclic binary codes; more recently I. N. Suparta [9, Section 3.2] has developed an elegant variant of these constructions. In each, n-digit cyclic binary codes are constructed inductively, using four $(n-2)$-digit binary codes.

Non-binary codes have been considered, for example, in [3, 4, 5, 6, 8, 9], but the question of constructing fairly uniform cyclic R-ary Gray codes for a general radix $R > 2$ has received far less attention than the binary case. In Section 3 we give a simple inductive construction which we later use in Section 4 to obtain **nearly-balanced** codes; that is, codes whose transition counts satisfy

$$\left| TC(j) - \frac{R^n}{n} \right| < \rho \quad \text{for all } 1 \leq j \leq n\,,$$

where ρ equals $R-1$ or $R-2$, whichever is even. The earlier constructions of well-balanced binary codes relied on delicate combinatorial arguments, whereas our use of R^n/n as a reference point has the advantage of allowing simpler proofs which involve only balancing inequalities.

2 Well-balanced cyclic 2-digit R-ary Codes

Theorem 1. *Let $R \geq 2$. If there exists a cyclic 2-digit R-ary Gray code with transition counts $\lfloor R^2/2 \rfloor + a$ and $\lceil R^2/2 \rceil + b$ for some constants a, b, then there exists a cyclic 2-digit $(R+2)$-ary Gray code with transition counts $\lfloor (R+2)^2/2 \rfloor + a$ and $\lceil (R+2)^2/2 \rceil\} + b$.*

Proof. The given cyclic R-ary code can be normalized so that it begins with R-1 0 and ends with the string 0 0. (This can be done because at some point there must be consecutive terms of the form $c\ d$; $c \pm 1\ d$. The required form is then obtained by translating and reflecting if necessary so that the last term is 0 0 and the first term is R-1 0. Since the distribution of transition counts remains the same under these operations, we may assume the original cyclic code has this form.) Therefore, the code corresponds to a Hamiltonian path from R-1 0 to 0 0 in the R-ary square to which the edge from 0 0 to R-1 0 can be added to form a Hamiltonian cycle.

Labeling the rows of the $(R+2)$-ary square as R+1 R . . . 1 0 and the columns as 0 1 . . . R+1, we first construct a path from the vertex R+1 0 to R-1 0 by beginning with the vertex R+1 0, proceeding horizontally through all vertices of the form R+1 i, from R+1 0 to R+1 R+1, and then vertically down to 0 R+1. From there we travel to 0 R and continue by proceeding vertically upward to R R, traveling horizontally over to R 0 and then ending with the edge to R-1 0. This initial path covers all vertices in which at least one component is either R or R+1. Its transition sequence is: $R+1$ twos, followed by $R+1$ ones, one 2, and then R ones, R twos and one 1, for a total of $2R+2$ copies of each of 1 and 2. Attaching the given Hamiltonian path from R-1 0 to 0 0 to the end of this initial segment results in a Hamiltonian path in the $(R+2)$-ary square, and the addition of the edge from 0 0 to R+1 0 results in a Hamiltonian cycle. (The quaternary code in Figure 1 was obtained in this way from the binary code, and the diagram in Figure 2 illustrates the general construction.)

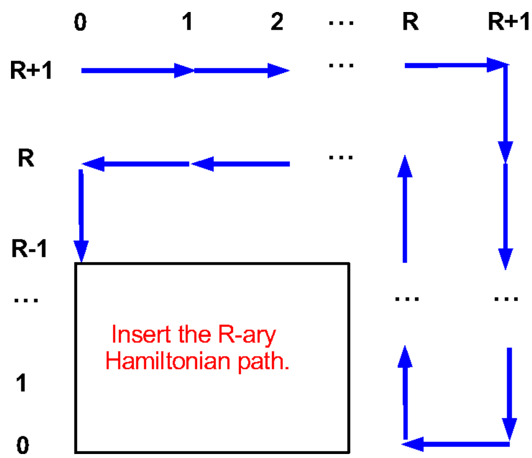

Figure 2: Construction of the initial segment.

Since

$$\lfloor R^2/2 \rfloor + 2R + 2 = \lfloor (R+2)^2/2 \rfloor \quad \text{and} \quad \lceil R^2/2 \rceil + 2R + 2 = \lceil (R+2)^2/2 \rceil \,,$$

the code produced is a cyclic 2-digit $(R+2)$-ary Gray code with the required transition counts. □

Theorem 2. *For any $R \geq 2$ there exists a cyclic 2-digit R-ary Gray code which is well-balanced, and so for even R the code is totally-balanced.*

Proof. Consider the statement: For every $n \geq 1$ there exist well-balanced cyclic 2-digit R-ary Gray codes for $R = 2n$ and $R = 2n+1$. The codes given in (1) satisfy this statement for $n = 1$. Application of Theorem 1 with $a = b = 0$ therefore yields the result by induction on n. □

3 Constructing $(\mathbf{n+1})$-digit codes from n-digit codes

One simple way to construct a Hamiltonian path on an $L \times R$ rectangular grid is to traverse successive vertices across rows; that is,

$$1\,0\ ,\ 1\,1\ ,\ \ldots\ ,\ 1\,R-1\ ;\ \ 2\,R-1\ ,\ \ldots\ ,\ 2\,1\ ,\ 2\,0\ \text{etc.}\ , \tag{3}$$

as illustrated in Figure 3. Notice that the terminal vertex is either L 1 or L R, according

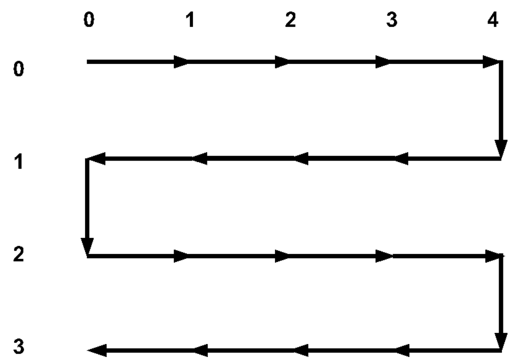

Figure 3: Our basic Hamiltonian path on the 4×5 grid.

to whether L is even or odd.

Our construction of nearly-balanced cyclic codes is based on a modification of this simple idea. The construction involves partitioning the row indices of an R-ary $(n+1)$-cube into blocks. Specifically, the rows are first indexed by a fixed cyclic n-digit R-ary Gray code, $a_1, a_2, \ldots, a_{R^n}$, and then are partitioned into L nonempty blocks of consecutive elements, say

$$B_1 := a_1, \ldots, a_{i_1}\ ,\ B_2 := a_{i_1+1}, \ldots, a_{i_2}\ ,\ \ldots\ ,\ B_L := a_{i_{L-1}+1}, \ldots, a_{R^n}\ .$$

This idea of using partitions is at least implicit in other published constructions, among them [2, 6, 7, 9, 10].

The digit changes from a_{i_k} to $a_{i_{k+1}}$ will be called the **connecting digits of the partition**, and the number of times any digit occurs as a connecting digit will be referred to as its **connecting multiplicity**. For notational convenience, we assume the code is in a standard form in which the coordinates of the n-strings have been permuted if necessary so that 1 is the L-th connecting digit, the transition digit from a_{R^n} to a_1.

We use this partition to construct a Hamiltonian path in the R-ary $(n+1)$-cube in the following way: First of all, the Hamiltonian path will "respect" the partition $B_1 \cup \ldots \cup B_L$; that is, for every column index k and every block B_j, the path must traverse the vertices $a_{i_j+1}\,k\ ,\ a_{i_j+2}\,k\ ,\ \ldots,\ a_{i_{j+1}}\,k$ consecutively either in that direction or in the reverse direction. This allows us to picture the $(n+1)$-cube as an $L \times R$ array in which the rows are indexed by the blocks $B_1, \ldots, B_L$.

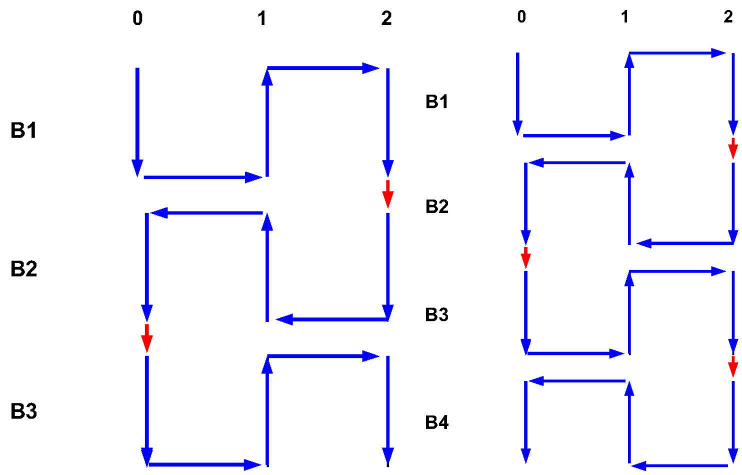

Figure 4: Ternary Hamiltonian paths for three and four blocks.

If R is odd, the construction given in (3) yields a Hamiltonian path on the R-ary $(n+1)$-cube (refer to Figure 4), and when the number L of partition-blocks is *even number*, the addition of the edge from $a_{R^n}\,0$ to $a_1\,0$ results in a Hamiltonian cycle.

When R is even, the path can be adjusted to get a Hamiltonian cycle on the R-ary $(n+1)$-cube in the following way: Within the given R-ary $(n+1)$-cube (with its rows labeled by the cyclic Gray code which has been partitioned by $B_1 \cup \ldots B_L$ and its columns labeled 0 1 ... R-1), consider the $R^n \times (R-1)$ grid consisting of the vertices whose last digit is nonzero. Since $R-1$ is odd, the Hamiltonian path given in (3) can be constructed on the blocks of this grid. Its initial vertex is $a_1\,1$ and the terminal vertex is either a_{R^n} R-1 or $a_{R^n}\,1$, depending on whether L is odd or even. In either event, the terminal vertex is adjacent to $a_{R^n}\,0$. A Hamiltonian cycle on the R-ary $(n+1)$-cube is obtained by appending this edge to the initial path, following through all vertices with zero second digit in the order:

$$a_{R^n}\,0\ ,\ a_{R^n-1}\,0\ ,\ \ldots\ ,\ a_1\,0\ ,$$

and then ending with the edge to $a_1\,1$. This is pictured for $R=4$ in Figure 5.

Regardless of whether R is even or odd, the construction gives a cyclic $(n+1)$-digit R-ary Gray code (provided the number of partition-blocks is even when R is odd). In what follows this will be referred to as the **code induced by the partition** $B_1 \cup \ldots B_L$.

By arranging the vertices of the (binary) n-cube in a grid whose rows are indexed by an $(n-2)$-cube and its columns by 00 10 11 01, most known n-digit binary codes can be seen to have this kind of partitioning behavior. When viewed from this perspective of partitions, the combinatorial argument for the binary constructions in [2, 7] required some restrictions on the usable partitions; for instance, at least two blocks can have only one element.

The following result gives the distribution of transition counts in codes induced by a partition.

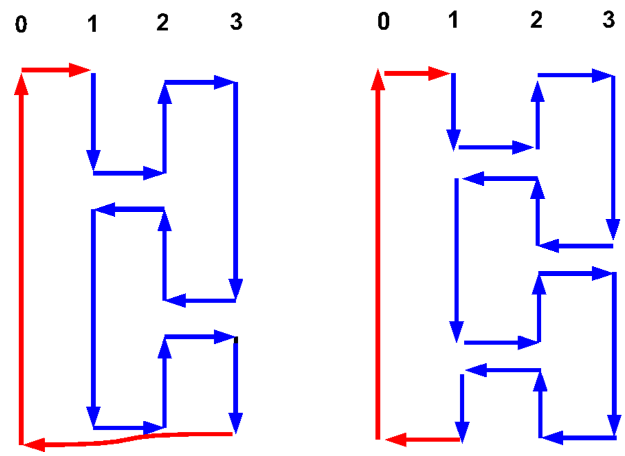

Figure 5: Construction of cyclic 4-ary codes.

Theorem 3. *Let $R \geq 3$ be an integer, and set ρ equal to either $R-2$ or $R-1$, whichever is even. Let $a_1, a_2, \ldots, a_{R^n}$ be any cyclic n-digit R-ary Gray code with transition counts $T_1, \ldots, T_n$ (where two digit positions have been transposed if necessary so that the digit change from a_{R^n} to a_1 is in the first digit). Let $B_1 \cup \ldots B_L$ be any partition of the code, and k_j be the connecting multiplicity of the digit j. When R is odd and L is even, the j-th transition count of the induced cyclic $(n+1)$-digit Gray code is*

$$TC(j) = \begin{cases} R\,T_j - k_j\,\rho & \text{if } j \leq n \\ L\,\rho & \text{if } j = n+1 \end{cases}. \tag{4}$$

When R is even the transition counts of the induced cyclic $(n+1)$-digit Gray code are

$$TC(j) = \begin{cases} R\,T_j - k_j\,\rho - 2 & \text{if } j = 1 \\ R\,T_j - k_j\,\rho & \text{if } 1 < j \leq n \\ L\,\rho + 2 & \text{if } j = n+1 \end{cases}. \tag{5}$$

Proof. First we consider the contribution from the initial Hamiltonian path. Setting $N := \rho + 1$ (which is odd), the Hamiltonian path on the $R^n \times N$ grid has (N-1)L horizontal lines, and so this initial part of the process of forming the Hamiltonian cycle accumulates (N-1)L changes in the digit $n+1$. Every edge within the partition-blocks is traversed in every column, the edge corresponding to the connecting digit from a_{R^n} to a_1 never occurs, and the edge corresponding to every other connecting digit occurs exactly once. Therefore, for every $1 < j \leq n$ the digit j changes

$$N\,(T_j - k_j) + k_j = N\,T_j - (N-1)k_j$$

times in this initial segment, and there is one fewer change for j=1.

When R is odd, the cycle is completed by adding one edge which corresponds to a change in the first digit. Since $N = R$ holds in this case, (4) is obtained.

For even R, two horizontal edges are added to complete the cycle, giving

$$TC(n+1) = (N-1)L + 2 = L\,\rho + 2\,.$$

As for additional vertical edges, every edge in the original code except for the one corresponding to transitioning from a_{R^n} to a_1 is added once. Since the transition from a_{R^n} to a_1 is a change in the first digit, this gives

$$TC(1) = (R-1)\,T_1 - (R-2)k_1 - 1 + T_1 - 1 = R\,T_1 - k_1\,\rho - 2\,,$$

and for all $2 \le j \le n$,

$$TC(j) = (R-1)\,T_j - (R-2)k_j + T_j = R\,T_j - k_j\,\rho\,.$$

This proves (5). $\square$

We end this section with the construction of a cyclic 3-digit ternary code induced from a partition of the 2-digit ternary code in (1). Since R is odd, the number of partition-blocks must be even. Also, the digit 1 occurs more frequently as a transition digit in the given 2-digit code than 2 (refer to (2)) and so a preferred partition would have 1 occurring at least as often as a connecting digit as 2 does. Consider any 4-block partition in which 2 occurs only once as a connecting digit. (For instance,

$$B_1 := \{20, 21\}\ ,\ B_2 := \{22\}\ ,\ B_3 := \{12, 02, 01\}\ ,\ B_4 := \{11, 10, 00\}$$

is such a partition, since the connecting digits are 2,1,1,1 .) From (4), the induced code has

$$TC(1) = 9\ ;\ TC(2) = 10\ ;\ TC(3) = 8\,.$$

It is not totally balanced but is nearly-balanced since $|TC(j) - R^n/n| \le 1$ holds for all j.

4 Nearly-balanced cyclic R-ary Gray Codes

Lemma 1. *Let $R \ge 3$ be an integer. If T is any transition count of either a well-balanced 2-digit R-ary code or a nearly-balanced n-digit R-ary code for $n \ge 3$, then there exists a unique integer k such that $0 < k < T$ and*

$$0 \le R\,T - k\,\rho - \frac{R^{n+1}}{n+1} < \rho\,. \tag{6}$$

Proof. For $k := \lfloor (RT - \frac{R^{n+1}}{n+1})/\rho \rfloor$, we see that (6) is satisfied. Rearranging (6),

$$R\,T - \frac{R^{n+1}}{n+1} - \rho < k\,\rho \le R\,T - \frac{R^{n+1}}{n+1}\,,$$

and so $0 < k < T$ can be proved by showing

$$R\,T - \frac{R^{n+1}}{n+1} - \rho > 0 \quad \text{and} \quad R\,T - \frac{R^{n+1}}{n+1} < T\rho\,;$$

that is,

$$RT - \frac{R^{n+1}}{n+1} - \rho > 0 \quad \text{and} \quad T\,(\rho - R) + \frac{R^{n+1}}{n+1} > 0\,. \tag{7}$$

Considering the case of a well-balanced 2-digit code, every transition count T satisfies

$$\frac{R^2-1}{2} \le T \le \frac{R^2+1}{2}\,,$$

giving

$$RT - \frac{R^3}{3} - \rho \ge \frac{R^3 - 9R + 6}{6} > 0$$

for $R \ge 3$, and since $R - \rho$ equals 2 or 1,

$$T\,(\rho - R) + \frac{R^3}{3} \ge \frac{R^2+1}{2}(-2) + \frac{R^3}{3} = \frac{R^2(R-3)-3}{3}$$

which is positive for all $R \ge 4$. Also, for $R = 3$, $T\,(\rho - R) + R^3/3 = -T + 9 \ge 4$ proving (7) for well-balanced 2-digit codes.

It remains to consider the transition counts of a nearly-balanced n-digit code with $n \ge 3$; that is,

$$\left| T - \frac{R^n}{n} \right| < \rho\,,$$

where we recall that ρ equals either $R-1$ or $R-2$, whichever is even. The first inequality in (7) can be easily checked for $R = n = 3$ since $\rho = 2$ and the transition counts satisfy $8 \le T \le 10$. Defining $a_n := R^{n+1}/n(n+1)$, we observe that for $R, n \ge 3$

$$\frac{a_{n+1}}{a_n} = R\big(1 - \frac{2}{n+2}\big) > 1\,,$$

and so $a_n \ge a_3 = R^4/12$ for all $n \ge 3$. Since T is a transition count of a nearly-balanced n-digit code, $|T - R^n/n| < \rho$; $T > \frac{R^n}{n} - \rho$ and $\rho \le R-1$ imply

$$\begin{aligned} RT - \frac{R^{n+1}}{n+1} - \rho &> R\Big(\frac{R^n}{n} - \rho\Big) - \frac{R^{n+1}}{n+1} - \rho = a_n - (R+1)\rho \\ &\ge a_3 - (R+1)(R-1) = \frac{R^2(R^2-12)+12}{12} \end{aligned}$$

which is positive for $R \ge 4$. The first inequality has already been proved for $R = n = 3$. For $R = 3$ and $n \ge 4$, we observe that

$$a_n - (R+1)\rho \ge a_4 - 4\cdot 2 = \frac{3^5}{20} - 8 > 0\,.$$

This proves the first inequality in (7) for all $R, n \ge 3$.

For the second inequality: $T < \rho + \frac{R^n}{n}$ and $\rho < R$ give

$$T\,(\rho - R) + \frac{R^{n+1}}{n+1} > \Big(\rho + \frac{R^n}{n}\Big)(\rho - R) + \frac{R^{n+1}}{n+1}$$
$$= \rho^2 + \Big(\frac{R^n}{n} - R\Big)\rho - \frac{R^{n+1}}{n(n+1)}\,,$$

a strictly increasing function of $\rho \geq R-2$. Evaluation at $\rho = R-2$ gives

$$(R-2)^2 - R(R-2) + R^n\Big(\frac{R}{n+1} - \frac{2}{n}\Big) = -2(R-2) + b_n\,,$$

where $b_n := R^n\Big(\frac{R}{n+1} - \frac{2}{n}\Big)$. We note that

$$\frac{b_{n+1}}{b_n} = R\,\frac{n}{n+2}\;\cdot\;\frac{(n+1)R - 2(n+2)}{nR - 2(n+1)}$$

where the restrictions $R \geq 3$ and $n \geq 3$ imply each of the two identified factors is greater than 1. Therefore, $b_n \geq b_3$ and so

$$-2(R-2) + b_n \geq -2(R-2) + b_3 = \frac{3R^4 - 8R^3 - 24R + 48}{12},$$

which is positive for all $R \geq 3$. This proves (7) and so the lemma. □

Theorem 4. *For every integer $R \geq 3$ and every $n \geq 2$ there exists a nearly-balanced cyclic n-digit R-ary Gray code.*

Proof. Let $R \geq 3$ be a fixed integer. Since every well-balanced code is nearly-balanced, Theorem 2 serves as the base case for a proof by induction on $n \geq 2$.

Let $T_1, \ldots, T_n$ be the transition counts of any 2-digit well-balanced code or any nearly-balanced n-digit code for $n \geq 3$. We will prove there is a partition of this code for which the induced cyclic $(n+1)$-digit code is nearly-balanced.

From Lemma 1 it follows that for each $j = 1, \ldots, n$ there exists a unique integer k_j with $1 \leq k_j < T_j$ such that

$$S_j := R\,T_j - k_j\,\rho - \frac{R^{n+1}}{n+1} \tag{8}$$

satisfies

$$0 \leq S_j < \rho\,. \tag{9}$$

Permute the coordinates of the n-digit code if necessary to obtain

$$\rho > S_1 \geq S_2 \geq \ldots \geq S_n \geq 0\;\;. \tag{10}$$

Since each $1 \leq k_j < T_j$, the digit j occurs more than k_j times as a transition digit in the code and so it can be partitioned into $L := \sum_{j=1}^n k_j$ blocks in such a way that each digit j occurs exactly k_j times as a connecting digit. Using (8) and $\sum_{j=1}^n T_j = R^n$,

$$\sum_{j=1}^{n} S_j = \frac{R^{n+1}}{n+1} - \rho\,L\,. \tag{11}$$

Let $M := \lfloor \sum_{j=1}^n S_j/\rho \rfloor$. Then (9) implies

$$0 \leq M < n\,. \tag{12}$$

Also, by definition of M, $\quad 0 \leq \sum_{j=1}^n S_j - M\,\rho < \rho$ holds and so (11) gives

$$0 \leq \frac{R^{n+1}}{n+1} - \rho\,(L+M) < \rho\,. \tag{13}$$

We also note that if

$$S_{M+1} = \ldots = S_n = 0,$$

then from (11) and (9),

$$\frac{R^{n+1}}{n+1} - \rho\,(L+M) = \sum_{j=1}^{n} S_j - M\,\rho = \sum_{j=1}^{M} S_j - M\,\rho < M\,\rho - M\,\rho = 0\,,$$

contrary to (13). From (9) we therefore have

$$0 < S_j < \rho \text{ for all } j \leq M+1 \quad \text{ and } \quad 0 \leq S_j < \rho \text{ for all } j > M+1\,. \tag{14}$$

We distinguish two cases, according to whether R is odd or even.

CASE 1: $R \geq 3$ is odd. We will prove there exists a refinement of the above partition for which the induced $(n+1)$-digit code is cyclic and nearly-balanced. Since R is odd, the number L of partition-blocks must be even for the code induced by the partition to be cyclic. Our refinement will have L' partition-blocks where L' equals either $L+M$ or $L+M+1$, whichever is even, and so $D := L' - L$ connecting digits will be added. Since D equals M or $M+1$ and $1 \leq k_j < T_j$, by (12) $\quad 0 \leq D \leq n$ and each of the first D digits can be chosen one more time as a connecting digit, giving L' partition-blocks with multiplicities $k'_1, \ldots, k'_n$ satisfying

$$k'_j = k_j + 1 \text{ if } j \leq D, \quad k'_j = k_j \text{ if } j > D\,.$$

According to (4), the first n transition counts of the code induced by this new partition have

$$TC(j) - \frac{R^{n+1}}{n+1} = \begin{cases} S_j - \rho & \text{if } j \leq D \\ S_j & \text{if } D < j \leq n \end{cases}\,,$$

which by (14) satisfy the nearly-balanced condition. Also, the last transition count is $TC(n+1) = L'\,\rho$, and (13) implies

$$\frac{R^{n+1}}{n+1} - \rho < (L+M)\,\rho \leq TC(n+1) \leq (L+M+1)\,\rho \leq \frac{R^{n+1}}{n+1} + \rho\,.$$

Therefore, the code is nearly-balanced unless

$$(L+M)\,\rho = R^{n+1}/(n+1)\,,$$

which cannot happen since ρ is even and R is odd. This completes the proof for odd R.

CASE 2: $R \geq 4$ is even. We refine the partition by adding one more connecting digit for each $j = 2, 3, \ldots, M+1$. This induces a code with $L+M$ partition-blocks whose first n transition digits satisfy

$$TC(j) - \frac{R^{n+1}}{n+1} = \begin{cases} S_1 - 2 & \text{if } j = 1 \\ S_j - \rho & \text{if } 1 < j \leq M+1 \\ S_j & \text{if } M+1 < j \leq n \end{cases} .$$

As in Case 1, (14) implies the nearly-balanced condition for all $TC(j)$ with $j \leq n$. Also, since $TC(n+1) = (L+M)\,\rho + 2$, from (13) we have

$$-\rho \leq -2 \leq \frac{R^{n+1}}{n+1} - TC(n+1) < \rho - 2 < \rho\,,$$

and so the code is nearly-balanced unless $\rho = 2$ (and so $R = 4$) and

$$TC(n+1) = \frac{R^{n+1}}{n+1} + 2\,.$$

For this special case, consider any modification of the last partition which has one less occurrence of 1 as a connecting digit. This changes only the first and last transition counts to

$$TC(1) + 2 = S_1 + \frac{R^{n+1}}{n+1} \quad \text{and} \quad TC(n+1) - \rho = TC(n+1) - 2 = \frac{R^{n+1}}{n+1}\,,$$

respectively, resulting in a nearly-balanced code. □

We next apply this theorem to the case when n divides R^n.

Corollary 1. *Let $R \geq 3$ and $n \geq 2$ be integers. If n divides R^n, then there exists a cyclic n-digit R-ary Gray code whose transition counts $TC(1), \ldots, TC(n)$ satisfy*

$$\left| TC(j) - \frac{R^n}{n} \right| \leq \rho - 1 \quad \textit{for all } 1 \leq j \leq n\,. \tag{15}$$

In particular, when n is a power of 2 there exist totally-balanced cyclic n-digit quaternary Gray codes.

Proof. By hypothesis, $TC(j) - \frac{R^n}{n}$ is an integer, and so the strict inequality in the definition of nearly-balanced implies (15).

When $R = 4$ and $n = 2^k$, n divides R^n and so every transition count of a nearly-balanced cyclic n-digit quaternary code is within $\rho - 1 = 1$ of the even integer $4^n/n$. Since the code is cyclic and R is even, every transition count must be even and so must equal $4^n/n$. □

We end with an application to binary codes.

Corollary 2. *For each* $n \geq 2$ *there exists a cyclic* $2n$*-digit binary Gray code whose transition counts* $TC(1), \ldots TC(2n)$ *are such that every average of the form* $\frac{TC(2i-1)+TC(2i)}{2}$ *equals either* $\lfloor 2^{2n}/(2n) \rfloor$ *or* $\lceil 2^{2n}/(2n) \rceil$.

Proof. Consider the following mapping of elements of $\mathbb{Z}_4$ to 2-digit binary strings:

$$0 \mapsto 00 \quad ; \quad 1 \mapsto 01 \quad ; \quad 2 \mapsto 11 \quad ; \quad 3 \mapsto 10\,.$$

This mapping extends to a mapping of quaternary n-strings to binary $2n$-strings under which Lee distance is preserved. Therefore, any cyclic n-digit quaternary Gray code is mapped to a cyclic $2n$-digit binary Gray code, and the transition counts $TC(1), \ldots TC(2n)$ of the binary code have the property that $TC(2i-1)+TC(2i)$ equals the transition count of the digit i in the 4-ary code for every $i = 1, \ldots n$. Using any nearly-balanced n-digit 4-ary Gray code, we therefore have

$$\left|\frac{TC(2i-1)+TC(2i)}{2} - \frac{2^{2n}}{2n}\right| = \frac{1}{2}\left|TC(2i-1)+TC(2i) - \frac{4^n}{n}\right| < 1\,,$$

from the definition of nearly-balanced. Since every transition count of a cyclic binary Gray code is even, each such average is an integer and the last inequality implies it must equal either $\lfloor 2^{2n}/(2n) \rfloor$ or $\lceil 2^{2n}/(2n) \rceil$, as claimed. □

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