Which Chessboards have a Closed Knight's Tour within the Cube?

Joe DeMaio

Department of Mathematics and Statistics
Kennesaw State University, Kennesaw, Georgia, 30144, USA
jdemaio@kennesaw.edu

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Abstract

A closed knight's tour of a chessboard uses legal moves of the knight to visit every square exactly once and return to its starting position. When the chessboard is translated into graph theoretic terms the question is transformed into the existence of a Hamiltonian cycle. There are two common tours to consider on the cube. One is to tour the six exterior $n \times n$ boards that form the cube. The other is to tour within the n stacked copies of the $n \times n$ board that form the cube. This paper is concerned with the latter. In this paper necessary and sufficient conditions for the existence of a closed knight's tour for the cube are proven.

1 Introduction

The closed knight's tour of a chessboard is a classic problem in mathematics. Can the knight use legal moves to visit every square on the board and return to its starting position? The unique movement of the knight makes its tour an intriguing problem which is trivial for other chess pieces. The knight's tour is an early example of the existence problem of Hamiltonian cycles. So early in fact that it predates Kirkman's [1] 1856 paper which posed the general problem and Hamilton's Icosian Game of the late 1850s [2]. Euler presented solutions for the standard 8×8 board [3] and the problem is easily generalized to rectangular boards. In 1991 Schwenk [4] completely answered the question: Which rectangular chessboards have a knight's tour?

Schwenk's Theorem: An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:

(a) m and n are both odd;
(b) $m \in \{1, 2, 4\}$;
(c) $m = 3$ and $n \in \{4, 6, 8\}$.

The problem of the closed knight's tour has been further generalized to many three-dimensional surfaces: the torus [5], the cylinder [6], the pillow [7], the Mobius strip, the Klein bottle, the exterior of the cube [8], the interior levels of the cube, etc. Watkins provides excellent coverage of these variations of the knight's tour in Across the Board: The Mathematics of Chessboard Problems [9]. However, the general analysis of these three-dimensional surfaces is to unfold them into the two-dimensional plane, apply Schwenk's Theorem as liberally as possible and tidy up any remaining cases as simply as possible. While this technique is successful at obtaining complete characterizations in some settings, it does not adequately tackle every surface and leaves the reader wondering what could be accomplished with a true three-dimensional technique.

There are two common tours to consider on the cube. One is to tour the six exterior $n \times n$ boards the form the cube. Qing and Watkins [8] recently showed that a knight's tour exists on the exterior of the cube for all n. The focus of this paper is the tour within the n stacked copies of the $n \times n$ board that form the cube.

In Watkins book three examples of closed knight's tours within the three-dimensional chess board of the cube are provided. In two (the cubes of side 6 and 8) cases constructions take the closed knight's tour for square boards and then piece the boards back together level by level to create a closed tour for the cube. Watkins does not provide a proof for the general case and indicates that the work lays in deciding which tours of the two-dimensional board to use and where to make the jumps from level to level. He thanks Stewart [10] for having worked out the details for the cube of side 8. The technique of touring the cube level by level does not adopt itself well to a general proof since deciding which boards to use is one component of the construction.

As a problem, Watkins assigns the exercise of constructing a closed knight's tour for the cube of side 4. One possible solution is shown in Figure 1 [9]. Watkins notes that "since there is not even an open tour of the 4×4 board ... this is perhaps a harder problem than finding a tour for the $8 \times 8 \times 8$ chessboard." I agree with Watkins. As seen with the closed tour of the cube of side 4, existence of a closed (or even open) tour of the board is not a requirement for the existence of a tour for the cube of side n.

4	23	30	9
29	10	3	24
22	1	12	31
11	32	21	2

1

27	8	13	18
14	17	28	7
5	26	19	16
20	15	6	25

2

36	55	62	41
61	42	35	56
54	33	44	63
43	64	53	34

3

59	40	45	50
46	49	60	39
37	58	51	48
52	47	38	57

4

Figure 1: KT_1, A Closed Tour of a Cube of Side 4

Kumar [11] notes that "little attention has been paid" to the knight's tour extension "in three-dimensional space." Kumar has constructed and investigated many closed and open knight's tours for parameters ≤ 8 but does not tackle the general case.

As it turns out the characterization of cubes that admit a closed knight's tour is very easy to state. Furthermore, once you divest yourself of the notion of tackling the cube by its two-dimensional levels, the proof falls out in a very natural inductive manner.

Theorem: For $n \geq 4$, the cube of side n contains a closed knight's tour if and only if n is even.

First of all, note that the cubes of sides $n = 1, 2, 3$ are too small to allow a knight to move from every square. For $n = 1, 2$ the knight cannot make a legal move. For $n = 3$, the knight cannot move to or from the center cell.

2 The nonexistence of a closed knight's tour within the cube of side $n \equiv 1 \mod 2$

There exists no closed knight's tour within the cube of side n where n is odd. This is a clear analogue of the fact that a closed knight's tour does not exist on the $n \times m$ board where both n and m are odd. It is not quite as immediate for the cube. Especially so as one considers the extra freedom granted in the cube as the knight extends its reach from 8 moves to 24 moves. For boards on an odd numbered level start with a black square in the upper left hand corner. For those boards on an even numbered level, start with a white square in the upper left hand corner. Now all legal moves of the knight alternate colors as demonstrated in Figure 2 with the $a - b$, $c - d$ and $e - f$ moves. The resulting graph of legal moves of the knight on the cube is now bipartite. When considering the cube as a whole, this coloring scheme seems very natural as all adjacent squares alternate color.

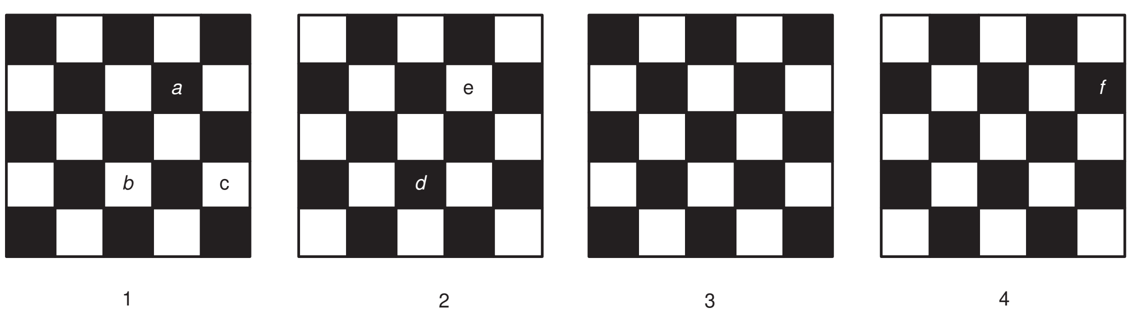

Figure 2

For the cube of side n there will exist $\left\lceil \frac{n^3}{2} \right\rceil$ black squares and $\left\lfloor \frac{n^3}{2} \right\rfloor$ white squares. If n is odd then $\left\lceil \frac{n^3}{2} \right\rceil \neq \left\lfloor \frac{n^3}{2} \right\rfloor$ and the corresponding bipartite graph will not contain a Hamiltonian cycle. Note that this argument easily extends to show that the $n \times m \times k$ board does not admit a closed knight's tour where n, m and k are all odd.

3 Construction of a closed knight's tour within the cube of side $n \equiv 0 \mod 4$

For $n = 4k$, take k copies of KT_1 of Figure 1 placed left to right. Any two copies of KT_1 can be combined to create a closed tour on the $4 \times 8 \times 4$ board by deleting the $2-3$ edge on level 1 of the left KT_1 and the $14-15$ edge on level 2 of the right KT_1. Next create the $2-15$ and $3-14$ edges as shown in Figure 3. Repeat this process left to right for the remaining copies of KT_1 and the result is a closed knight's tour for the $4 \times n \times 4$ board which we shall denote KT_2.

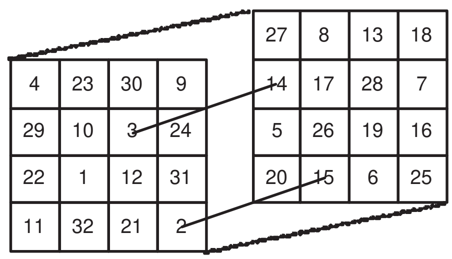

Figure 3

Now create $k-1$ additional copies of KT_2 placed below each other. On level 2 in the leftmost KT_1 of each KT_2, delete the $5-6$ edge on the back copy of KT_2 and the $7-8$ edge on the front copy of KT_2 and create the $5-8$ and $6-7$ edges as shown in Figure 4. This creates a closed knight's tour for the $n \times n \times 4$ board, denoted KT_3.

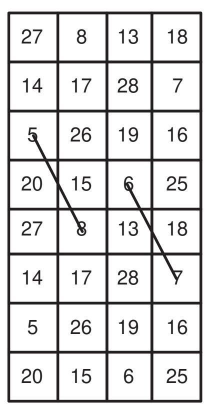

Figure 4

Finally take $k-1$ copies of KT_3 and stack them atop one another. To connect two copies of KT_3 delete the $46-47$ edge of level 4 in the leftmost KT_1 in the bottom copy and the $10-11$ edge of level 1 in the top copy of KT_3 in the leftmost KT_1. Now create the $10-47$ and $11-46$ edges. This results in a closed knight's tour for the cube of side $n = 4k$ for all positive integers k. Of course this method can be used to construct a closed knight's tour for the $n \times m \times k$ board for $n, m, k \equiv 0 \mod 4$.

4 Construction of a closed knight's tour within the cube of side $n \equiv 2 \mod 4$

First a base case of a closed knight's tour of side $n = 6$ is provided from [11].

47	50	53	38	41	62
52	39	48	63	54	37
49	46	51	40	61	42
66	69	64	57	72	55
45	58	67	70	43	60
68	65	44	59	56	71

1

104	81	102	85	88	83
101	94	105	82	75	86
80	103	100	87	84	89
95	106	93	76	99	74
92	79	108	97	90	77
107	96	91	78	73	98

2

5	16	7	34	3	36
18	33	4	1	8	27
15	6	17	28	35	2
22	19	32	11	26	9
31	14	21	24	29	12
20	23	30	13	10	25

3

122	133	138	115	120	131
137	114	121	132	139	116
134	123	136	117	130	119
113	144	111	124	127	140
110	135	142	129	118	125
143	112	109	126	141	128

4

185	204	193	216	187	206
194	199	186	205	192	215
203	184	211	198	207	188
200	195	202	189	214	191
183	210	197	212	181	208
196	201	182	209	190	213

5

174	165	176	147	150	163
177	146	173	164	167	148
172	175	166	149	162	151
145	178	155	170	159	168
154	171	180	157	152	161
179	156	153	160	169	158

6

Figure 5: A Closed Tour of a Cube of Side 6

Extending this cube of side 6 to a cube of side $n \equiv 2 \mod 4$ will not be as simple as extending the cube of side 4 to a cube of side $n \equiv 0 \mod 4$. We cannot just take copies of the cube of side 6 to use as an extension since the formal induction employed is to show that the existence of a tour within the cube of side $n \equiv 2 \mod 4$ implies the existence of a tour within the cube of side $n+4$. Other closed tours of rectangular prisms will be required.

Consider the closed tour on the $3 \times 6 \times 4$ board of Figure 6. Take two copies of Figure 6 placed front to back. Now, delete the $37 - 38$ edge on level 1 in the front copy and the $8 - 9$ edge on level 2 in the back copy. Using those same vertices, create the $8 - 38$ edge and the $9 - 37$ edge. This provides us with a closed knight's tour for the $6 \times 6 \times 4$ board.

1	6	47	28	33	38
46	57	4	37	12	31
5	2	43	32	29	34

1

52	49	72	7	40	27
71	64	53	26	19	8
54	51	48	9	42	39

2

55	58	67	10	13	22
68	3	44	23	30	35
45	56	59	36	11	14

3

66	63	60	21	18	15
61	50	69	16	41	24
70	65	62	25	20	17

4

Figure 6: A Closed Tour of the $3 \times 6 \times 4$ Board

The first step in constructing a closed knight's tour for the cube of side $n = 4k + 2$ is to stack $k - 1$ copies of the $6 \times 6 \times 4$ board on top of the cube of side 6 of Figure 5. Delete the $174 - 175$ edge of Figure 5 and the $5 - 6$ edge of the back copy of the $6 \times 6 \times 4$ board. Create the $5 - 174$ and $6 - 175$ edges to form a closed knight's tour on the $6 \times 6 \times 10$ board. Attach the remaining $k - 2$ Figure 6s by deleting in adjacent pairs (front or back, but matching) of the $6 \times 6 \times 4$ board, the $65 - 66$ edge of level 4 of the bottom Figure 6 and the $5 - 6$ edge of level 1 of the top Figure 6 and creating $5 - 66$ and $6 - 65$ edges, thus creating the closed tour for the $6 \times 6 \times n$ box.

The second step is to extend this construction to width n. Consider the open tour of Figure 7. Note that k copies of this open tour can be extended to an open $6 \times 4k$ tour by deleting the $22 - 23$ edge and creating the $1 - 22$ and $23 - 24$ edges in adjacent copies.

1	4	23	20
24	21	2	5
3	6	19	22
16	13	8	11
7	10	15	18
14	17	12	9

1	4	23	20	1	4	23	20
24	21	2	5	24	21	2	5
3	6	19	22	3	6	19	22
16	13	8	11	16	13	8	11
7	10	15	18	7	10	15	18
14	17	12	9	14	17	12	9

Figure 7: An Open Tour of the 6×4 Board and Extension

Create six copies of a $6 \times (n - 6)$ open tour as indicated in Figure 7. In the base cube of side 6 from Figure 5, delete the $41 - 42$, $88 - 89$, $2 - 3$, $119 - 120$, $187 - 188$ and $150 - 151$ edges on levels 1 though 6 and then using one copy of the $6 \times (n - 6)$ open tour per level create the $1 - 42$, $24 - 41$, $1 - 89$, $24 - 88$, $1 - 2$, $3 - 24$, $1 - 119$, $24 - 120$, $1 - 188$, $24 - 187$, $1 - 151$ and $24 - 150$ edges. Next create an additional $n - 6$ copies of a $6 \times (n - 6)$ open tour of Figure 7. These copies will be attached to the $n - 6$ copies of Figure 6 that were stacked on top of the base cube of side 6 from Figure 5. To do so delete the $33 - 34$, $39 - 40$, $13 - 14$ and $17 - 18$ edges on levels 1 through 4. Take four copies of a $6 \times (n-6)$ open tour of Figure 7 per Figure 6, delete the $33-34$, $39-40$, $13-14$ and $17-18$ edges and create the $1-34$, $24-33$, $1-39$, $24-40$, $1-14$, $13-24$, $1-17$ and $18-24$ edges.

This now forms a closed knight's tour for the $6 \times n \times n$ rectangular prism. This tour will form the back wall of the cube of side $n \equiv 2 \mod 4$. Now we play this game again to create the left wall of the cube of side $n \equiv 2 \mod 4$ as shown in Figure 8. Once the left wall is completed, a cube of side $n-6 \equiv 0 \mod 4$ and a board of size $(n-6) \times (n-6) \times 6$ will be inserted to complete the cube of side $n \equiv 2 \mod 4$.

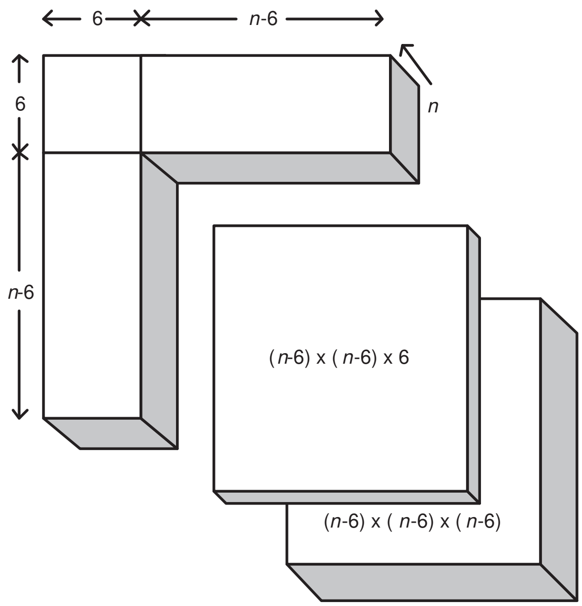

Figure 8: Construction of a Cube of Side $n \equiv 2 \mod 4$

Once again, create six copies of a $6 \times (n-6)$ open tour as indicated in Figure 7. In the base cube of side 6 from Figure 5, delete the $59-60$, $77-78$, $12-13$, $125-126$, $208-209$ and $160-161$ edges on levels 1 through 6. Next create the $1-59$, $24-60$, $1-78$, $24-77$, $1-13$, $12-24$, $1-126$, $24-125$, $1-209$, $24-208$, $1-160$ and $24-161$ edges. Take four copies of a $6 \times (n-6)$ open tour as indicated in Figure 7 per board, delete the $31-32$, $8-9$, $35-36$ and $24-25$ edges in each copy of the $6 \times 6 \times 4$ board of Figure 6 and create the $1-32$, $24-31$, $1-9$, $8-24$, $1-36$, $24-35$, $1-25$ and $24-24$ edges.

This construction yields the left and back walls of our cube of length, height and width n, going in 6 squares. Now use a cube of side $n-6$. Since $n \equiv 2 \mod 4$ then $n-6 \equiv 0 \mod 4$ and we can take a cube from our previous construction. Take this cube and note the $3-4$ edge on level 1 in the very first KT_1. Furthermore note the $9-10$ edge in the open 6×4 tour of Figure 7. Delete these two edges and create the $3-9$ and $4-10$ edges. All that is left is to extend this cube up 6 squares. To do so construct a closed tour of the $(n-6) \times (n-6) \times 2$ board.

1	14	23	28
24	27	2	13
15	4	25	22
26	21	16	3

1

10	5	32	19
31	20	9	6
8	11	18	29
17	30	7	12

2

Figure 9: A Closed Tour of the $4 \times 4 \times 2$ Board

Take Figure 9 and extend it widthwise by creating multiple copies. Delete the $2 - 3$ edge on level 1 of the left copy and the $30 - 31$ edge of level 2 of the right copy. Create the $2 - 31$ and $3 - 30$ edges. Now take multiple copies of this new construction and extend it lengthwise by deleting on level 1 on the leftmost side of the back copy the $21 - 22$ edge and on level 1 on the leftmost side of the front copy the $23 - 24$ edge. Now create the $21 - 24$ and $22 - 23$ edges. Finally stack 3 copies of this new construction by deleting in adjacent copies the $10 - 11$ edge on level 2 of the bottom copy and the $14 - 15$ edge on level 1 of the top copy and creating the $10 - 15$ and $11 - 14$ edges. Attach this closed tour to the cube of side $n - 6$ by deleting any $15 - 16$ edge of level 1 of this construction. Note that this level 1 is sitting atop a level 4 of a KT_1 in the construction of the cube of side $n - 6$. Delete the $51 - 52$ edge in this KT_1 and create the $15 - 51$ and $16 - 52$ edges, thus creating the closed knight's tour on the cube of side $n \equiv 2 \mod 4$.

5 Future Work

The next step in this work is to extend this characterization of the cubes which admit a closed knight's tour to a characterization of the general rectangular prism. My conjecture is that like the cube, once the dimensions of the rectangular prism grow to be sufficiently large only the prism with an odd number of squares will not admit a closed knight's tour.

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