# Distance domination and distance irredundance in graphs


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**Abstract**

A set $D \subseteq V$ of vertices is said to be a *(connected) distance $k$-dominating set* of $G$ if the distance between each vertex $u \in V - D$ and $D$ is at most $k$ (and $D$ induces a connected graph in $G$). The minimum cardinality of a (connected) distance $k$-dominating set in $G$ is the *(connected) distance $k$-domination number* of $G$, denoted by $\gamma_k(G)$ ($\gamma_k^c(G)$, respectively). The set $D$ is defined to be a *total $k$-dominating set* of $G$ if every vertex in $V$ is within distance $k$ from some vertex of $D$ other than itself. The minimum cardinality among all total $k$-dominating sets of $G$ is called the *total $k$-domination number* of $G$ and is denoted by $\gamma_k^t(G)$. For $x \in X \subseteq V$, if $N^k[x] - N^k[X - x] \neq \emptyset$, the vertex $x$ is said to be *$k$-irredundant in $X$*. A set $X$ containing only $k$-irredundant vertices is called *$k$-irredundant*. The *$k$-irredundance number of $G$*, denoted by $ir_k(G)$, is the minimum cardinality taken over all maximal $k$-irredundant sets of vertices of $G$. In this paper we establish lower bounds for the distance $k$-irredundance number of graphs and trees. More precisely, we prove that $\frac{5k+1}{2} ir_k(G) \geq \gamma_k^c(G) + 2k$ for each connected graph $G$ and $(2k+1) ir_k(T) \geq \gamma_k^c(T) + 2k \geq |V| + 2k - kn_1(T)$ for each tree $T = (V, E)$ with $n_1(T)$ leaves. A class of examples shows that the latter bound is sharp. The second inequality generalizes a result of Meierling and Volkmann [9] and Cyman, Lemańska and Raczek [2] regarding $\gamma_k$ and the first generalizes a result of Favaron and Kratsch [4] regarding $ir_1$. Furthermore, we shall show that $\gamma_k^c(G) \leq \frac{3k+1}{2}\gamma_k^t(G) - 2k$ for each connected graph $G$, thereby generalizing a result of Favaron and Kratsch [4] regarding $k = 1$.



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# 1 Terminology and introduction

In this paper we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set $V$ and edge set $E$. The number of vertices $|V|$ is called the *order* of $G$ and is denoted by $n(G)$. For two distinct vertices $u$ and $v$ the *distance* $d(u, v)$ between $u$ and $v$ is the length of a shortest path between $u$ and $v$. If $X$ and $Y$ are two disjoint subsets of $V$, then the distance between $X$ and $Y$ is defined as $d(X, Y) = \min\{d(x, y) \mid x \in X,\ y \in Y\}$. The *open k-neighborhood* $N^k(X)$ of a subset $X \subseteq V$ is the set of vertices in $V \setminus X$ of distance at most $k$ from $X$ and the *closed k-neighborhood* is defined by $N^k[X] = N^k(X) \cup X$. If $X = \{v\}$ is a single vertex, then we denote the (closed) $k$-neighborhood of $v$ by $N^k(v)$ ($N^k[v]$, respectively). The (closed) 1-neighborhood of a vertex $v$ or a set $X$ of vertices is usually denoted by $N(v)$ or $N(X)$, respectively ($N[v]$ or $N[X]$, respectively). Now let $U$ be an arbitrary subset of $V$ and $u \in U$. We say that $v$ is a *private k-neighbor of u with respect to U* if $d(u, v) \leq k$ and $d(u', v) > k$ for all $u' \in U - \{u\}$, that is $v \in N^k[u] - N^k[U - \{u\}]$. The *private k-neighborhood of u with respect to U* will be denoted by $PN^k[u, U]$ ($PN^k[u]$ if $U = V$).

For a vertex $v \in V$ we define the *degree* of $v$ as $d(v) = |N(v)|$. A vertex of degree one is called a *leaf* and the number of leaves of $G$ will be denoted by $n_1(G)$.

A set $D \subseteq V$ of vertices is said to be a *(connected) distance k-dominating set* of $G$ if the distance between each vertex $u \in V - D$ and $D$ is at most $k$ (and $D$ induces a connected graph in $G$). The minimum cardinality of a (connected) distance $k$-dominating set in $G$ is the *(connected) distance k-domination number* of $G$, denoted by $\gamma_k(G)$ ($\gamma_k^c(G)$, respectively). The distance 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A set $D \subseteq V$ of vertices is defined to be a *total k-dominating set* of $G$ if every vertex in $V$ is within distance $k$ from some vertex of $D$ other than itself. The minimum cardinality among all total $k$-dominating sets of $G$ is called the *total k-domination number* of $G$ and is denoted by $\gamma_k^t(G)$. We note that the parameters $\gamma_k^c(G)$ and $\gamma_k^t(G)$ are only defined for connected graphs and for graphs without isolated vertices, respectively.

For $x \in X \subseteq V$, if $PN^k[x] \neq \emptyset$, the vertex $x$ is said to be *k-irredundant in X*. A set $X$ containing only $k$-irredundant vertices is called *k-irredundant*. The *k-irredundance number of G*, denoted by $ir_k(G)$, is the minimum cardinality taken over all maximal $k$-irredundant sets of vertices of $G$.

In 1975, Meir and Moon [10] introduced the concept of a $k$-dominating set (called a '$k$-covering' in [10]) in a graph, and established an upper bound for the $k$-domination number of a tree. More precisely, they proved that $\gamma_k(T) \leq |V(T)|/(k+1)$ for every tree $T$. This leads immediately to $\gamma_k(G) \leq |V(G)|/(k+1)$ for an arbitrary graph $G$. In 1991, Topp and Volkmann [11] gave a complete characterization of the class of graphs $G$ that fulfill the equality $\gamma_k(G) = |V(G)|/(k+1)$.

The concept of $k$-irredundance was introduced by Hattingh and Henning [5] in 1995. With $k = 1$, the definition of an $k$-irredundant set coincides with the notion of an irredundant set, introduced by Cockayne, Hedetniemi and Miller [1] in 1978. Since then a lot of research has been done in this field and results have been presented by many authors (see [5]).

In 1991, Henning, Oellermann and Swart [8] motivated the concept of total distance domination in graphs which finds applications in many situations and structures which give rise to graphs.

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [6], [7].

In this paper we establish lower bounds for the distance $k$-irredundance number of graphs and trees. More precisely, we prove that $\frac{5k+1}{2} ir_k(G) \geq \gamma_k^c(G) + 2k$ for each connected graph $G$ and $(2k+1) ir_k(T) \geq \gamma_k(T) + 2k \geq |V| + 2k - kn_1(T)$ for each tree $T = (V, E)$ with $n_1(T)$ leaves. A class of examples shows that the latter bound is sharp. Since $\gamma_k(G) \geq ir_k(G)$ for each connected graph $G$, the latter generalizes a result of Meierling and Volkmann [9] and Cyman, Lemanska and Raczek [2] regarding $\gamma_k$ and the former generalizes a result of Favaron and Kratsch [4] regarding $ir_1$. In addition, we show that if $G$ is a connected graph, then $\gamma_k^c(G) \leq (2k+1)\gamma_k(G) - 2k$ and $\gamma_k^c(G) \leq \frac{3k-1}{2}\gamma_k^t(G) - 2k$ thereby generalizing results of Duchet and Meyniel [3] for $k = 1$ and Favaron and Kratsch [4] for $k = 1$, respectively.

## 2 Results

First we show the inequality $\gamma_k^c \leq (2k+1)\gamma_k - 2k$ for connected graphs.

**Theorem 2.1.** *If $G$ is a connected graph, then*

$$\gamma_k^c(G) \leq (2k+1)\gamma_k(G) - 2k.$$

*Proof.* Let $G$ be a connected graph and let $D$ be a distance $k$-dominating set. Then $G[D]$ has at most $|D|$ components. Since $D$ is a distance $k$-dominating set, we can connect two of these components to one component by adding at most $2k$ vertices to $D$. Hence, we can construct a connected $k$-dominating set $D' \supseteq D$ in at most $|D| - 1$ steps by adding at most $(|D| - 1)2k$ vertices to $D$. Consequently,

$$\gamma_k^c(G) \leq |D'| \leq |D| + (|D| - 1)2k = (2k+1)|D| - 2k$$

and if we choose $D$ such that $|D| = \gamma_k(G)$, the proof of this theorem is complete. $\square$

The results given below follow directly from Theorem 2.1.

**Corollary 2.2 (Duchet & Meyniel [3] 1982).** *If $G$ is a connected graph, then*

$$\gamma^c(G) \leq 3\gamma(G) - 2.$$

**Corollary 2.3 (Meierling & Volkmann [9] 2005; Cyman, Lemańska & Raczek [2] 2006).** *If $T$ is a tree with $n_1$ leaves, then*

$$\gamma_k(T) \geq \frac{|V(T)| - kn_1 + 2k}{2k+1}.$$

*Proof.* Since $\gamma_k^c(T) \geq |V(T)| - kn_1$ for each tree $T$, the proposition is immediate. $\square$

The following lemma is a preparatory result for Theorems 2.5 and 2.7.

**Lemma 2.4.** *Let $G$ be a connected graph and let $I$ be a maximal $k$-irredundant set such that $ir_k(G) = |I|$. If $I_1 = \{v \in I \mid v \in PN^k[v]\}$ is the set of vertices that have no $k$-neighbor in $I$, then*

$$\gamma_k^c(G) \leq (2k+1)ir_k(G) - 2k + (k-1)\frac{|I - I_1|}{2}.$$

*Proof.* Let $G$ be a connected graph and let $I \subseteq V$ be a maximal $k$-irredundant set. Let

$$I_1 := \{v \in I \mid v \in PN^k[v]\}$$

be the set of vertices in $I$ that have no $k$-neighbors in $I$ and let

$$I_2 := I - I_1$$

be the complement of $I_2$ in $I$. For each vertex $v \in I_2$ let $u_v \in PN^k[v]$ be a $k$-neighbor of $v$ such that the distance between $v$ and $u_v$ is minimal and let

$$B := \{u_v \mid v \in I_2\}$$

be the set of these $k$-neighbors. Note that $|B| = |I_2|$. If $w$ is a vertex such that $w \notin N^k[I \cup B]$, then $I \cup \{w\}$ is a $k$-irredundant set of $G$ that strictly contains $I$, a contradiction. Hence $I \cup B$ is a $k$-dominating set of $G$.

Note that $G[I \cup B]$ has at most $|I \cup B| = |I_1| + 2|I_2|$ components. From $I \cup B$ we shall construct a connected $k$-dominating set $D \supseteq I \cup B$ by adding at most

$$|I_2|(k-1) + (|I_1| + \left\lfloor \frac{|I_2|}{2} \right\rfloor - 1)2k + \left\lfloor \frac{|I_2|}{2} \right\rfloor (k-1)$$

vertices to $I \cup B$.

We can connect each vertex $v \in I_2$ with its corresponding $k$-neighbor $u_v \in B$ by adding at most $k-1$ vertices to $I \cup B$.

Recall that each vertex $v \in I_2$ has a $k$-neighbor $w \neq v$ in $I_2$. Therefore we need to add at most $k-1$ vertices to $I \cup B$ to connect such a pair of vertices.

By combining the two observations above, we can construct a $k$-dominating set $D' \supseteq I \cup B$ from $I \cup B$ with at most $|I_1| + \lfloor |I_2|/2 \rfloor$ components by adding at most $(k-1)|I_2| + (k-1)\lceil |I_2|/2 \rceil$ vertices to $I \cup B$. Since $D'$ is a $k$-dominating set of $G$, these components can be joined to a connected $k$-dominating set $D$ by adding at most $(|I_1| + \lfloor |I_2|/2 \rfloor - 1)2k$ vertices to $D'$.

All in all we have shown that there exists a connected $k$-dominating set $D$ of $G$ such that

$$\begin{aligned} |D| &\leq |I_1| + 2|I_2| + (k-1)|I_2| + (k-1)\left\lceil \frac{|I_2|}{2} \right\rceil + 2k(|I_1| + \left\lfloor \frac{|I_2|}{2} \right\rfloor - 1) \\ &\leq (2k+1)|I| - 2k + (k-1)\frac{|I_2|}{2}. \end{aligned}$$

Hence, if we choose the set $I$ such that $|I| = ir_k(G)$, the proof of this lemma is complete. $\square$

Since $|I_2| \leq |I|$ for each $k$-irredundant set $I$, we derive the following theorem.

**Theorem 2.5.** *If $G$ is a connected graph, then*

$$\gamma_k^c(G) \leq \frac{5k+1}{2} ir_k(G) - 2k.$$

The next result follows directly from Theorem 2.5.

**Corollary 2.6 (Favaron & Kratsch [4] 1991).** *If $G$ is a connected graph, then*

$$\gamma^c(G) \leq 3ir(G) - 2.$$

For acyclic graphs Lemma 2.4 can be improved as follows.

**Theorem 2.7.** *If $T$ is a tree, then*

$$\gamma_k^c(T) \leq (2k+1)ir_k(T) - 2k.$$

*Proof.* Let $T$ be a tree and let $I \subseteq V$ be a maximal $k$-irredundant set. Let

$$I_1 := \{v \in I \mid v \in PN^k[v]\}$$

be the set of vertices in $I$ that have no $k$-neighbors in $I$ and let

$$I_2 := I - I_1$$

be the complement of $I_2$ in $I$. For each vertex $v \in I_2$ let $u_v \in PN^k[v]$ be a $k$-neighbor of $v$ such that the distance between $v$ and $u_v$ is minimal and let

$$B := \{u_v \mid v \in I_2\}$$

be the set of these $k$-neighbors. Note that $|B| = |I_2|$. If $w$ is a vertex such that $w \notin N^k[I \cup B]$, then $I \cup \{w\}$ is a $k$-irredundant set of $G$ that strictly contains $I$, a contradiction. Hence $I \cup B$ is a $k$-dominating set of $G$.

Note that $T[I \cup B]$ has at most $|I \cup B| = |I_1| + 2|I_2|$ components. From $I \cup B$ we shall construct a connected $k$-dominating set $D \supseteq I \cup B$ by adding at most

$$(2k-1)|I_2| + 2k(|I_1| - 1)$$

vertices to $I \cup B$. To do this we need the following definitions. For each vertex $v \in I_2$ let $P_v$ be the (unique) path between $v$ and $u_v$ and let $x_v$ be the predecessor of $u_v$ on $P_v$. Let $I_2 = S \cup L_1 \cup L_2$ be a partition of $I_2$ such that

$$S = \{v \in I_2 \mid d(v, u_v) = 1\}$$

is the set of vertices of $I_2$ that are connected by a 'short' path with $u_v$,

$$L_1 = \{v \in I_2 \mid N^k(x_v) \cap I_1 \neq \emptyset\}$$

is the set of vertices of $I_2$ that are connected by a 'long' path with $u_v$ and the vertex $x_v$ has a $k$-neighbor in $I_1$ and

$$L_2 = I_2 - (S \cup L_1)$$

is the complement of $S \cup L_1$ in $I_2$. In addition, let $L = L_1 \cup L_2$. We construct $D$ following the procedure given below.

*Step 0:* Set $\mathcal{I} := I_2$, $\mathcal{S} := S$ and $\mathcal{L} := L$.

*Step 1:* We consider the vertices in $\mathcal{S}$.

*Step 1.1:* If there exists a vertex $v \in \mathcal{S}$ such that $d(v, w) \leq k$ for a vertex $w \in \mathcal{L}$, we can connect the vertices $v$, $u_v$, $w$ and $u_w$ to one component by adding at most $2(k-1)$ vertices to $I \cup B$.
Set $\mathcal{I} := \mathcal{I} - \{v, w\}$, $\mathcal{S} := \mathcal{S} - \{v\}$ and $\mathcal{L} := \mathcal{L} - \{w\}$ and repeat Step 1.1.

*Step 1.2:* If there exists a vertex $v \in \mathcal{S}$ such that $d(v, w) \leq k$ for a vertex $w \in \mathcal{S}$ with $v \neq w$, we can connect the vertices $v$, $u_v$, $w$ and $u_w$ to one component by adding at most $k-1$ vertices to $I \cup B$.
Set $\mathcal{I} := \mathcal{I} - \{v, w\}$ and $\mathcal{S} := \mathcal{S} - \{v, w\}$ and repeat Step 1.2.

*Step 1.3:* If there exists a vertex $v \in \mathcal{S}$ such that $d(v, w) \leq k$ for a vertex $w \in I_2 - (\mathcal{S} \cup \mathcal{L})$, we can connect the vertices $v$ and $u_v$ to $w$ by adding at most $k-1$ vertices to $I \cup B$.
Set $\mathcal{I} := \mathcal{I} - \{v\}$ and $\mathcal{S} := \mathcal{S} - \{v\}$ and repeat Step 1.3.

Note that after completing Step 1 the set $\mathcal{S}$ is empty and there are at most $|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3$ components left, where $r_i$ denotes the number of times Step 1.i was repeated for $i = 1, 2, 3$. Furthermore, we have added at most $(k-1)(2r_1 + r_2 + r_3)$ vertices to $I \cup B$.

*Step 2:* We consider the vertices in $L_1$.

If there exists a vertex $v \in L_1 \cap \mathcal{L}$, let $w \in I_1$ be a $k$-neighbor of $x_v$. We can connect the vertices $v$, $u_v$ and $w$ to one component by adding at most $2(k-1)$ vertices to $I \cup B$.
Set $\mathcal{I} := \mathcal{I} - \{v\}$ and $\mathcal{L} := \mathcal{L} - \{v\}$ and repeat Step 2.

Note that after completing Step 2 we have $\mathcal{L} \subseteq L_2$ and there are at most $|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s$ components left, where $s$ denotes the number of times Step 2 was repeated and the numbers $r_i$ are defined as above. Furthermore, we have added at most $(k-1)(2r_1 + r_2 + r_3 + 2s)$ vertices to $I \cup B$.

*Step 3:* We consider the vertices in $L_2$. Recall that for each vertex $v \in L_2$ the vertex $x_v$ has a $k$-neighbor $w \in I_2$ besides $v$.

Let $v$ be a vertex in $L_2 \cap \mathcal{L}$ such that $x_v$ has a $k$-neighbor $w \in I_2 - \mathcal{I}$. We can connect the vertices $v$, $u_v$ and $w$ by adding at most $2(k-1)$ vertices to $I \cup B$. Set $\mathcal{I} := \mathcal{I} - \{v\}$ and $\mathcal{L} := \mathcal{L} - \{v\}$ and repeat Step 3.

Note that after completing Step 3 the sets $\mathcal{I}$ and $\mathcal{L}$ are empty and there are at most $|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s - 2t$ components left, where $t$ denotes the number of times Step 3 was repeated and the numbers $r_i$ and $s$ are defined as above. Furthermore, we have added at most $(k-1)(2r_1 + r_2 + r_3 + 2s + 2t)$ vertices to $I \cup B$.

*Step 4:* We connect the remaining components to one component.

Let $D'$ be the set of vertices that consists of $I \cup B$ and all vertices added in Steps 1 to 3. Since $D'$ is a $k$-dominating set of $G$, the remaining at most $|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s - 2t$ components can be connected to one component by adding at most $(|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s - 2t - 1)2k$ vertices to $D'$.

After completing Step 4 we have constructed a connected $k$-dominating set $D \supseteq I \cup B$ by adding at most

$$(k-1)(2r_1 + r_2 + r_3 + 2s + 2t) + (|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s - 2t - 1)2k$$

vertices to $I \cup B$.

We shall show now that the number of vertices we have have added is less or equal than $(2k-1)|I_2| + 2k(|I_1| - 1)$. Note that $|I_2| = 2r_1 + 2r_2 + r_3 + s + t$. Then

$$\begin{aligned}
&(k-1)(2r_1 + r_2 + r_3 + 2s + 2t) + (|I_1| + 2|I_2| - 3(r_1 + r_2) - 2r_3 - 2s - 2t - 1)2k \\
&\quad - (2k-1)|I_2| - 2k(|I_1| - 1) \\
&= (2k+1)|I_2| - 3k(2r_1 + 2r_2 + r_3 + s + t) - k(r_3 + s + t) \\
&\quad + (k-1)(2r_1 + r_2 + r_3 + 2s + 2t) \\
&= -(k-1)(2r_1 + 2r_2 + r_3 + s + t) - k(r_3 + s + t) + (k-1)(2r_1 + r_2 + r_3 + 2s + 2t) \\
&= -(k-1)r_2 - kr_3 - s - t \\
&\leq 0.
\end{aligned}$$

If we choose $|I|$ such that $|I| = ir_k(T)$, it follows that

$$\begin{aligned}
\gamma_k^c(T) \leq |D| &\leq |I_1| + 2|I_2| + 2k|I_1| + (2k-1)|I_2| - 2k \\
&= (2k+1)|I| - 2k \\
&= (2k+1)ir_k(T) - 2k
\end{aligned}$$

which completes the proof of this theorem. □

As an immediate consequence we get the following corollary.

**Corollary 2.8.** *If $T$ is a tree with $n_1$ leaves, then*

$$ir_k(G) \geq \frac{|V(T)| - kn_1 + 2k}{2k+1}.$$

*Proof.* Since $\gamma_k^c(T) \geq |V(T)| - kn_1$ for each tree $T$, the result follows directly from Theorem 2.7. □

Note that, since $\gamma_k(G) \geq ir_k(G)$ for each graph $G$, Corollary 2.8 is also a generalization of Corollary 2.3. The following theorem provides a class of examples that shows that the bound presented in Theorem 2.7 is sharp.

**Theorem 2.9 (Meierling & Volkmann [9] 2005; Cyman, Lemanska & Raczek [2] 2006).** *Let $\mathcal{R}$ denote the family of trees in which the distance between each pair of distinct leaves is congruent $2k$ modulo $(2k+1)$. If $T$ is a tree with $n_1$ leaves, then*

$$\gamma_k(T) = \frac{|V(T)| - kn_1 + 2k}{2k+1}$$

*if and only if $T$ belongs to the family $\mathcal{R}$.*

**Remark 2.10.** *The graph in Figure 1 shows that the construction presented in the proof of Theorem 2.7 does not work if we allow the graph to contain cycles. It is easy to see that $I = \{v_1, v_2\}$ is an $ir_2$-set of $G$ and that $D = \{u_1, u_2, x_1, x_2, x_3\}$ is a $\gamma_2^c$-set of $G$. Following the construction in the proof of Theorem 2.7, we have $I_1 = \emptyset$, $I_2 = \{v_1, v_2\}$ and $B = \{u_1, u_2\}$ and consequently, $D' = I_2 \cup B \cup \{x_1, x_2, x_3\}$. But $|D'| = 7 \not\leq 6 = (2 \cdot 2 + 1)|I| - 2 \cdot 2$ and $D$ contains none of the vertices of $I$.*

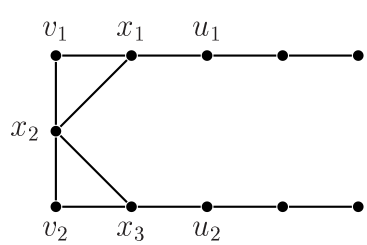


Figure 1.

Nevertheless, we think that the following conjecture is valid.

**Conjecture 2.11.** *If $G$ is a connected graph, then*

$$\gamma_k^c(G) \leq (2k+1)ir_k(G) - 2k.$$

Now we analyze the relation between the connected distance domination number and the total distance domination number of a graph.

**Theorem 2.12.** *If $G$ is a connected graph, then*

$$\gamma_k^c(G) \leq \frac{3k+1}{2}\gamma_k^t(G) - 2k.$$

*Proof.* Let $G$ be a connected graph and let $D$ be a total $k$-dominating set of $G$ of size $\gamma_k^t(G)$. Each vertex $x \in D$ is in distance at most $k$ of a vertex $y \in D - \{x\}$. Thus we get a dominating set of $G$ with at most $\lfloor |D|/2 \rfloor$ components by adding at most $\lceil |D|/2 \rceil (k-1)$ vertices to $D$. As in the proof of Lemma 2.4, the resulting components can be joined to a connected $k$-dominating set $|D'|$ by adding at most $(\lfloor |D|/2 \rfloor - 1)2k$ vertices. Consequently,

$$\gamma_k^c(G) \leq |D'| \leq |D| + \left\lceil \frac{|D|}{2} \right\rceil (k-1) + (\left\lfloor \frac{|D|}{2} \right\rfloor - 1)2k \leq \frac{3k+1}{2}|D| - 2k = \frac{3k+1}{2}\gamma_k^t(G) - 2k$$

and the proof is complete. □

For distance $k = 1$ we obtain the following result.

**Corollary 2.13 (Favaron & Kratsch [4] 1991).** *If $G$ is a connected graph, then*

$$\gamma^c(G) \leq 2\gamma^t(G) - 2.$$

The following example shows that the bound presented in Theorem 2.12 is sharp.

**Example 2.14.** *Let $P$ be the path on $n = (3k+1)r$ vertices with $r \in \mathbb{N}$. Then $\gamma_k^c(P) = n - 2k$, $\gamma_k^t(P) = 2r$ and thus, $\gamma_k^c(P) = \frac{3k+1}{2}\gamma_k^t(P) - 2k$.*

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