A combinatorial representation with Schröder paths of biorthogonality of Laurent biorthogonal polynomials

Shuhei Kamioka*

Department of Applied Mathematics and Physics, Graduate School of Informatics
Kyoto University, Kyoto 606-8501, Japan
kamioka@amp.i.kyoto-u.ac.jp

Submitted: Apr 12, 2006; Accepted: May 3, 2007; Published: May 11, 2007
Mathematics Subject Classifications: 05A15, 42C05, 05E35

Abstract

Combinatorial representation in terms of Schröder paths and other weighted plane paths are given of Laurent biorthogonal polynomials (LBPs) and a linear functional with which LBPs have orthogonality and biorthogonality. Particularly, it is clarified that quantities to which LBPs are mapped by the corresponding linear functional can be evaluated by enumerating certain kinds of Schröder paths, which imply orthogonality and biorthogonality of LBPs.

1 Introduction and preliminaries

Laurent biorthogonal polynomials, or LBPs for short, appeared in problems related to Thron type continued fractions (T-fractions), two-point Padé approximants and moment problems (see, e.g., [6]), and are studied by many authors (e.g. [6, 4, 5, 12, 11]). We recall fundamental properties of LBPs.

Remark. In this paper, ℓ and m, n are used for integers and nonnegative integers, respectively.

Let $\mathbb{K}$ be a field. (Commonly $\mathbb{K} = \mathbb{C}$.) LBPs are monic polynomials $P_n(z) \in \mathbb{K}[z]$, $n \geq 0$, such that $\deg P_n(z) = n$ and $P(0) \neq 0$, which satisfy the orthogonality property with a linear functional $\mathcal{L} : \mathbb{K}[z^{-1}, z] \to \mathbb{K}$

$$\mathcal{L}\left[z^{\ell} P_n(z^{-1})\right] = h_n \delta_{\ell,n}, \quad 0 \leq \ell \leq n,\ n \geq 0, \tag{1}$$

*JSPS Research Fellow.

where h_n are some nonzero constants. Such a linear functional is uniquely determined up to a constant factor, and then we normalize it by $\mathcal{L}[1] = 1$ in what follows. It is well known that LBPs satisfy a three-term recurrence equation of the form

$$\begin{cases} P_0(z) = 1, \quad P_1(z) = z - c_0, \\ P_n(z) = (z - c_{n-1})P_{n-1}(z) - a_{n-2} z P_{n-2}(z), \quad n \geq 2 \end{cases} \tag{2}$$

where the coefficients a_n and c_n are some nonzero constants. The LBPs $P_n(z)$ have unique biorthogonal partners, namely monic polynomials $Q_n(z) \in \mathbb{K}[z]$, $n \geq 0$, such that $\deg Q_n(z) = n$, which satisfy the orthogonality property

$$\mathcal{L}\big[z^{-\ell} Q_n(z)\big] = h_n \delta_{\ell,n}, \quad 0 \leq \ell \leq n,\ n \geq 0, \tag{3}$$

or, equivalently, do the biorthogonality one

$$\mathcal{L}\big[P_m(z^{-1}) Q_n(z)\big] = h_m \delta_{m,n}, \quad m, n \geq 0. \tag{4}$$

In this paper, we consider the case $Q_n(0) \neq 0$, that is, we assume that the biorthogonal partners $Q_n(z)$ are also LBPs with respect to the functional $\mathcal{L}'$ defined by $\mathcal{L}'[z^\ell] = \mathcal{L}[z^{-\ell}]$.

Our aim in this paper is a combinatorial interpretation of LBPs and their properties. Especially, we explain orthogonality and biorthogonality of LBPs in terms of Schröder paths and other weighted plane paths. This paper is organized as follows. In the rest of this Section 1, we introduce and define several combinatorial concepts used throughout this paper, e.g., Schröder paths and enumerators for them. In Section 2, we introduce Favard paths for LBPs, or Favard-LBP paths for short, with which we interpret the three-term recurrence equation (2) of LBPs. They play an auxiliary role to prove claims in the following sections concerned with orthogonality and biorthogonality of LBPs. In Section 3, we give to the quantity

$$\mathcal{L}\big[z^\ell P_n(z^{-1})\big], \quad \ell \in \mathbb{Z},\ n \geq 0 \tag{5}$$

a combinatorial representation derived by enumerating some kinds of Schröder paths. We then show that the LBPs $P_n(z)$ can be regarded as generating functions of some quantities obtained by enumerating Favard-LBP paths, and that the corresponding linear functional $\mathcal{L}$ can be done by doing Schröder paths. Section 4 is devoted for a similar subject, but we consider the quantity

$$\mathcal{L}\big[z^\ell Q_n(z)\big], \quad \ell \in \mathbb{Z},\ n \geq 0, \tag{6}$$

and combinatorially interpret the biorthogonal partners $Q_n(z)$. Finally, in Section 5, we clarify that we can evaluate the quantity

$$\mathcal{L}\big[z^\ell P_m(z^{-1}) Q_n(z)\big], \quad \ell \in \mathbb{Z},\ m, n \geq 0 \tag{7}$$

by enumerating Schröder paths. As a result, we shall be able to understand from a combinatorial viewpoint the LBPs $P_n(z)$, the linear functional $\mathcal{L}$, the biorthogonal partners $Q_n(z)$ and the orthogonality and the biorthogonality satisfied by them.

This combinatorial approach to orthogonal functions is due to Viennot [10]. He gave to general (classical) orthogonal polynomials, following Flajolet's interpretation of Jacobi type continued fractions (J-fractions) [3], a combinatorial interpretation using Motzkin paths. Specifically, he showed, for general orthogonal polynomials $p_n(z)$ which are orthogonal with respect to a linear functional f, that the quantity

$$f\left[z^\ell p_m(z)p_n(z)\right], \quad \ell, m, n \geq 0$$

can be evaluated by enumerating Motzkin paths of length ℓ, starting at level m and ending at level n, which implies the orthogonality $f[p_m(z)p_n(z)] = \kappa_m \delta_{m,n}$. Kim [7] presented an extension of Motzkin paths and generalized Viennot's result for biorthogonal polynomials.

First of all, we introduce combinatorial concepts fundamental throughout this paper. We consider plane paths each of whose points (or vertices) lies on the point lattice

$$\mathbb{L} = \{(x,y),\ (x+1/2,y) \mid x,y \in \mathbb{Z},\ y \geq 0\} \subset \mathbb{R}^2 \tag{8}$$

and each of whose elementary steps (or edges) is directed. (See Figure 1, 2, etc., for example.) We identify two paths if they coincide with translation. We use the symbol $\Pi_\diamondsuit^\heartsuit$ for the finite set of plane paths characterized by the scripts $\heartsuit$ and $\diamondsuit$. Moreover, for a plane path $\pi = s_1 s_2 \cdots s_n$, where each s_i is its elementary step, we denote by $s_i(\pi)$ the i-th elementary step s_i, and denote by $s_{i,j}(\pi)$ the part $s_i \cdots s_j$ if $i \leq j$ or the empty path ϕ if $i > j$, namely the path consisting only of one point. Additionally, we denote by $|\pi|$ the number n of the elementary steps of π.

Valuations, weight and enumerators A *valuation* v is a map from a set of elementary steps to the field $\mathbb{K}$. Then, *weight* of a path π is the product

$$\mathrm{wgt}(v;\pi) = \prod_{i=1}^{|\pi|} v(s_i(\pi)), \tag{9}$$

and an *enumerator* for paths in $\Pi_\diamondsuit^\heartsuit$ is the sum of weight

$$\mu_\diamondsuit^\heartsuit(v) = \sum_{\pi \in \Pi_\diamondsuit^\heartsuit} \mathrm{wgt}(v;\pi). \tag{10}$$

Note that the enumerator $\mu_\diamondsuit^\heartsuit(v)$ is a generalization of the cardinality of the set $\Pi_\diamondsuit^\heartsuit$ of plane paths, which is obtained by letting $\mathbb{K} = \mathbb{Q}$ and letting the valuation v be the constant 1.

Schröder paths Commonly, a Schröder path is a lattice path in the xy-plane from $(0,0)$ to (n,n), $n \geq 0$, consisting of the three kinds of elementary steps $(1,0)$, $(0,1)$ and $(1,1)$, and not going above the line $\{y = x\}$. The number of such paths are counted by the large Schröder numbers (the sequence A006318 in [9]). See for Schröder paths and the Schröder numbers, e.g., [8, 1] and [2, pp.80–81].

In this paper, instead, we use the following definition of Schröder paths, in which we consider direction of paths: rightward and leftward. A *rightward Schröder path* of length $\ell \geq 0$ is a plane path on $\mathbb{L}$,

- starting at $(x, 0)$ and ending at $(x+\ell, 0)$,
- not going under the horizontal line $\{y = 0\}$,
- consisting of the three kinds of elementary steps: up-diagonal $a_k^{\mathrm{R}} = (1/2, 1)$, down-diagonal $b_k^{\mathrm{R}} = (1/2, -1)$ and horizontal $c_k^{\mathrm{R}} = (1, 0)$,

where the subscript k of each elementary step indicates the level of its starting point. See Figure 1 for example. The definition of a *leftward Schröder path* of length $\ell \geq 1$ is same as that of rightward one, except for it ending at $(x-\ell, 0)$ and consisting of the three kinds of elementary steps: $a_k^{\mathrm{L}} = (-1/2, 1)$, $b_k^{\mathrm{L}} = (-1/2, -1)$ and $c_k^{\mathrm{L}} = (-1, 0)$. We regard, for convenience, the empty path ϕ as a rightward path. We denote by Π_ℓ^{S}, $\ell \geq 0$, the set of such rightward Schröder paths, and do by $\Pi_{-\ell}^{\mathrm{S}}$, $\ell \geq 1$, that of such leftward ones.

We deal with Schröder paths starting by a horizontal step c_0^{R} or c_0^{L}. Let us denote the set of such paths by Π^{SH}. Additionally, we use the following notation for their sets, for any $\ell \in \mathbb{Z}$, and use the notation

$$\Pi_\ell^{\mathrm{SH}} = \Pi_\ell^{\mathrm{S}} \cap \Pi^{\mathrm{SH}}. \tag{11}$$

Valuations, weight and enumerators for Schröder paths Let $\alpha = (\alpha_k)_{k=0}^\infty$ and $\gamma = (\gamma_k)_{k=0}^\infty$ be such two sequences on $\mathbb{K}$ that every term of them is nonzero. We then define a valuation $v = (\alpha, \gamma)$ by

$$\begin{aligned} v(a_k^{\mathrm{R}}) = \alpha_k, \quad v(b_k^{\mathrm{R}}) = 1, \quad v(c_k^{\mathrm{R}}) = \gamma_k, \\ v(a_k^{\mathrm{L}}) = \alpha_k^*, \quad v(b_k^{\mathrm{L}}) = 1, \quad v(c_k^{\mathrm{L}}) = \gamma_k^* \end{aligned} \tag{12}$$

where $\alpha^* = (\alpha_k^*)_{k=0}^\infty$ and $\gamma^* = (\gamma_k^*)_{k=0}^\infty$ are given by

$$V^*: \quad \alpha_k^* = \frac{\alpha_k}{\gamma_k \gamma_{k+1}}, \quad \gamma_k^* = \frac{1}{\gamma_k}. \tag{13}$$

We can regard this (13) as the transformation of valuations which maps $v = (\alpha, \gamma)$ to $v^* = (\alpha^*, \gamma^*)$. We then represent it as V^*, that is, in this case $v^* = V^*(v)$. In what follows, for any superscript $\heartsuit$, we denote by $\alpha^\heartsuit$ and $\gamma^\heartsuit$ sequences $(\alpha_k^\heartsuit)_{k=0}^\infty$ and $(\gamma_k^\heartsuit)_{k=0}^\infty$, respectively, and denote by $v^\heartsuit$ the valuation $(\alpha^\heartsuit, \gamma^\heartsuit)$.

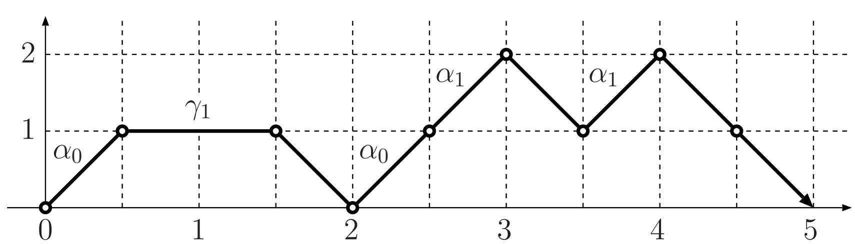

Figure 1: A rightward Schröder path $\omega = a_0^{\mathrm{R}} c_1^{\mathrm{R}} b_1^{\mathrm{R}} a_0^{\mathrm{R}} a_1^{\mathrm{R}} b_2^{\mathrm{R}} a_1^{\mathrm{R}} b_2^{\mathrm{R}} b_1^{\mathrm{R}}$ of length 5, $\mathrm{wgt}(v; \omega) = (\alpha_0)^2 (\alpha_1)^2 \gamma_1$.

Using valuations of this kind we weight Schröder paths by (9) and then evaluate enumerators by (10). For example, a few of them are

$$\begin{aligned}
\mu^{\mathrm{SH}}_{-2}(v) &= \gamma^*_0(\alpha^*_0 + \gamma^*_0),\\
\mu^{\mathrm{SH}}_{-1}(v) &= \gamma^*_0,\\
\mu^{\mathrm{S}}_{0}(v) &= 1,\\
\mu^{\mathrm{S}}_{1}(v) &= \alpha_0 + \gamma_0,\\
\mu^{\mathrm{S}}_{2}(v) &= \alpha_0\alpha_1 + \alpha_0\gamma_1 + (\alpha_0)^2 + 2\alpha_0\gamma_0 + (\gamma_0)^2.
\end{aligned}$$

Clearly, we have the following.

Lemma 1. *Enumerators for Schröder paths satisfy the equalities*

$$\begin{cases} \mu^{\mathrm{S}}_{\ell}(v) = \gamma_0 \mu^{\mathrm{SH}}_{\ell-1}(v), & \ell \le 0,\\ \mu^{\mathrm{SH}}_{\ell}(v) = \gamma_0 \mu^{\mathrm{S}}_{\ell-1}(v), & \ell \ge 1. \end{cases} \tag{14}$$

Since the transformation V^* of valuations is an involution, then we have the following.

Lemma 2. *If $v^* = V^*(v)$, then enumerators for Schröder paths satisfy the equalities*

$$\mu^{\mathrm{S}}_{\ell}(v) = \mu^{\mathrm{S}}_{-\ell}(v^*), \quad \mu^{\mathrm{SH}}_{\ell}(v) = \mu^{\mathrm{SH}}_{-\ell}(v^*), \quad \ell \in \mathbb{Z}. \tag{15}$$

Linear functionals To combinatorially interpret LBPs, it shall be inevitable to define a linear functional in terms of Schröder paths as

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell}\right] = \begin{cases} \mu^{\mathrm{SH}}_{\ell}(v), & \ell \le -1,\\ \mu^{\mathrm{S}}_{\ell}(v), & \ell \ge 0, \end{cases} \tag{16}$$

with respect to which LBPs shall be orthogonal. We have the following from Lemmas 1 and 2.

Lemma 3. *If $v^* = V^*(v)$, then linear functionals satisfy the equality*

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell}\right] = \gamma^*_0 \mathcal{L}^{\mathrm{S}}(v^*)\left[z^{-\ell-1}\right], \quad \ell \in \mathbb{Z}. \tag{17}$$

2 Favard paths for Laurent biorthogonal polynomials

Favard paths, appeared in [10], are plane paths introduced to interpret general orthogonal polynomials, especially to do three-term recurrence equation satisfied by them. We use a similar approach to interpret LBPs and their recurrence equation.

A *Favard path for Laurent biorthogonal polynomials*, or a *Favard-LBP path* for short, of height n and width ℓ is a plane path on $\mathbb{L}$,

- starting at $(x, 0)$ and ending at $(x + \ell, n)$, and

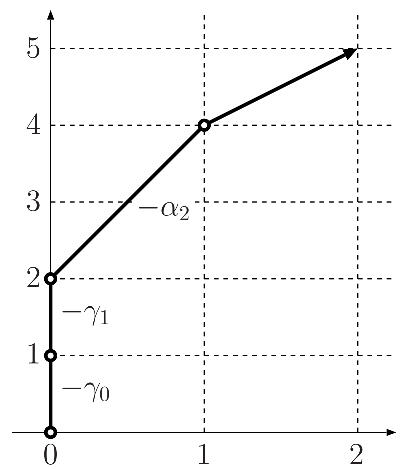

Figure 2: A Favard-LBP path $\eta = c_0^{\mathrm{F}} c_1^{\mathrm{F}} a_2^{\mathrm{F}} b_4^{\mathrm{F}}$ of height 5 and width 2, $\mathrm{wgt}(v; \eta) = -\alpha_2\gamma_0\gamma_1$.

- consisting of the three kinds of elementary steps: up-up-diagonal $a_k^{\mathrm{F}} = (1, 2)$, up-diagonal $b_k^{\mathrm{F}} = (1, 1)$, and up $c_k^{\mathrm{F}} = (0, 1)$,

where the subscript k of each elementary step indicates the level of its starting point. See Figure 2 for example. We denote by $\Pi_{n,\ell}^{\mathrm{F}}$ the set of such Favard-LBP paths.

To weight Favard-LBP paths we extend the valuation v for Schröder paths by

$$v(a_k^{\mathrm{F}}) = -\alpha_k, \quad v(b_k^{\mathrm{F}}) = 1, \quad v(c_k^{\mathrm{F}}) = -\gamma_k, \tag{18}$$

with which we may evaluate the enumerators $\mu_{n,\ell}^{\mathrm{F}}(v)$ for Favard-LBP paths. Moreover, we consider the generating functions of the enumerators

$$G_n^{\mathrm{F}}(v; z) = \sum_{k=0}^{n} \mu_{n,k}^{\mathrm{F}}(v) z^k, \quad n \geq 0. \tag{19}$$

The structure of Favard-LBP paths obviously implies the following recurrence.

Proposition 4. *Enumerators for Favard-LBP paths satisfy the equality*

$$\mu_{n,\ell}^{\mathrm{F}}(v) = \mu_{n-1,\ell-1}^{\mathrm{F}}(v) - \gamma_{n-1}\mu_{n-1,\ell}^{\mathrm{F}}(v) - \alpha_{n-2}\mu_{n-2,\ell-1}^{\mathrm{F}}(v), \quad n \geq 1, \tag{20}$$

where $\mu_{-1,\ell}^{\mathrm{F}}(v) = 0$ for each ℓ.

Thus, the generating functions satisfy the recurrence equation

$$\begin{cases} G_0^{\mathrm{F}}(v; z) = 1, \quad G_1^{\mathrm{F}}(v; z) = z - \gamma_0, \\ G_n^{\mathrm{F}}(v; z) = (z - \gamma_{n-1}) G_{n-1}^{\mathrm{F}}(v; z) - \alpha_{n-2} z G_{n-2}^{\mathrm{F}}(v; z), \quad n \geq 2, \end{cases} \tag{21}$$

whose form is identical to that (2) of LBPs. Then, we can interpret LBPs in terms of Favard-LBP paths. This fact will be explicitly noted in Theorem 8 in the next section.

3 First orthogonality

In this section, we give a combinatorial representation to the quantity

$$\mathcal{L}\big[z^\ell P_n(z^{-1})\big], \quad \ell \in \mathbb{Z},\ n \geq 0,$$

where $P_n(z)$ are the LBPs which satisfy the orthogonality (1) with the unique linear functional $\mathcal{L}$, and do the recurrence equation (2). For this, instead, we evaluate the quantity

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^\ell G_n^{\mathrm{F}}(v^*; z^{-1})\big], \quad \ell \in \mathbb{Z},\ n \geq 0, \tag{22}$$

where v and $v^* = V^*(v)$ are valuations for Schröder paths. We then shall understand from a combinatorial viewpoint the LBPs $P_n(z)$, the linear functional $\mathcal{L}$ and the orthogonality (1) of the LBPs.

We consider such a Schröder path $\omega = s_1 \cdots s_\nu \in \Pi_\ell^{\mathrm{S}}$ (resp. $\omega = s_0 s_1 \cdots s_\nu \in \Pi_\ell^{\mathrm{SH}}$) that it has at least $m+n$ steps (resp. $m+n+1$ steps) and its m steps $s_1, \ldots, s_m$ and n ones $s_{\nu-n+1}, \ldots, s_\nu$ are all up-diagonal and down-diagonal, respectively. See Figure 3 for example. We denote by $\Pi_{\ell;m,n}^{\mathrm{S}}$ (resp. by $\Pi_{\ell;m,n}^{\mathrm{SH}}$) the set of such paths.

The next theorem is a main subject of this section.

Theorem 5 (First orthogonality). *Let v be a valuation for Schröder paths and let $v^* = V^*(v)$. Then, generating functions of enumerators for Favard-LBP paths satisfy the equality*

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^\ell G_n^{\mathrm{F}}(v^*; z^{-1})\big] = \begin{cases} \mu_{\ell-n;n,0}^{\mathrm{SH}}(v), & \ell \leq -1, \\ \left[\displaystyle\prod_{i=0}^{n-1}\left(-\frac{1}{\gamma_i}\right)\right]\mu_{\ell;n,0}^{\mathrm{S}}(v), & \ell \geq 0. \end{cases} \tag{23}$$

Particularly, they satisfy the orthogonality property

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^\ell G_n^{\mathrm{F}}(v^*; z^{-1})\big] = \left[\prod_{i=0}^{n-1}\left(-\frac{\alpha_i}{\gamma_i}\right)\right]\delta_{\ell,n}, \quad 0 \leq \ell \leq n. \tag{24}$$

Hereafter we call this theorem, especially the formula (23), *first orthogonality.*

To prove the first orthogonality we introduce a new but simple kind of plane paths. An *S×F path* (ω, η) is an ordered pair of a Schröder path ω and a Favard-LBP path η,

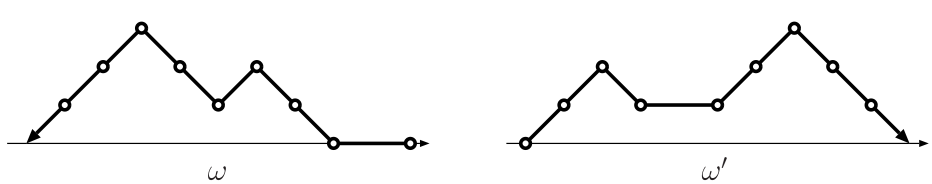

Figure 3: Schröder paths $\omega \in \Pi_{-5;1,3}^{\mathrm{SH}}$ and $\omega' \in \Pi_{5;2,2}^{\mathrm{S}}$.

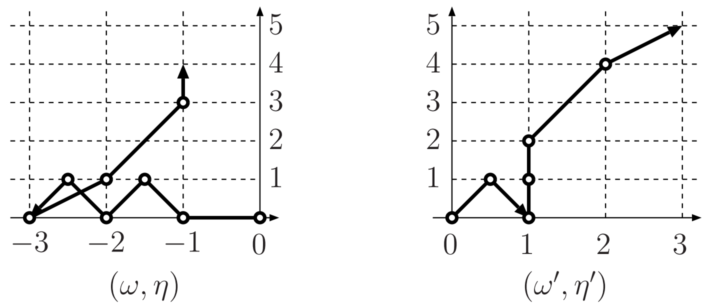

Figure 4: S×F paths $(\omega, \eta) \in \Pi^{\mathrm{S}\times\mathrm{F}}_{-1,4}$ and $(\omega', \eta') \in \Pi^{\mathrm{S}\times\mathrm{F}}_{3,5}$.

where $\omega \in \Pi^{\mathrm{SH}}$ if ω is leftward. Graphically, it is a path derived by coupling the ending point of ω and the starting point of η. See Figure 4 for example. We denote by $\Pi^{\mathrm{S}\times\mathrm{F}}_{i,j}$, $(i, j) \in \mathbb{L}$, the set of S×F paths from $(0, 0)$ to (i, j). Note that it can be represented as

$$\Pi^{\mathrm{S}\times\mathrm{F}}_{i,j} = \left(\bigcup_{k=0}^{i} \Pi^{\mathrm{S}}_{i-k} \times \Pi^{\mathrm{F}}_{j,k} \right) \cup \left(\bigcup_{k=i+1}^{j} \Pi^{\mathrm{SH}}_{i-k} \times \Pi^{\mathrm{F}}_{j,k} \right). \tag{25}$$

The first step to prove the first orthogonality is the next.

Lemma 6. *The following equality holds,*

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell} G^{\mathrm{F}}_n(v^*; z^{-1})\right] = \sum_{(\omega,\eta)\in\Pi^{\mathrm{S}\times\mathrm{F}}_{\ell,n}} \mathrm{wgt}(v; \omega) \cdot \mathrm{wgt}(v^*; \eta). \tag{26}$$

Proof. We have from the definition (16) of linear functionals

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell} G^{\mathrm{F}}_n(v^*; z^{-1})\right] = \sum_{k=0}^{\ell} \mu^{\mathrm{S}}_{\ell-k}(v) \cdot \mu^{\mathrm{F}}_{n,k}(v^*) + \sum_{k=\ell+1}^{n} \mu^{\mathrm{SH}}_{\ell-k}(v) \cdot \mu^{\mathrm{F}}_{n,k}(v^*).$$

This and (25) lead (26). □

Prior to the second step, we classify S×F paths into two groups: proper and improper ones. A *proper* S×F path is a path in the sets

$$\widetilde{\Pi}^{\mathrm{S}\times\mathrm{F}}_{i,j} = \begin{cases} \Pi^{\mathrm{SH}}_{i-j;j,0} \times \Pi^{\mathrm{F}}_{j,j}, & i \le -1, \\ \Pi^{\mathrm{S}}_{i;j,0} \times \Pi^{\mathrm{F}}_{j,0}, & i \ge 0. \end{cases} \tag{27}$$

See Figure 5 for example. Note that $\Pi^{\mathrm{F}}_{j,j} = \{\tilde{\eta}_{j,j}\}$ and $\Pi^{\mathrm{F}}_{j,0} = \{\tilde{\eta}_{j,0}\}$, $j \ge 0$, where $\tilde{\eta}_{j,j} = b^{\mathrm{F}}_0 \cdots b^{\mathrm{F}}_{j-1}$, the path consisting only of up-diagonal steps, and $\tilde{\eta}_{j,0} = c^{\mathrm{F}}_0 \cdots c^{\mathrm{F}}_{j-1}$, the one doing only of up ones. (In the case $j = 0$, $\tilde{\eta}_{0,0}$ is the empty path ϕ.) Meanwhile, an *improper* S×F path is a path which is not proper, and belongs to the complement $\Pi^{\mathrm{S}\times\mathrm{F}}_{i,j} \setminus \widetilde{\Pi}^{\mathrm{S}\times\mathrm{F}}_{i,j}$. That is characterized as follows. An S×F path $(\omega, \eta) \in \Pi^{\mathrm{S}\times\mathrm{F}}_{i,j}$ is improper if and only if ω is rightward (resp. ω is leftward) and

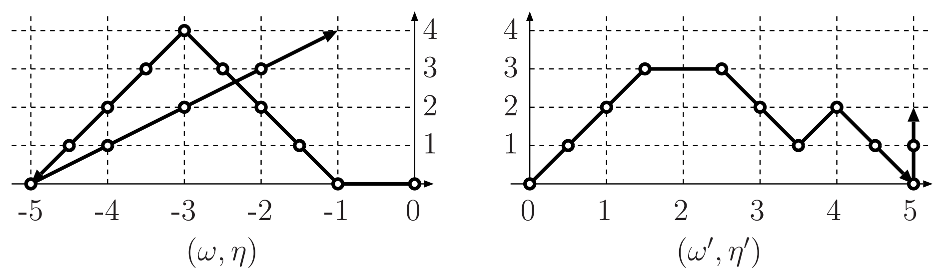

Figure 5: Proper S×F paths $(\omega,\eta)\in\Pi^{\mathrm{S\times F}}_{-1,4}$ and $(\omega',\eta')\in\Pi^{\mathrm{S\times F}}_{5,2}$.

- ω has at least one down-diagonal step or horizontal step in $s_{1,\min\{j,|\omega|\}}(\omega)$ (resp. in $s_{2,\min\{j+1,|\omega|\}}(\omega)$), or
- η has at least one up-diagonal step (resp. up step) or up-up-diagonal step in $s_{1,\min\{j,|\eta|\}}(\eta)$.

The second step to prove the first orthogonality is the next.

Lemma 7. *There exists an involution $T^{\mathrm{S\times F}}_{\ell,n}$ on $\Pi^{\mathrm{S\times F}}_{\ell,n}\setminus\widetilde{\Pi}^{\mathrm{S\times F}}_{\ell,n}$ of improper S×F paths, satisfying for any pair (ω,η) and $(\omega',\eta')=T^{\mathrm{S\times F}}_{\ell,n}((\omega,\eta))$*

$$\mathrm{wgt}(v;\omega)\cdot\mathrm{wgt}(v^*;\eta)=-\mathrm{wgt}(v;\omega')\cdot\mathrm{wgt}(v^*;\eta'). \tag{28}$$

Proof. We show such an involution as a transformation which takes an improper S×F path (ω,η) as the input and outputs one (ω',η') after transforming the input a little.

Definition 1 (Involution $T^{\mathrm{S\times F}}_{\ell,n}$). For a given input $(\omega,\eta)\in\Pi^{\mathrm{S\times F}}_{\ell,n}\setminus\widetilde{\Pi}^{\mathrm{S\times F}}_{\ell,n}$, output $(\omega',\eta')\in\Pi^{\mathrm{S\times F}}_{\ell,n}\setminus\widetilde{\Pi}^{\mathrm{S\times F}}_{\ell,n}$ as follows.

(i) **Case $\omega\in\cup_{\ell\le-2}\Pi^{\mathrm{SH}}_{\ell}$, or $\omega\in\Pi^{\mathrm{SH}}_{-1}$ and $s_1(\eta)=a^{\mathrm{F}}_0$ or c^{F}_0:**
Let $\nu\ge1$ be the minimal integer satisfying $(s_{\nu+1}(\omega),s_\nu(\eta))\ne(a^{\mathrm{L}}_{\nu-1},b^{\mathrm{F}}_{\nu-1})$. Then, output (ω',η') following the next table.

	$s_{\nu+1}(\omega)$	$s_\nu(\eta)$	ω'	η'
(iP1)	$b^{\mathrm{L}}_{\nu-1}$	$b^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu-1}(\omega)s_{\nu+2,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-2}(\eta)a^{\mathrm{F}}_{\nu-2}s_{\nu+1,\lvert\eta\rvert}(\eta)$
(iP2)	any	$a^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu}(\omega)a^{\mathrm{L}}_{\nu-1}b^{\mathrm{L}}_{\nu}s_{\nu+1,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)b^{\mathrm{F}}_{\nu-1}b^{\mathrm{F}}_{\nu}s_{\nu+1,\lvert\eta\rvert}(\eta)$
(iH1)	$c^{\mathrm{L}}_{\nu-1}$	$b^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu}(\omega)s_{\nu+2,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)c^{\mathrm{F}}_{\nu-1}s_{\nu+1,\lvert\eta\rvert}(\eta)$
(iH2)	any	$c^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu}(\omega)c^{\mathrm{L}}_{\nu-1}s_{\nu+1,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)b^{\mathrm{F}}_{\nu-1}s_{\nu+1,\lvert\eta\rvert}(\eta)$

This table means, for example, that, if $(s_{\nu+1}(\omega),s_\nu(\eta))=(b^{\mathrm{L}}_{\nu-1},b^{\mathrm{F}}_{\nu-1})$, then output $(\omega',\eta')=(s_{1,\nu-1}(\omega)s_{\nu+2,|\omega|}(\omega),s_{1,\nu-2}(\eta)a^{\mathrm{F}}_{\nu-2}s_{\nu+1,|\eta|}(\eta))$, where "any" means no restriction. See Figure 6 for example.

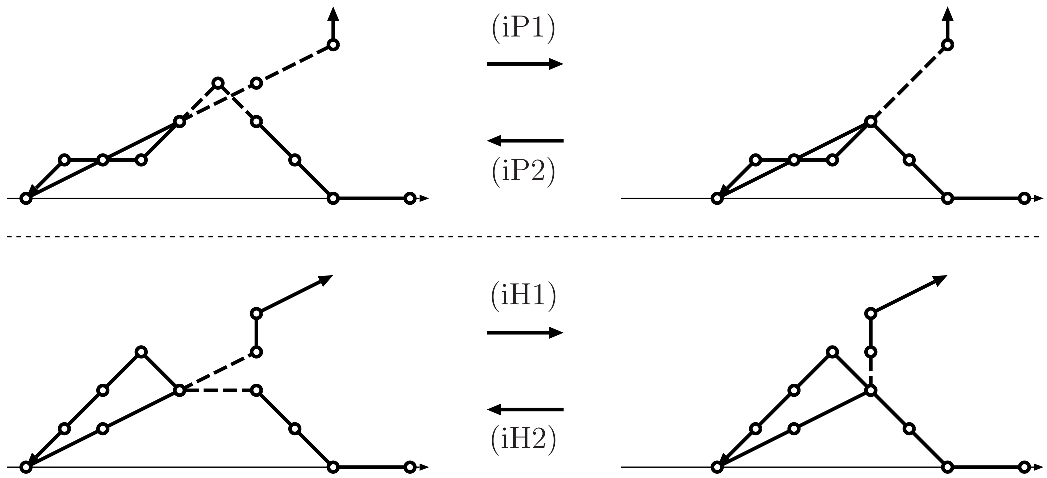

Figure 6: Transformations by $T^{\mathrm{S}\times\mathrm{F}}_{-1,5}$, Case (i).

(ii) **Case $\omega \in \Pi^{\mathrm{SH}}_{-1}$ and $s_1(\eta) = b^{\mathrm{F}}_0$, or $\omega \in \Pi^{\mathrm{S}}_0$ and $s_1(\eta) = c^{\mathrm{F}}_0$:**
Output (ω', η') following the next table.

	ω	$s_1(\eta)$	ω'	η'
(ii1)	c^{L}_0	b^{F}_0	ϕ	$c^{\mathrm{F}}_0 s_{2,\lvert\eta\rvert}(\eta)$
(ii2)	ϕ	c^{F}_0	c^{L}_0	$b^{\mathrm{F}}_0 s_{2,\lvert\eta\rvert}(\eta)$

See Figure 7 for example.

(iii) **Case $\omega \in \Pi^{\mathrm{S}}_0$ and $s_1(\eta) = a^{\mathrm{F}}_0$ or b^{F}_0, or $\omega \in \cup_{\ell\geq 1}\Pi^{\mathrm{S}}_\ell$:**
Let $\nu \geq 1$ be the minimal integer satisfying $(s_\nu(\omega), s_\nu(\eta)) \neq (a^{\mathrm{L}}_{\nu-1}, c^{\mathrm{F}}_{\nu-1})$. Then, output (ω', η') following the next table.

	$s_\nu(\omega)$	$s_\nu(\eta)$	ω'	η'
(iiiP1)	any	$a^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu-1}(\omega)a^{\mathrm{R}}_{\nu-1}b^{\mathrm{R}}_{\nu}s_{\nu,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)c^{\mathrm{F}}_{\nu-1}c^{\mathrm{F}}_{\nu}s_{\nu+1,\lvert\eta\rvert}(\nu)$
(iiiP2)	$b^{\mathrm{R}}_{\nu-1}$	$c^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu-2}(\omega)s_{\nu+1,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-2}(\eta)a^{\mathrm{F}}_{\nu-2}s_{\nu+1,\lvert\eta\rvert}(\eta)$
(iiiH1)	any	$b^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu-1}(\omega)c^{\mathrm{R}}_{\nu-1}s_{\nu,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)c^{\mathrm{F}}_{\nu-1}s_{\nu+1,\lvert\eta\rvert}(\eta)$
(iiiH2)	$c^{\mathrm{R}}_{\nu-1}$	$c^{\mathrm{F}}_{\nu-1}$	$s_{1,\nu-1}(\omega)s_{\nu+1,\lvert\omega\rvert}(\omega)$	$s_{1,\nu-1}(\eta)b^{\mathrm{F}}_{\nu-1}s_{\nu+1,\lvert\eta\rvert}(\eta)$

See Figure 8 for example.

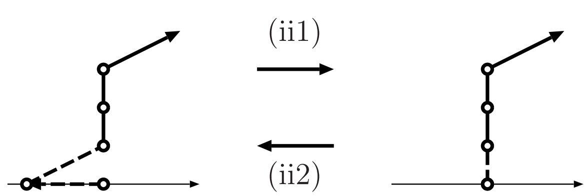

Figure 7: Transformations by $T^{\mathrm{S}\times\mathrm{F}}_{1,4}$, Case (ii).

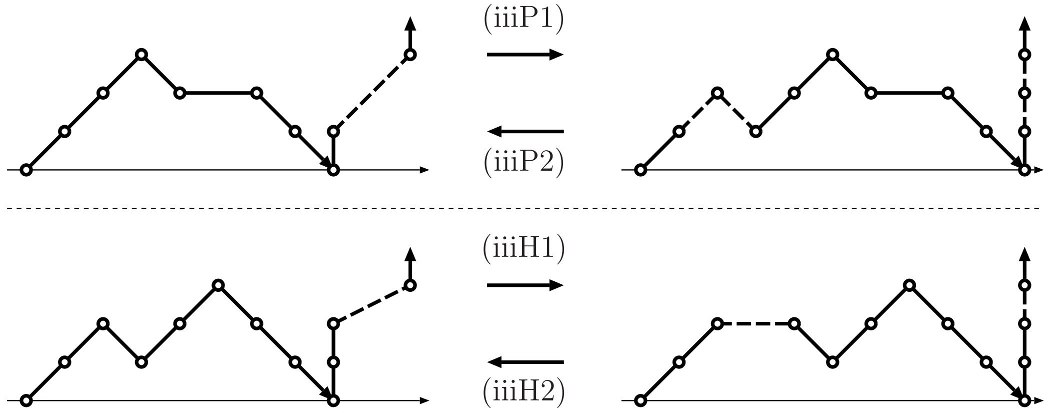

Figure 8: Transformations by $T^{\mathrm{S}\times\mathrm{F}}_{5,4}$, Case (iii).

In this transformation, (iP1) and (iP2), (iH1) and (iH1), (ii1) and (ii2), (iiiP1) and (iiiP2), and (iiiH1) and (iiiH2) are inverse to each other, respectively. That is, for example, if $T^{\mathrm{S}\times\mathrm{F}}_{\ell,n}((\omega,\eta))$ outputs (ω',η') by (iP1), then $T^{\mathrm{S}\times\mathrm{F}}_{\ell,n}((\omega',\eta'))$ outputs (ω,η) by (iP2). Hence, $T^{\mathrm{S}\times\mathrm{F}}_{\ell,n}$ is an involution. Finally, the equality (28) is easily validated using (13). For example, in the case (iiiP1), (ω',η') is made from (ω,η) only by inserting $a^{\mathrm{R}}_{\nu-1}b^{\mathrm{R}}_{\nu}$ (weighing $\alpha_{\nu-1}$) into ω and replacing $a^{\mathrm{F}}_{\nu-1}$ (weighing $-\alpha^*_{\nu-1}$) in η with $c^{\mathrm{F}}_{\nu-1}c^{\mathrm{F}}_{\nu}$ (weighing $\gamma^*_{\nu-1}\gamma^*_{\nu}$), in which $1\cdot(-\alpha^*_{\nu-1}) = -(\alpha_{\nu-1}\cdot\gamma^*_{\nu-1}\gamma^*_{\nu})$ holds from (13), and then (28) holds. We have completed the proof. $\square$

We make up a proof of the first orthogonality using these lemmas.

Proof of Theorem 5. Lemmas 6 and 7 lead

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^{\ell}G^{\mathrm{F}}_n(v^*;z^{-1})\big] = \sum_{(\omega,\eta)\in\widetilde{\Pi}^{\mathrm{S}\times\mathrm{F}}_{\ell,n}} \mathrm{wgt}(v;\omega)\cdot\mathrm{wgt}(v^*;\eta), \tag{29}$$

since in the summation of the right hand side of (26) only proper S×F paths survive while improper ones cancel out. Thus, we have from (27), if $\ell\le -1$,

$$\text{r.h.s. of (29)} = \mathrm{wgt}(v^*;\tilde{\eta}_{n,n})\sum_{\omega\in\Pi^{\mathrm{SH}}_{\ell-n;n,0}}\mathrm{wgt}(v;\omega) = \mu^{\mathrm{SH}}_{\ell-n;n,0}(v),$$

and, if $\ell\ge 0$, with (13)

$$\text{r.h.s. of (29)} = \mathrm{wgt}(v^*;\tilde{\eta}_{n,0})\sum_{\omega\in\Pi^{\mathrm{S}}_{\ell;n,0}}\mathrm{wgt}(v;\omega) = \left[\prod_{i=0}^{n-1}\left(-\frac{1}{\gamma_i}\right)\right]\mu^{\mathrm{S}}_{\ell;n,0}(v).$$

Finally, the orthogonality property (24) follows the fact that $\Pi^{\mathrm{S}}_{\ell;n,0}$ is empty if $0\le\ell\le n-1$ and $\Pi^{\mathrm{S}}_{n;n,0} = \{a^{\mathrm{R}}_0\cdots a^{\mathrm{R}}_{n-1}b^{\mathrm{R}}_n\cdots b^{\mathrm{R}}_1\}$. $\square$

The first orthogonality gives us a combinatorial representation of the LBPs $P_n(z)$ and the linear functional $\mathcal{L}$ in terms of Favard-LBP paths and Schröder paths, respectively.

Theorem 8. *Let $P_n(z) \in \mathbb{K}[z]$ be the LBPs satisfying the three-term recurrence equation (2) whose nonzero coefficients are $a = (a_k)_{k=0}^{\infty}$ and $c = (c_k)_{k=0}^{\infty}$, and let $\mathcal{L} : \mathbb{K}[z^{-1}, z] \to \mathbb{K}$ be the unique linear functional with which the LBPs $P_n(z)$ have the orthogonality (1). Let $v_P = (a, c)$ be a valuation for Schröder paths. Then $P_n(z)$ and $\mathcal{L}$ are represented as*

$$P_n(z) = G_n^{\mathrm{F}}(v_P; z), \quad n \geq 0 \tag{30}$$

$$\mathcal{L} = \mathcal{L}^{\mathrm{S}}(V^*(v_P)). \tag{31}$$

As a corollary we have the following.

Corollary 9. *If $a_n + c_{n+1} = 0$ for some $n \geq 0$, then the constant term $Q_{n+1}(0)$ of the biorthogonal partner $Q_{n+1}(z)$ vanishes.*

Proof. Since $\deg\left(c_{n+1}P_{n+1}(z) + a_n z P_n(z)\right) \leq n$, we have from the recurrence (2), and the orthogonalities (1), (4) and (3)

$$0 = \mathcal{L}\big[P_{n+2}(z^{-1})Q_{n+1}(z)\big] = Q_{n+1}(0)\mathcal{L}\big[z^{-1}P_{n+1}(z^{-1})\big].$$

Here, $\mathcal{L}[z^{-1}P_{n+1}(z^{-1})]$ is explicitly calculated, with (30), (31) and the first orthogonality (23), as

$$\mathcal{L}\big[z^{-1}P_{n+1}(z^{-1})\big] = \mu_{-n-2;n+1,0}^{\mathrm{SH}}(V^*(v_P)) = c_0\left(\prod_{i=0}^{n} a_i\right) \neq 0.$$

Hence, $Q_{n+1}(0) = 0$. □

Moreover, the nonzero constants h_n appearing in the orthogonality (1) are

$$h_n = \prod_{i=0}^{n-1}\left(-\frac{a_i}{c_{i+1}}\right), \quad n \geq 0. \tag{32}$$

4 Second orthogonality

In this section, we give a combinatorial representation to the quantity

$$\mathcal{L}\big[z^{\ell}Q_n(z)\big], \quad \ell \in \mathbb{Z},\ n \geq 0,$$

where $Q_n(z)$ are the unique biorthogonal partners of the LBPs $P_n(z)$ which are characterized by the orthogonality (3). For this, instead, we find such a valuation $\bar{v}$ that the generating functions $G_n^{\mathrm{F}}(\bar{v}; z)$ satisfy the orthogonality

$$\mathcal{L}(v)\big[z^{-\ell}G_n^{\mathrm{F}}(\bar{v}; z)\big] = \left[\prod_{i=0}^{n-1}\left(-\frac{\alpha_i}{\gamma_i}\right)\right]\delta_{\ell,n}, \quad 0 \leq \ell \leq n,\ n \geq 0,$$

and evaluate the quantity

$$\mathcal{L}(v)\big[z^\ell G_n^{\rm F}(\bar{v};z)\big],\quad \ell\in\mathbb{Z},\ n\geq 0.$$

We then shall understand from a combinatorial viewpoint the partners $Q_n(z)$ and their orthogonality (3). We consider only the case that $Q_n(0)\neq 0$, namely that $Q_n(z)$ are also LBPs. Thus, from Corollary 9, we assume in what follows that the coefficients a_n and c_n of the recurrence equation (2) of the LBPs $P_n(z)$ satisfy $a_n+c_{n+1}\neq 0$ for each $n\geq 0$, and also assume that the valuation $v=(\alpha,\gamma)$ for Schröder paths satisfies $\alpha_n+\gamma_n\neq 0$ for each $n\geq 0$ so that $v^*=V^*(v)$ satisfies $\alpha_n^*+\gamma_{n+1}^*\neq 0$.

Lemmas 1 and 2 can be generalized for paths in $\Pi^{\rm S}_{\ell;m,n}$ and $\Pi^{\rm SH}_{\ell;m,n}$ like

$$\mu^{\rm S}_{\ell;m,n}(v)=\gamma_0\mu^{\rm SH}_{\ell-1;m,n}(v),\quad \ell\leq 0.$$

We then have as a corollary of the first orthogonality (23) with Lemma 3

$$\mathcal{L}^{\rm S}(v)\big[z^\ell G_n^{\rm F}(v;z)\big]=\begin{cases}\left[\displaystyle\prod_{i=0}^{n-1}(-\gamma_i)\right]\mu^{\rm SH}_{\ell;n,0}(v), & \ell\leq -1,\\[2ex] \mu^{\rm S}_{\ell+n;n,0}(v), & \ell\geq 0.\end{cases}\tag{33}$$

The valuation v appearing here is not a desired one, however it looks to be close to that. Thus, we call the equality (33) *imperfect orthogonality*, and we will use it to derive a desired $\bar{v}$ afterwards.

We consider Schröder paths $\omega=s_1\cdots s_\nu\in\Pi^{\rm S}_{\ell;m,n}$ and $\omega=s_0s_1\cdots s_\nu\in\Pi^{\rm SH}_{\ell;m,n}$, $s_0=c_0^{\rm R}$ or $c_0^{\rm L}$ satisfying the following conditions: the elementary step $\{$(i) s_{m+1}, (ii) $s_{\nu-n}\}$, if ω has, is $\{$(a) not up-diagonal, (b) not down-diagonal, (c) horizontal$\}$. We represent the sets of such paths as in the next table, in which the superscripts $\heartsuit$ are any of S and SH.

	(a)	(b)	(c)
(i)	$\Pi^{\heartsuit}_{\ell;\binom{\neg a}{m},n}$	$\Pi^{\heartsuit}_{\ell;\binom{\neg b}{m},n}$	$\Pi^{\heartsuit}_{\ell;\binom{c}{m},n}$
(ii)	$\Pi^{\heartsuit}_{\ell;m,\binom{\neg a}{n}}$	$\Pi^{\heartsuit}_{\ell;m,\binom{\neg b}{n}}$	$\Pi^{\heartsuit}_{\ell;m,\binom{c}{n}}$

We also deal with paths which satisfy combinations of the above conditions. For example,

$$\Pi^{\rm S}_{\ell;\binom{\neg b}{m},\binom{\neg a}{n}}=\Pi^{\rm S}_{\ell;\binom{\neg b}{m},0}\cap\Pi^{\rm S}_{\ell;0,\binom{\neg a}{n}}.$$

Moreover, we take into consideration the existence of peaks and valleys in a Schröder path. Namely, we call two consecutive elementary steps $a_k^{\rm R}b_{k+1}^{\rm R}$ and $a_k^{\rm L}b_{k+1}^{\rm L}$ peaks of level k. Similarly, we call $b_k^{\rm R}a_{k-1}^{\rm R}$ and $b_k^{\rm L}a_{k-1}^{\rm L}$ valleys of level k. Let $\Pi^{\rm SnP}$ and $\Pi^{\rm SnV}$ be the sets of Schröder paths without peaks and without valleys, respectively. We use the following notation to represent subsets of them, for $\heartsuit=$ S or SH and for any subscript $\diamondsuit$

$$\Pi^{\heartsuit\rm nP}_{\diamondsuit}=\Pi^{\heartsuit}_{\diamondsuit}\cap\Pi^{\rm SnP},\quad \Pi^{\heartsuit\rm nV}_{\diamondsuit}=\Pi^{\heartsuit}_{\diamondsuit}\cap\Pi^{\rm SnV}.$$

To find a desired valuation $\bar{v}$, we consider *enumerator-conserving* transformations of Schröder paths.

Lemma 10. *The following equalities of enumerators hold for $\ell \geq 0$,*

$$\mu^{\mathrm{S}}_{\ell;\binom{\neg b}{m},\binom{\neg a}{n}}(v) = \mu^{\mathrm{SnP}}_{\ell;m,n}(v^{\mathrm{nP}}), \tag{34a}$$

$$\mu^{\mathrm{SnP}}_{\ell+1;m,n}(v^{\mathrm{nP}}) = \mu^{\mathrm{SHnV}}_{\ell+1;m,n}(v^{\mathrm{nV}}) = \gamma_0^{\mathrm{nV}} \mu^{\mathrm{SnV}}_{\ell;m,n}(v^{\mathrm{nV}}), \tag{34b}$$

$$\mu^{\mathrm{SnV}}_{\ell;m,n}(v^{\mathrm{nV}}) = \mu^{\mathrm{S}}_{\ell;m,n}(\bar{v}), \tag{34c}$$

where $\alpha^{\mathrm{nV}}_{-1} = 0$ and v^{nP}, v^{nV} and $\bar{v}$ are the valuations determined by

$$\alpha^{\mathrm{nP}}_k = \alpha_k, \quad \gamma^{\mathrm{nP}}_k = \alpha_k + \gamma_k, \tag{35a}$$

$$\alpha^{\mathrm{nV}}_k = \alpha^{\mathrm{nP}}_k \frac{\gamma^{\mathrm{nP}}_{k+1}}{\gamma^{\mathrm{nP}}_k}, \quad \gamma^{\mathrm{nV}}_k = \gamma^{\mathrm{nP}}_k, \tag{35b}$$

$$\bar{\alpha}_k = \alpha^{\mathrm{nV}}_k, \quad \bar{\alpha}_{k-1} + \bar{\gamma}_k = \gamma^{\mathrm{nV}}_k, \tag{35c}$$

respectively.

Proof of (34a). We consider the transformation $T^{\mathrm{S}\to\mathrm{SnP}}$ of plane paths defined by the next recursive algorithm.

Algorithm 2 (Transformation $T^{\mathrm{S}\to\mathrm{SnP}}$). For a given input π, output π' as follows.

(i) If $\pi = \phi$, then output $\pi' = \phi$.

(ii) Else if $s_{1,2}(\pi) = a^{\mathrm{R}}_k b^{\mathrm{R}}_{k+1}$, then output $\pi' = c^{\mathrm{R}}_k T^{\mathrm{S}\to\mathrm{SnP}}(s_{3,|\pi|}(\pi))$.

(iii) Otherwise, output $\pi' = s_1(\pi) T^{\mathrm{S}\to\mathrm{SnP}}(s_{2,|\pi|}(\pi))$.

As shown in the example in Figure 9, this $T^{\mathrm{S}\to\mathrm{SnP}}$ replaces every peak with a horizontal step of the same level, and hence it maps $\Pi^{\mathrm{S}}_{\ell;\binom{\neg b}{m},\binom{\neg a}{n}}$ onto $\Pi^{\mathrm{SnP}}_{\ell;m,n}$. Additionally, it is weight-conserving with the equalities (35a) of valuations, namely for any path $\omega' \in \Pi^{\mathrm{SnP}}_{\ell;m,n}$

$$\sum_{\omega \in (T^{\mathrm{S}\to\mathrm{SnP}})^{-1}(\omega')} \mathrm{wgt}(v;\omega) = \mathrm{wgt}(v^{\mathrm{nP}};\omega')$$

holds. Thus, we obtain (34a) by summing this equality over $\omega' \in \Pi^{\mathrm{SnP}}_{\ell;m,n}$. □

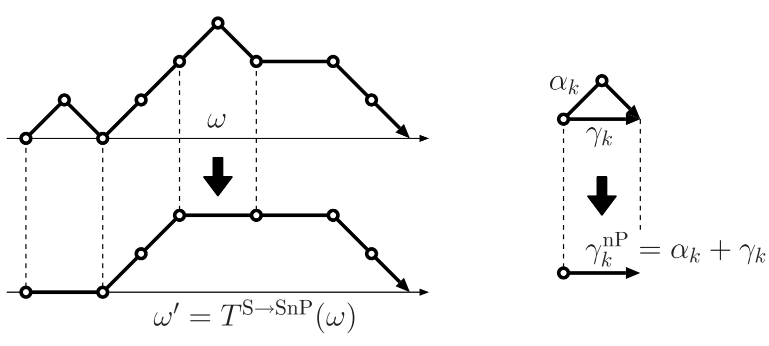

Figure 9: A transformation by $T^{\mathrm{S}\to\mathrm{SnP}}$.

Thus, the transformation $T^{\mathrm{S}\to\mathrm{SnP}}$ yields the equality (34a) of enumerators with the equality (35a) of valuations. In this sense we call it enumerator-conserving. We can prove (34b) and (34c) in similar ways, but we use the transformations $T^{\mathrm{SnP}\to\mathrm{SnV}}$ and $T^{\mathrm{S}\to\mathrm{SnV}}$, respectively, defined as follows.

Algorithm 3 (Transformation $T^{\mathrm{SnP}\to\mathrm{SnV}}$). For a given input π, output π' as follows.

(i) If $\pi = \phi$, then output $\pi' = \phi$.

(ii) Else if $s_{|\pi|-1,|\pi|}(\pi) = a^{\mathrm{R}}_{k-1}c^{\mathrm{R}}_{k}$, then output $\pi' = T^{\mathrm{SnP}\to\mathrm{SnV}}(s_{1,|\pi|-2}(\pi)c^{\mathrm{R}}_{k-1})a^{\mathrm{R}}_{k-1}$.

(iii) Otherwise, output $\pi' = T^{\mathrm{SnP}\to\mathrm{SnV}}(s_{1,|\pi|-1}(\pi))s_{|\pi|}(\pi)$.

Algorithm 4 (Transformation $T^{\mathrm{S}\to\mathrm{SnV}}$). For a given input π, output π' as follows.

(i) If $\pi = \phi$, then output $\pi' = \phi$.

(ii) Else if $s_{1,2}(\pi) = b^{\mathrm{R}}_{k}a^{\mathrm{R}}_{k-1}$, then output $\pi' = c^{\mathrm{R}}_{k}T^{\mathrm{S}\to\mathrm{SnV}}(s_{3,|\pi|}(\pi))$.

(iii) Otherwise, output $\pi' = s_1(\pi)T^{\mathrm{S}\to\mathrm{SnP}}(s_{2,|\pi|}(\pi))$.

$T^{\mathrm{SnP}\to\mathrm{SnV}}$ maps $\Pi^{\mathrm{SnP}}_{\ell;m,n}$ onto $\Pi^{\mathrm{SnV}}_{\ell;m,n}$ by replacing the part of the form $a^{\mathrm{R}}_{k_1}\cdots a^{\mathrm{R}}_{k_2-1}c^{\mathrm{R}}_{k_2}$, $k_1 < k_2$, with $c^{\mathrm{R}}_{k_1}a^{\mathrm{R}}_{k_1}\cdots a^{\mathrm{R}}_{k_2-1}$, while $T^{\mathrm{S}\to\mathrm{SnV}}$ maps $\Pi^{\mathrm{S}}_{\ell;m,n}$ onto $\Pi^{\mathrm{SnV}}_{\ell;m,n}$ by doing every valley with a horizontal step of the same level. They are also enumerator-conserving with the equalities (35b) and (35c) of valuations, respectively. See Figures 10 and 11 for example.

Thus, combining the equalities in (34) and (35), we have

$$\mu^{\mathrm{S}}_{\ell;\binom{-b}{m},\binom{-a}{n}}(v) = \bar{\gamma}_0\mu^{\mathrm{S}}_{\ell-1;m,n}(\bar{v}), \quad \ell \geq 1, \tag{36}$$

where $\bar{v}$ is the valuation given by

$$\bar{V}: \quad \bar{\alpha}_k = \frac{\alpha_{k+1}+\gamma_{k+1}}{\alpha_k+\gamma_k}\alpha_k, \quad \bar{\gamma}_k = \frac{\alpha_k+\gamma_k}{\alpha_{k-1}+\gamma_{k-1}}\gamma_{k-1} \tag{37}$$

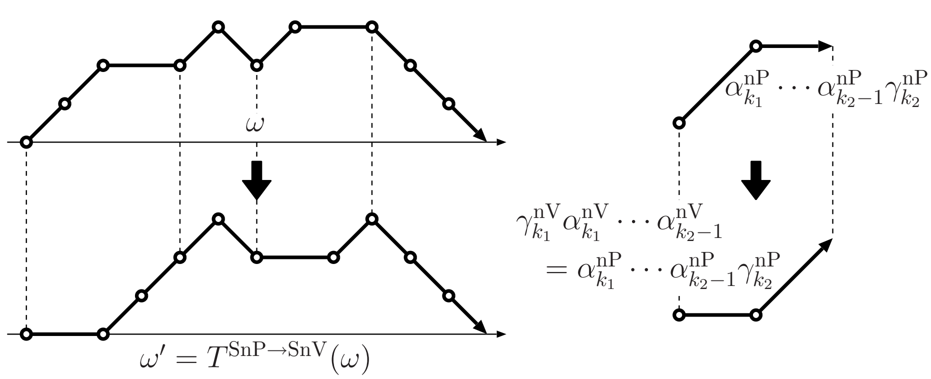

Figure 10: A transformation by $T^{\mathrm{SnP}\to\mathrm{SnV}}$.

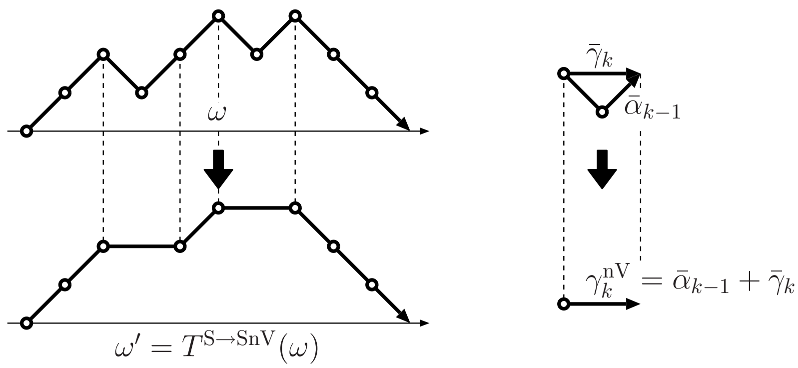

Figure 11: A transformation by $T^{\mathrm{S}\to\mathrm{SnV}}$.

with $\alpha_{-1} = 0$ and $\gamma_{-1} \neq 0$. We represent this transformation (37) of valuations as $\bar{V}$, namely in this case $\bar{v} = \bar{V}(v)$. Then, the transformation

$$\bar{V}^* = \bar{V} \circ V^* \tag{38}$$

of valuations is an involution, which implies with Lemmas 1 and 2 and (36)

$$\mu^{\mathrm{SH}}_{\ell;m,n}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{\ell-1;\binom{\neg b}{m},\binom{\neg a}{n}}(\bar{v}), \quad \ell \leq -1. \tag{39}$$

Additionally, it holds that

$$\mu^{\mathrm{S}}_{0;m,n}(v) = \mu^{\mathrm{S}}_{0;\binom{\neg b}{m},\binom{\neg a}{n}}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{-1;m,n}(\bar{v}) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{-1;\binom{\neg b}{m},\binom{\neg a}{n}}(\bar{v}) = \begin{cases} 1, & m = n = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

As a whole, we have

Proposition 11. *Let v and $\bar{v}$ be valuations for Schröder paths satisfying $\bar{v} = \bar{V}(v)$. Then, the following equalities of enumerators hold,*

$$\begin{cases} \mu^{\mathrm{SH}}_{\ell;m,n}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{\ell-1;\binom{\neg b}{m},\binom{\neg a}{n}}(\bar{v}), & \ell \leq -1, \\ \mu^{\mathrm{S}}_{0;m,n}(v) = \mu^{\mathrm{S}}_{0;\binom{\neg b}{m},\binom{\neg a}{n}}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{-1;m,n}(\bar{v}) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{-1;\binom{\neg b}{m},\binom{\neg a}{n}}(\bar{v}), & \\ \mu^{\mathrm{S}}_{\ell;\binom{\neg b}{m},\binom{\neg a}{n}}(v) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{\ell-1;m;n}(\bar{v}), & \ell \geq 1. \end{cases} \tag{41}$$

Particularly, in the case of $m = n = 0$ we have

$$\begin{cases} \mu^{\mathrm{SH}}_{\ell}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{\ell-1}(\bar{v}), & \ell \leq -1, \\ \mu^{\mathrm{S}}_{0}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{-1}(\bar{v}), & \\ \mu^{\mathrm{S}}_{\ell}(v) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{\ell-1}(\bar{v}), & \ell \geq 1, \end{cases} \tag{42}$$

which is equivalent, in terms of linear functionals, to

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^{\ell}\big] = \bar{\gamma}_0 \mathcal{L}^{\mathrm{S}}(\bar{v})\big[z^{\ell-1}\big], \quad \ell \in \mathbb{Z}. \tag{43}$$

Hence, we have from the imperfect orthogonality (33) with the last equality of (41)

$$\mathcal{L}^{\mathrm{S}}(v)\big[z^{\ell} G_n^{\mathrm{F}}(\bar{v}; z)\big] = \mu^{\mathrm{S}}_{\ell+n;\binom{\neg b}{n},0}(v) = \mu^{\mathrm{S}}_{\ell+n;0,\binom{\neg a}{n}}(v), \quad \ell \geq 1, \tag{44}$$

where in the last equality we use the symmetry of flipping a Schröder path in the horizontal direction.

On the other hand, the above three enumerator-conserving transformations $T^{\mathrm{S}\to\mathrm{SnP}}$, $T^{\mathrm{SnP}\to\mathrm{SnV}}$ and $T^{\mathrm{S}\to\mathrm{SnV}}$ also yield the following.

Lemma 12. *The following equalities of enumerators hold for $\ell \geq 0$,*

$$\begin{cases} (\alpha_n + \gamma_n)\mu^{\mathrm{S}}_{\ell;\binom{\neg b}{m},n}(v) = \mu^{\mathrm{SnP}}_{\ell+1;m,\binom{c}{n}}(v^{\mathrm{nP}}) & \text{if } \ell = m = n \text{ does not hold,} \\ (\alpha_\ell + \gamma_\ell)\mu^{\mathrm{S}}_{\ell;\ell,\ell}(v) = \mu^{\mathrm{SnP}}_{\ell+1;\ell,\binom{c}{\ell}}(v^{\mathrm{nP}}), \end{cases} \tag{45a}$$

$$\mu^{\mathrm{SnP}}_{\ell+1;m,\binom{c}{n}}(v^{\mathrm{nP}}) = \mu^{\mathrm{SHnV}}_{\ell+1;m,\binom{\neg b}{n}}(v^{\mathrm{nV}}) = \gamma_0^{\mathrm{nV}} \mu^{\mathrm{SnV}}_{\ell;m,\binom{\neg b}{n}}(v^{\mathrm{nV}}), \tag{45b}$$

$$\mu^{\mathrm{SnV}}_{\ell;m,\binom{\neg b}{n}}(v^{\mathrm{nV}}) = \mu^{\mathrm{S}}_{\ell;m,\binom{\neg b}{n}}(\bar{v}), \tag{45c}$$

where v^{nP}, v^{nV} and $\bar{v}$ are the valuations given by (35) with $\alpha_{-1}^{\mathrm{nV}} = 0$.

Proof. Suppose that $\ell = m = n$ does not hold. The transformation $T^{\mathrm{S}\to\mathrm{SnP}}$ maps the set

$$\Pi' = \left\{ \omega \in \Pi^{\mathrm{S}}_{\ell+1;\binom{\neg b}{m},n};\ s_{|\omega|-n-1,|\omega|-n}(\omega) = a_n^{\mathrm{R}} b_{n+1}^{\mathrm{R}} \text{ or } s_{|\omega|-n}(\omega) = c_n^{\mathrm{R}} \right\}$$

onto $\Pi^{\mathrm{SnP}}_{\ell+1;m,\binom{c}{n}}$, and is enumerator-conserving with (35a) as $\mu'(v) = \mu^{\mathrm{SnP}}_{\ell+1;m,\binom{c}{n}}(v^{\mathrm{nP}})$. The trivial surjection from Π' onto $\Pi^{\mathrm{S}}_{\ell;\binom{\neg b}{m},n}$

$$\Pi' \ni \omega \quad \mapsto \quad \begin{cases} s_{1,|\omega|-n-2}(\omega) s_{|\omega|-n+1,|\omega|}(\omega) & \text{if } s_{|\omega|-n-1,|\omega|-n}(\omega) = a_n^{\mathrm{R}} b_{n+1}^{\mathrm{R}}, \\ s_{1,|\omega|-n-1}(\omega) s_{|\omega|-n+1,|\omega|}(\omega) & \text{if } s_{|\omega|-n}(\omega) = c_n^{\mathrm{R}} \end{cases}$$

leads $\mu'(v) = (\alpha_n + \gamma_n)\mu^{\mathrm{S}}_{\ell;\binom{\neg b}{m},n}(v)$. We then have the first equality of (45a). Similarly, we can obtain the second one of (45a). The equalities (45b) and (45c) are obtained using $T^{\mathrm{SnP}\to\mathrm{SnV}}$ and $T^{\mathrm{S}\to\mathrm{SnV}}$, respectively. $\square$

In a way similar to that used to obtain Proposition 11 from Lemma 10, we have the following.

Proposition 13. *Let v and $\bar{v}$ be valuations for Schröder paths satisfying $\bar{v} = \bar{V}(v)$. Then, the following equalities of enumerators hold,*

- *if* $\ell \leq -1$ *and* $-\ell - 1 = m = n$ *does not hold, then*

$$\frac{\gamma_{n-1}\gamma_n}{\alpha_{n-1} + \gamma_{n-1}} \mu^{\mathrm{SH}}_{\ell;m,\binom{\neg b}{n}}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{\ell;\binom{\neg b}{m},n}(\bar{v}), \tag{46a}$$

- *if* $\ell \geq 0$ *and* $\ell = m = n$ *does not hold, then*

$$(\alpha_n + \gamma_n)\mu^{\mathrm{S}}_{\ell;\binom{\neg b}{m},n}(v) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{\ell;m,\binom{\neg b}{n}}(\bar{v}), \tag{46b}$$

- *otherwise*

$$\begin{cases} \dfrac{\gamma_{-\ell-2}\gamma_{-\ell-1}}{\alpha_{-\ell-2} + \gamma_{-\ell-2}} \mu^{\mathrm{SH}}_{\ell;-\ell-1,-\ell-1}(v) = \bar{\gamma}_0 \mu^{\mathrm{SH}}_{\ell;-\ell-1,-\ell-1}(\bar{v}), & \ell \leq -1, \\ (\alpha_\ell + \gamma_\ell)\mu^{\mathrm{S}}_{\ell;\ell,\ell}(v) = \bar{\gamma}_0 \mu^{\mathrm{S}}_{\ell;\ell,\ell}(\bar{v}), & \ell \geq 0, \end{cases} \tag{46c}$$

where $\alpha_{-1} = 0$ *and* $\gamma_{-1} \neq 0$*.*

Hence, we have from the imperfect orthogonality (33) with the equalities (43), (46a) and the first of (46c)

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^\ell G^{\mathrm{F}}_n(\bar{v}; z)\right] = -\left[\prod_{i=0}^{n}(-\gamma_i)\right] \mu^{\mathrm{SH}}_{\ell-1;0,\binom{\neg b}{n}}(v), \quad \ell \leq 0. \tag{47}$$

Thus, $\bar{v} = \bar{V}(v)$ is a desired valuation, that is, we have the following by combining the equalities (44) and (47).

Theorem 14 (Second orthogonality). *Let* v *be such a valuation for Schröder paths that* $\alpha_n + \gamma_n \neq 0$ *for each* $n \geq 0$*, and let* $\bar{v} = \bar{V}(v)$*. Then, generating functions of enumerators for Favard-LBP paths satisfy the equality*

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^\ell G^{\mathrm{F}}_n(\bar{v}; z)\right] = \begin{cases} -\left[\displaystyle\prod_{i=0}^{n}(-\gamma_i)\right] \mu^{\mathrm{SH}}_{\ell-1;0,\binom{\neg b}{n}}(v), & \ell \leq 0, \\ \mu^{\mathrm{S}}_{\ell+n;0,\binom{\neg a}{n}}(v), & \ell \geq 1. \end{cases} \tag{48}$$

Particularly, they satisfy the orthogonality property

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{-\ell} G^{\mathrm{F}}_n(\bar{v}; z)\right] = \left[\prod_{i=0}^{n-1}\left(-\frac{\alpha_i}{\gamma_i}\right)\right] \delta_{\ell,n}, \quad 0 \leq \ell \leq n. \tag{49}$$

Hereafter we call this theorem, especially the formula (48), *second orthogonality.*

This theorem gives us a combinatorial representation of the biorthogonal partners $Q_n(z)$ in terms of Favard-LBP paths.

Theorem 15. *Let $P_n(z) \in \mathbb{K}[z]$ be the LBPs satisfying the three-term recurrence equation (2) whose nonzero coefficients $a = (a_k)_{k=0}^{\infty}$ and $c = (c_k)_{k=0}^{\infty}$ satisfy the condition $a_n + c_{n+1} \neq 0$ for each $n \geq 0$. Let $v_P = (a, c)$ be a valuation for Schröder paths. Then the biorthogonal partners $Q_n(z) \in \mathbb{K}[z]$ of the LBPs $P_n(z)$ are represented as*

$$Q_n(z) = G_n^{\mathrm{F}}(v_Q; z), \quad n \geq 0, \tag{50}$$

where the valuation v_Q is given by $v_Q = \bar{V}^*(v_P)$.

Here we also know the following with Corollary 9.

Corollary 16. *The biorthogonal partners $Q_n(z)$ are again LBPs if and only if the recurrence coefficients a_n and c_n of $P_n(z)$ satisfy $a_n + c_{n+1} \neq 0$ for each $n \geq 0$.*

5 Biorthogonality

Finally, in this section, we give a combinatorial representation to the quantity

$$\mathcal{L}\left[z^{\ell} P_m(z^{-1}) Q_n(z)\right], \quad \ell \in \mathbb{Z},\ m, n \geq 0,$$

which shall imply the biorthogonality (4). For this, instead, we evaluate the quantity

$$\Sigma_{\ell;m,n}(v) = \mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell} G_m^{\mathrm{F}}(v^*; z^{-1}) G_n^{\mathrm{F}}(\bar{v}; z)\right], \quad \ell \in \mathbb{Z},\ m, n \geq 0, \tag{51}$$

where v, $v^* = V^*(v)$ and $\bar{v} = \bar{V}(v)$ are valuations for Schröder paths.

Case $m \leq n$: Expanding $G_m^{\mathrm{F}}(v^*; z)$ in the right-hand side of (51) and using the second orthogonality (48), we have

$$\Sigma_{\ell;m,n}(v) = \Sigma_1 - \left[\prod_{i=0}^{n} (-\gamma_i)\right] \Sigma_2, \tag{52}$$

where Σ_1 and Σ_2 are

$$\Sigma_1 = \sum_{i=0}^{\ell-1} \mu^{\mathrm{S}}_{\ell+n-i;0,\binom{\neg a}{n}}(v) \cdot \mu^{\mathrm{F}}_{m,i}(v^*), \quad \Sigma_2 = \sum_{i=\ell}^{m} \mu^{\mathrm{SH}}_{\ell-i-1;0,\binom{\neg b}{n}}(v) \cdot \mu^{\mathrm{F}}_{m,i}(v^*). \tag{53}$$

Here, Σ_1 is evaluated as

$$\Sigma_1 = \begin{cases} 0, & \ell \leq 0, \\ \left[\displaystyle\prod_{i=0}^{m-1} \left(-\frac{1}{\gamma_i}\right)\right] \mu^{\mathrm{S}}_{\ell+n;m,\binom{\neg a}{n}}(v), & \ell \geq 1. \end{cases} \tag{54}$$

Proof of (54). We can rewrite Σ_1 in (52) as

$$\Sigma_1 = \sum_{(\omega,\eta)\in\Pi_1} \mathrm{wgt}(v;\omega)\cdot\mathrm{wgt}(-v^*;\eta),$$

where Π_1 is the set of S×F paths from $(0,0)$ to $(\ell+n,m)$

$$\Pi_1 = \bigcup_{i=0}^{\ell-1}\Big(\Pi^{\mathrm{S}}_{\ell+n-i;0,\binom{\neg a}{n}} \times \Pi^{\mathrm{F}}_{m,i}\Big).$$

If $\ell \le 0$, then Π_1 is empty and $\Sigma_1 = 0$. Let us consider the case $\ell \ge 1$. For any $(\omega,\eta)\in\Pi_1$, the Schröder path ω is rightward and its length is at least $n+1$. Additionally, if its length is $n+1$, then it is any of $\Pi^{\mathrm{S}}_{n+1;0,\binom{\neg a}{n}} = \{a^{\mathrm{R}}_0\cdots a^{\mathrm{R}}_{n-1}a^{\mathrm{R}}_n b^{\mathrm{R}}_{n+1} b^{\mathrm{R}}_n\cdots b^{\mathrm{R}}_1,\ a^{\mathrm{R}}_0\cdots a^{\mathrm{R}}_{n-1}c^{\mathrm{R}}_n b^{\mathrm{R}}_n\cdots b^{\mathrm{R}}_1\}$. Moreover, its first m steps and its last n ones are disjoint. Thus, the set $\Pi_1 \setminus \widetilde{\Pi}^{\mathrm{S\times F}}_{\ell+n,m}$ is closed under the transformation $T^{\mathrm{S\times F}}_{\ell+n,m}$. Hence, in a way similar to that used to obtain the first orthogonality (23), we have the second case of (54). □

Similarly, Σ_2 is evaluated as

$$\Sigma_2 = \begin{cases} \mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{n}}(v), & \ell \le 0,\\ 0, & \ell \ge 1.\end{cases} \tag{55}$$

As a whole, we have

$$\Sigma_{\ell;m,n}(v) = \begin{cases} -\left[\prod_{i=0}^{n}(-\gamma_i)\right]\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{n}}(v), & \ell \le 0,\\ \left[\prod_{i=0}^{m-1}\left(-\dfrac{1}{\gamma_i}\right)\right]\mu^{\mathrm{S}}_{\ell+n;m,\binom{\neg a}{n}}(v), & \ell \ge 1.\end{cases} \tag{56}$$

Case $m > n$: First, we rewrite $\Sigma_{\ell;m,n}(v)$ in (51), using (43) and Lemma 3, as

$$\Sigma_{\ell;m,n}(v) = \mathcal{L}^{\mathrm{S}}(\bar{v}^*)\big[z^{-\ell}G^{\mathrm{F}}_m(v^*;z)G^{\mathrm{F}}_n(\bar{v};z^{-1})\big] = \Sigma_{-\ell;n,m}(\bar{v}^*),$$

where $\bar{v}^* = \bar{V}^*(v)$. Thus, we have from the formula (56) and Proposition 13

$$\Sigma_{\ell;m,n}(v) = \begin{cases} -\left[\prod_{i=0}^{n}(-\gamma_i)\right]\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{n}}(v), & \ell \le -1,\\ \left[\prod_{i=0}^{m-1}\left(-\dfrac{1}{\gamma_i}\right)\right]\mu^{\mathrm{S}}_{\ell+n;m,\binom{\neg a}{n}}(v), & \ell \ge 0.\end{cases} \tag{57}$$

As a result, we have the following.

Theorem 17 (Biorthogonality). *Let v be such a valuation for Schröder paths that $\alpha_n+\gamma_n\neq 0$ for each $n\geq 0$, and let $v^*=V^*(v)$ and $\bar{v}=\bar{V}(v)$. Then, generating functions of enumerators for Favard-LBP paths satisfy the equality*

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell}G_m^{\mathrm{F}}(v^*;z^{-1})G_n^{\mathrm{F}}(\bar{v};z)\right]=\begin{cases}-\left[\displaystyle\prod_{i=0}^{n}(-\gamma_i)\right]\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{n}}(v), & \ell\in\mathbb{Z}^-_{m,n},\\ \left[\displaystyle\prod_{i=0}^{m-1}\left(-\frac{1}{\gamma_i}\right)\right]\mu^{\mathrm{S}}_{\ell+n;m,\binom{\neg a}{n}}(v), & \ell\in\mathbb{Z}^+_{m,n},\end{cases}\tag{58}$$

where $\mathbb{Z}^{\pm}_{m,n}\subset\mathbb{Z}$ are the sets of integers

$$\mathbb{Z}^-_{m,n}=\begin{cases}\mathbb{Z}_{\leq 0}, & m\leq n,\\ \mathbb{Z}_{\leq -1}, & m>n,\end{cases}\qquad \mathbb{Z}^+_{m,n}=\mathbb{Z}\setminus\mathbb{Z}^-_{m,n}.$$

Particularly, they satisfy the biorthogonality property

$$\mathcal{L}^{\mathrm{S}}(v)\left[G_m^{\mathrm{F}}(v^*;z^{-1})G_n^{\mathrm{F}}(\bar{v};z)\right]=\left[\prod_{i=0}^{m-1}\left(-\frac{\alpha_i}{\gamma_i}\right)\right]\delta_{m,n}.\tag{59}$$

This biorthogonality, letting $m=0$, naturally induces the second orthogonality of Theorem 14. Similarly, it does the first one of Theorem 5 by letting $n=0$, which, however, is seen unobvious at a glance in the case $\ell=m=0$ and in the one $\ell\leq -1$. At the last, let us confirm this. Substituting $n=0$ in the biorthogonality (58), we have

$$\mathcal{L}^{\mathrm{S}}(v)\left[z^{\ell}G_m^{\mathrm{F}}(v^*;z^{-1})\right]=\begin{cases}\gamma_0\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{0}}(v), & \ell\in\mathbb{Z}^-_{m,0},\\ \left[\displaystyle\prod_{i=0}^{m-1}\left(-\frac{1}{\gamma_i}\right)\right]\mu^{\mathrm{S}}_{\ell;m,0}(v), & \ell\in\mathbb{Z}^+_{m,0}.\end{cases}$$

- Case $\ell=m=0$: Since $0\in\mathbb{Z}^-_{m,0}$, then we need $\gamma_0\mu^{\mathrm{SH}}_{-1;0,\binom{\neg b}{0}}(v)=1$. Note that $\Pi^{\mathrm{SH}}_{-1;0,\binom{\neg b}{0}}=\{c_0^{\mathrm{L}}\}$. Thus, we have $\mu^{\mathrm{SH}}_{-1;0,\binom{\neg b}{0}}(v)=\gamma_0^*$, which satisfies the need.
- Case $\ell\leq -1$: Since $\ell\in\mathbb{Z}^-_{m,0}$, then we need $\gamma_0\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{0}}(v)=\mu^{\mathrm{SH}}_{\ell-m;m,0}(v)$. Note that any path in $\Pi^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{0}}$ is leftward, its length is at least 2 and it ends by a horizontal step c_0^{L}. Thus, deleting this last step, we have $\mu^{\mathrm{SH}}_{\ell-m-1;m,\binom{\neg b}{0}}(v)=\gamma_0^*\mu^{\mathrm{SH}}_{\ell-m;m,0}(v)$, which satisfies the need.

References

[1] J. Bonin, L. Shapiro and R. Simion, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, J. Statist. Plann. Inference **34** (1993), 35–55.

[2] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, 1974.

[3] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32** (1980) 125–161.

[4] E. Hendriksen and H. van Rossum, Orthogonal Laurent polynomials, Nederl. Akad. Wetensch. Indag. Math. **48** (1986) 17–36.

[5] M.E.H. Ismail and D.R. Masson, Generalized orthogonality and continued fractions, J. Approx. Theory **83** (1995) 1–40.

[6] W.B. Jones and W.J. Thron, Survey of continued fraction methods of solving moment problems and related topics, in: Analytic Theory of Continued Fractions, Lecture Notes in Mathematics, Vol. 932, Springer, Berlin, 1982, pp.4–37.

[7] D. Kim, A combinatorial approach to biorthogonal polynomials, SIAM J. Discrete Math. **5** (1992), no. 3, 413–421.

[8] E. Schröder, Vier combinatorische probleme, Z. Math. Phys. **15** (1870), 361–376.

[9] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at www.research.att.com/~njas/sequences/, 2006.

[10] G. Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in: Orthogonal Polynomials and Applications, Lecture Notes in Mathematics, Vol. 1171, Springer, Berlin, 1985, pp.139–157.

[11] L. Vinet and A. Zhedanov, Spectral transformations of the Laurent biorthogonal polynomials. I. q-Appel polynomials, J. Comput. Appl. Math. **131** (2001) 253–266.

[12] A. Zhedanov, The "classical" Laurent biorthogonal polynomials, J. Comput. Appl. Math. **98** (1998) 121–147.