Bipartite-uniform hypermaps on the sphere

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Abstract

A hypermap is (hypervertex-) bipartite if its hypervertices can be 2-coloured in such a way that "neighbouring" hypervertices have different colours. It is bipartite-uniform if within each of the sets of hypervertices of the same colour, hyperedges and hyperfaces, all the elements have the same valency. The flags of a bipartite hypermap are naturally 2-coloured by assigning the colour of its adjacent hypervertices. A hypermap is bipartite-regular if the automorphism group acts transitively on each set of coloured flags. If the automorphism group acts transitively on the set of all flags, the hypermap is regular. In this paper we classify the bipartite-uniform hypermaps on the sphere (up to duality). Two constructions of bipartite-uniform hypermaps are given. All bipartite-uniform spherical hypermaps are shown to be constructed in this way. As a by-product we show that every bipartite-uniform hypermap $\mathcal{H}$ on the sphere is bipartite-regular. We also compute their irregularity group and index, and also their closure cover $\mathcal{H}^{\Delta}$ and covering core $\mathcal{H}_{\Delta}$.

1 Introduction

A map generalises to a hypermap when we remove the requirement that an edge must join two vertices at most. A hypermap $\mathcal{H}$ can be regarded as a bipartite map where one of the two monochromatic sets of vertices represent the hypervertices and the other the hyperedges of $\mathcal{H}$. In this perspective hypermaps are cellular embeddings of hypergraphs on compact connected surfaces (two-dimensional compact connected manifolds) without boundary – in this paper we deal only with the boundary-free case.

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Usually classifications in map/hypermap theory are carried out by genus, by number of faces, by embedding of graphs, by automorphism groups or by some fixed properties such as edge-transitivity. Since Klein and Dyck [13, 11] – where certain 3-valent regular maps of genus 3 were studied in connection with constructions of automorphic functions on surfaces – most classifications of maps (and hypermaps) involve regularity or orientably-regularity (direct-regularity). The orientably-regular maps on the torus (in [10]), the orientably-regular embeddings of complete graphs (in [15]), the orientably-regular maps with automorphism groups isomorphic to $PSL(2,q)$ (in [21]) and the bicontactual regular maps (in [26]), are examples to name but a few. The just-edge-transitive maps of Jones [18] and the classification by Siran, Tucker and Watkins [22] of the edge-transitive maps on the torus, on the other hand, include another kind of "regularity" other than regularity or orientably-regularity. According to Graver and Wakins [17], an edge transitive map is determined by 14 types of automorphism groups. Among these, 11 correspond to "restricted regularity" [1]. Jones's "just-edge-transitive" maps correspond to $\Delta^{\hat{0}\hat{2}}$-regular maps of "rank 4", where $\Delta^{\hat{0}\hat{2}}$ is the normal closure of $\langle R_1, R_0R_2\rangle$ of index 4 in the free product $\Delta = C_2 * C_2 * C_2$ generated by the 3 reflections R_0, R_1 and R_2 on the sides of a hyperbolic triangle with zero internal angles; "rank 4" means that it is not Θ-regular for no normal subgroup Θ of Δ of index < 4. Moreover, the automorphism group of the toroidal edge-transitive maps realise 7 of the above 14 family-types [22]; they all correspond to restrictedly regular maps, namely of ranks 1 [the regular maps], 2 [the just-orientably-regular (or chiral) maps, the just-bipartite-regular maps, the just-face-bipartite-regular maps and the just-Petrie-bipartite-regular maps] and 4 [the just-$\Delta^{+\hat{0}}$-regular maps and the just-$\Delta^{+\hat{2}}$-regular maps] (see [1]).

In this paper we classify the "bipartite-uniform" hypermaps on the sphere. They all turn out to be "bipartite-regular". A hypermap $\mathcal{H}$ is *bipartite* if its hypervertices can be 2-coloured in such a way that "neighbouring" hypervertices have different colours. It is *bipartite-uniform* if the hypervertices of one colour, the hypervertices of the other colour, the hyperedges and the hyperfaces have common valencies l_1, l_2, m and n respectively. The flags of a bipartite hypermap are naturally 2-coloured by assigning the colour of their adjacent hypervertices. A bipartite hypermap is *bipartite-regular* if the automorphism group acts transitively on each set of coloured flags. If the automorphism group acts transitively on the whole set of flags the hypermap is *regular*. Bipartite-regularity corresponds to $\Delta^{\hat{0}}$-regularity [1] where $\Delta^{\hat{0}}$, a normal subgroup of index 2 in Δ, is the normal closure of the subgroup generated by R_1 and R_2.

We also compute the irregularity group and the irregularity index of the bipartite-regular hypermaps $\mathcal{H}$ on the sphere as well as their closure cover $\mathcal{H}^{\Delta}$ (the smallest regular hypermap that covers $\mathcal{H}$) and their covering core $\mathcal{H}_{\Delta}$ (the largest regular hypermap covered by $\mathcal{H}$). Regular hypermaps on the sphere (see §1.4) are up to a S_3-duality (see §1.3) regular maps and these are the five Platonic solids plus the two infinite families of type $(2;2;n)$ and $(n;n;1)$, and their duals. An interesting well known fact, which comes from the "universality" of the sphere, is that uniform hypermaps on the sphere are regular. According to [1] this translates to "Δ-uniformity in the sphere implies Δ-regularity". We may now ask for which normal subgroups Θ of finite index in Δ do we still have "Θ-uniformity in the sphere implies Θ-regularity", once the meaning of Θ-uniformity is understood? As a byproduct of the classification we show in this paper that bipartite-uniformity (that is, $\Delta^{\hat{0}}$-uniformity) still implies bipartite-regularity (that is, $\Delta^{\hat{0}}$-regularity). $\Delta^{\hat{0}}$ is just one of the seven normal subgroups with index 2 in Δ. The others are $\Delta^{\hat{1}} = \langle R_0, R_2\rangle^{\Delta}$, $\Delta^{\hat{2}} = \langle R_0, R_1\rangle^{\Delta}$, $\Delta^{0} = \langle R_0, R_1R_2\rangle^{\Delta}$, $\Delta^{1} = \langle R_1, R_0R_2\rangle^{\Delta}$, $\Delta^{2} = \langle R_2, R_0R_1\rangle^{\Delta}$ and $\Delta^{+} = \langle R_1R_2, R_2R_0\rangle$ (see [4] for more details). As the notation indicates they are grouped into three families, within which they differ by a dual operation. This duality says that the result is still valid if we replace $\Delta^{\hat{0}}$ by $\Delta^{\hat{1}}$ or $\Delta^{\hat{2}}$. For $\Theta = \Delta^0, \Delta^1, \Delta^2$, and Δ^+, Θ-uniformity is the same as uniformity, and since regularity implies Θ-regularity, on the sphere Θ-uniformity implies Θ-regularity for any subgroup Θ of index 2 in Δ. At the end, as a final comment, we show that on each orientable surface we can find always bipartite-chiral (that is, irregular bipartite-regular) hypermaps.

1.1 Hypermaps

A *hypermap* is combinatorially described by a four-tuple $\mathcal{H} = (\Omega_{\mathcal{H}}; h_0, h_1, h_2)$ where $\Omega_{\mathcal{H}}$ is a non-empty finite set and h_0, h_1, h_2 are fixed-point free involutory permutations of $\Omega_{\mathcal{H}}$ generating a permutation group $\langle h_0, h_1, h_2\rangle$ acting transitively on $\Omega_{\mathcal{H}}$. The elements of $\Omega_{\mathcal{H}}$ are called *flags*, the permutations h_0, h_1 and h_2 are called *canonical generators* and the group $\mathrm{Mon}(\mathcal{H}) = \langle h_0, h_1, h_2\rangle$ is the *monodromy group of* $\mathcal{H}$. One says that $\mathcal{H}$ is a *map* if $(h_0h_2)^2 = 1$. The *hypervertices* (or 0*-faces*) of $\mathcal{H}$ correspond to $\langle h_1, h_2\rangle$-orbits on $\Omega_{\mathcal{H}}$. Likewise, the *hyperedges* (or 1*-faces*) and *hyperfaces* (or 2*-faces*) correspond to $\langle h_0, h_2\rangle$ and $\langle h_0, h_1\rangle$-orbits on $\Omega_{\mathcal{H}}$, respectively. If a flag ω belongs to the corresponding orbit determining a k-face f we say that ω belongs to f, or that f contains ω.

We fix $\{i, j, k\} = \{0, 1, 2\}$. The *valency* of a k-face $f = w\langle h_i, h_j\rangle$, where $\omega \in \Omega_{\mathcal{H}}$, is the least positive integer n such that $(h_ih_j)^n \in \mathrm{Stab}(w)$. Since $h_i \neq 1$ and $h_j \neq 1$, h_ih_j generates a normal subgroup with index two in $\langle h_i, h_j\rangle$. It follows that $|\langle h_i, h_j\rangle| = 2|\langle h_ih_j\rangle|$ and so the valency of a k-face is equal to half of its cardinality. $\mathcal{H}$ is *uniform* if its k-faces have the same valency n_k, for each $k \in \{0, 1, 2\}$. We say that $\mathcal{H}$ has *type* $(l; m; n)$ if l, m and n are, respectively, the least common multiples of the valencies of the hypervertices, hyperedges and hyperfaces. The *characteristic* of a hypermap is the Euler characteristic of its underlying surface, the imbedding surface of the underlying hypergraph (see Lemma 3 for a combinatorial definition).

A *covering* from a hypermap $\mathcal{H} = (\Omega_{\mathcal{H}}; h_0, h_1, h_2)$ to another hypermap $\mathcal{G} = (\Omega_{\mathcal{G}}; g_0, g_1, g_2)$ is a function $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ such that $h_i\psi = \psi g_i$ for all $i \in \{0, 1, 2\}$. The transitive action of $\mathrm{Mon}(\mathcal{G})$ on $\Omega_{\mathcal{G}}$ implies that ψ is onto. By von Dyck's theorem ([16, pg 28]) the assignment $h_i \mapsto g_i$ extends to a group epimorphism $\Psi : \mathrm{Mon}(\mathcal{H}) \to \mathrm{Mon}(\mathcal{G})$ called the *canonical epimorphism.* The covering ψ is an *isomorphism* if it is injective. If there exists a covering ψ from $\mathcal{H}$ to $\mathcal{G}$, we say that $\mathcal{H}$ *covers* $\mathcal{G}$ or that $\mathcal{G}$ *is covered by* $\mathcal{H}$; if ψ is an isomorphism we say that $\mathcal{H}$ and $\mathcal{G}$ are *isomorphic* and write $\mathcal{H} \cong \mathcal{G}$. An *automorphism* of $\mathcal{H}$ is an isomorphism $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{H}}$ from $\mathcal{H}$ to itself; that is, a function ψ that commutes with the canonical generators. The set of automorphisms of $\mathcal{H}$ is represented by $\mathrm{Aut}(\mathcal{H})$. As a direct consequence of the Euclidean Division Algorithm we have:

Lemma 1. *Let $\psi : \Omega_{\mathcal{H}} \to \Omega_{\mathcal{G}}$ be a covering from $\mathcal{H}$ to $\mathcal{G}$ and $\omega \in \Omega_{\mathcal{H}}$. Then the valency of the k-face of $\mathcal{G}$ that contains $\omega\psi$ divides the valency of the k-face of $\mathcal{H}$ that contains ω.*

Of the two groups $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$ the first acts transitively on $\Omega = \Omega_{\mathcal{H}}$ (by definition) and the second, due to the commutativity of the automorphisms with the canonical generators, acts *semi-regularly* on Ω; that is, the non-identity elements of $\mathrm{Aut}(\mathcal{H})$ act without fixed points. A transitive semi-regular action is called a *regular* action. These two actions give rise to the following inequalities:

$$|\mathrm{Mon}(\mathcal{H})| \geq |\Omega| \geq |\mathrm{Aut}(\mathcal{H})| \,.$$

Moreover, each of the above equalities implies the other. An equality in the first of these inequalities implies that $Mon(\mathcal{H})$ acts semi-regularly (hence regularly) on Ω, while an equality on the second implies that $Aut(\mathcal{H})$ acts transitively (hence regularly) on Ω. If $\mathrm{Mon}(\mathcal{H})$ acts regularly on Ω, or equivalently if $\mathrm{Aut}(\mathcal{H})$ acts regularly on Ω, the hypermap $\mathcal{H}$ is regular.

Each hypermap $\mathcal{H}$ gives rise to a permutation representation $\rho_{\mathcal{H}} : \Delta \to \mathrm{Mon}(\mathcal{H})$, $R_i \mapsto h_i$, where Δ is the free product $C_2 * C_2 * C_2$ with presentation $\Delta = \langle R_0, R_1, R_2 \mid {R_0}^2 = {R_1}^2 = {R_2}^2 = 1\rangle$. The group Δ acts naturally and transitively on $\Omega_{\mathcal{H}}$ via $\rho_{\mathcal{H}}$. The stabiliser $H = \mathrm{Stab}_{\Delta}(\omega)$ of a flag $\omega \in \Omega_{\mathcal{H}}$ under the action of Δ is called the *hypermap subgroup* of $\mathcal{H}$; this is unique up to conjugation in Δ. The valency of a k-face containing ω is the least positive integer n such that $(R_iR_j)^n \in H$; more generally, the valency of a k-face containing the flag $\sigma = \omega \cdot g = \omega(g)\rho_{\mathcal{H}} \in \Omega_{\mathcal{H}}$, where $g \in \Delta$, is the least positive integer n such that $(R_iR_j)^n \in \mathrm{Stab}_{\Delta}(\sigma) = \mathrm{Stab}_{\Delta}(\omega \cdot g) = \mathrm{Stab}_{\Delta}(\omega)^g = H^g$.

Denote by $\mathrm{Alg}(\mathcal{H}) = (\Delta/_rH; a_0, a_1, a_2)$ where $a_i : \Delta/_rH \to \Delta/_rH$, $Hg \mapsto HgH_{\Delta}R_i = HgR_i$. It is easy to see that $\mathrm{Alg}(\mathcal{H}) \cong \mathcal{H}$. We say that $\mathrm{Alg}(\mathcal{H})$ is the *algebraic presentation* of $\mathcal{H}$. Moreover, it is well known that:

1. A hypermap $\mathcal{H}$ is regular if and only if its hypermap subgroup H is normal in Δ.

2. A regular hypermap is necessarily uniform.

Since $\mathrm{Alg}(\mathcal{H})$ and $\mathcal{H}$ are isomorphic, we will not differentiate one from the other.

Following [1], if $H < \Theta$ for a given $\Theta \lhd \Delta$, we say that $\mathcal{H}$ is Θ-*conservative*. A Δ^+-conservative hypermap is better known as an orientable hypermap. An automorphism of an orientable hypermap either preserves the two Δ^+-orbits or permutes them. Those that preserve Δ^+-orbits are called *orientation-preserving automorphisms*. The set of orientation-preserving automorphisms is a subgroup of $\mathrm{Aut}(\mathcal{H})$ and is denoted by $\mathrm{Aut}^+(\mathcal{H})$. If $\mathcal{H}$ is $\Delta^{\hat{0}}$-conservative (resp. $\Delta^{\hat{1}}$-conservative, resp. $\Delta^{\hat{2}}$-conservative) we say that $\mathcal{H}$ is *bipartite*, *vertex-bipartite* or 0-*bipartite* (resp. *edge-bipartite* or 1-*bipartite*, resp. *face-bipartite* or 2-*bipartite*).

Lemma 2. *If $\mathcal{H}$ is bipartite and $\omega \in \Omega_{\mathcal{H}}$, then the valencies of the hyperedge and the hyperface that contain ω must be even.*

Proof. If m and n are the valencies of the hyperedge and the hyperface that contain $\omega = Hd$, $d \in \Delta$, then $(R_2R_0)^m, (R_0R_1)^n \in H^d \subseteq \Delta^{\hat{0}}$. Therefore m and n must be even. $\square$

If $H \lhd \Delta^+$, we say that $\mathcal{H}$ is *orientably-regular*. If $H \lhd \Delta^{\hat{0}}$ (resp. $H \lhd \Delta^{\hat{1}}$ and $H \lhd \Delta^{\hat{2}}$), we say that $\mathcal{H}$ is *vertex-bipartite-regular* (resp. *edge-bipartite-regular* and *face-bipartite-regular*). If $\mathcal{H}$ is vertex-bipartite-regular (resp. edge-bipartite-regular, resp. face-bipartite-regular) but not regular, we say that $\mathcal{H}$ is *vertex-bipartite-chiral* (resp. *edge-bipartite-chiral*, resp. *face-bipartite-chiral*). We will use *bipartite-regular* and *bipartite-chiral* in place of vertex-bipartite-regular and vertex-bipartite-chiral for short.

A bipartite-uniform hypermap is a bipartite hypermap such that all the hypervertices in the same $\Delta^{\hat{0}}$-orbit have the same valency, as do all the hyperedges and all the hyperfaces. The *bipartite-type* of a bipartite-uniform hypermap $\mathcal{H}$ is a four-tuple $(l_1, l_2; m; n)$ (or $(l_2, l_1; m; n)$) where l_1 and l_2 ($l_1 \leq l_2$) are the valencies (not necessarily distinct) of the hypervertices of $\mathcal{H}$, m is the valency of the hyperedges of $\mathcal{H}$ and n is the valency of the hyperfaces of $\mathcal{H}$. We note that if $\mathcal{H}$ is a bipartite-uniform hypermap of bipartite-type $(l_1, l_2; m; n)$, then m and n must be even by Lemma 2.

1.2 Euler formula for uniform hypermaps

Using the well known Euler formula for maps one easily gets the following well known result:

Lemma 3 (Euler formula for hypermaps). *Let $\mathcal{H}$ be a hypermap with V hypervertices, E hyperedges and F hyperfaces. If $\mathcal{H}$ has underlying surface $\mathcal{S}$ with Euler characteristic χ, then $\chi = V + E + F - \frac{|\Omega_{\mathcal{H}}|}{2}$.* (See for example [28] and the references therein.)

If $\mathcal{H}$ is uniform of type (l, m, n), then $V = \frac{|\Omega_{\mathcal{H}}|}{2l}$, $E = \frac{|\Omega_{\mathcal{H}}|}{2m}$ and $F = \frac{|\Omega_{\mathcal{H}}|}{2n}$. Replacing the values of V, E and F in the last formula, we get:

Corollary 4 (Euler formula for uniform hypermaps).

$$\chi = \frac{|\Omega_{\mathcal{H}}|}{2}\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right).$$

1.3 Duality

A non-inner automorphism ψ of Δ (that is, an automorphism not arising from a conjugation) gives rise to an operation on hypermaps by transforming a hypermap $\mathcal{H} = (\Delta/_r H, H_\Delta R_0, H_\Delta R_1, H_\Delta R_2)$, with hypermap-subgroup H, into its operation-dual

$$\begin{aligned} D_\psi(\mathcal{H}) &= (\Delta/_r H\psi; (H\psi)_\Delta R_0, (H\psi)_\Delta R_1, (H\psi)_\Delta R_2) \\ &= (\Delta/_r H\psi; H_\Delta \psi R_0, H_\Delta \psi R_1, H_\Delta \psi R_2) \end{aligned}$$

with hypermap-subgroup $H\psi$ (see [14, 19, 20] for more details). Note that if ψ is inner, then $D_\psi(\mathcal{H})$ is isomorphic to $\mathcal{H}$. In particular, each permutation $\sigma \in S_{\{0,1,2\}} \backslash \{id\}$ induces a non-inner automorphism $\sigma^\circ : \Delta \longrightarrow \Delta$ by assigning $R_i \mapsto R_{i\sigma}$, for $i = 0, 1, 2$. This automorphism induces an operation $\mathcal{D}_\sigma$ on hypermaps by assigning the hypermap-subgroup H of $\mathcal{H}$ to a hypermap-subgroup $H\sigma^\circ$. Such an operator transforms each hypermap $\mathcal{H} = (\Omega_\mathcal{H}; h_0, h_1, h_2)$ into its σ-dual $D_\sigma(\mathcal{H}) \cong (\Omega_\mathcal{H}; h_{0\sigma^{-1}}, h_{1\sigma^{-1}}, h_{2\sigma^{-1}})$. We note that the k-faces of $\mathcal{H}$ are the $k\sigma$-faces of $\mathcal{D}_\sigma(\mathcal{H})$. From this note and the definition of σ-duality one easily get the following properties of $\mathcal{D}_\sigma$.

Lemma 5 (Properties of $\mathcal{D}_\sigma$). *Let $\mathcal{H}$, $\mathcal{G}$ be two hypermaps and $\sigma, \tau \in S_{\{0,1,2\}}$. Then (1) $\mathcal{D}_1(\mathcal{H}) = \mathcal{H}$, where $1 = id \in S_{\{0,1,2\}}$; (2) $\mathcal{D}_\tau(\mathcal{D}_\sigma(\mathcal{H})) = \mathcal{D}_{\sigma\tau}(\mathcal{H})$; (3) If $\mathcal{H}$ covers $\mathcal{G}$, then $\mathcal{D}_\sigma(\mathcal{H})$ covers $\mathcal{D}_\sigma(\mathcal{G})$; (4) If $\mathcal{H} \cong \mathcal{G}$, then $\mathcal{D}_\sigma(\mathcal{H}) \cong \mathcal{D}_\sigma(\mathcal{G})$; (5) If $\mathcal{H}$ is uniform, then $\mathcal{D}_\sigma(\mathcal{H})$ is uniform; (6) If $\mathcal{H}$ is k-bipartite-uniform, then $\mathcal{D}_\sigma(\mathcal{H})$ is $k\sigma$-bipartite-uniform; (7) If $\mathcal{H}$ is regular, then $\mathcal{D}_\sigma(\mathcal{H})$ is regular; (8) If $\mathcal{H}$ is k-bipartite-regular, then $\mathcal{D}_\sigma(\mathcal{H})$ is $k\sigma$-bipartite-regular; (9) Both $\mathcal{H}$ and $\mathcal{D}_\sigma(\mathcal{H})$ have same underlying surface.*

1.4 Spherical uniform hypermaps

A hypermap $\mathcal{H}$ is *spherical* if its underlying surface is a sphere (i.e if its Euler characteristic is 2). By taking $l \le m \le n$ and $\chi = 2$ in the Euler formula one easily sees that $l < 3$. A simple analysis to the above inequality leads us to the following table of possible types (up to duality):

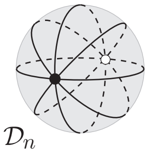
l	m	n	V	E	F	$\lvert\Omega_\mathcal{H}\rvert$	$\mathrm{Mon}(\mathcal{H})$	$\mathcal{H}$	$\mathrm{Aut}^+(\mathcal{H})$
1	k	k	k	1	1	$2k$	D_k	$\mathcal{D}_{(02)}(\mathcal{D}_k)$	C_k
2	2	k	k	k	2	$4k$	$D_k \times C_2$	$\mathcal{P}_k$	C_k
2	3	3	6	4	4	24	S_4	$\mathcal{D}_{(01)}(\mathcal{T})$	A_4
2	3	4	12	8	6	48	$S_4 \times C_2$	$\mathcal{D}_{(01)}(\mathcal{C})$	S_4
2	3	5	30	20	12	120	$A_5 \times C_2$	$\mathcal{D}_{(01)}(\mathcal{D})$	A_5

Table 1: Possible values (up to duality) for type $(l; m; n)$.

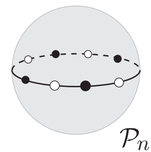
Lemma 6. *All uniform hypermaps on the sphere are regular.*

This result arises because each type $(l; m; n)$ in Table 1 determines a cocompact subgroup $H = \langle (R_1R_2)^l, (R_2R_0)^m, (R_0R_1)^n \rangle^\Delta$ with index $|\Omega_\mathcal{H}|$ in the free product $\Delta = C_2 * C_2 * C_2$ generated by R_0, R_1 and R_2.

Let $\mathcal{T}$, $\mathcal{C}$, $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{I}$ denote the 2-skeletons of the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. These are, up to isomorphism, the unique uniform hypermaps of type $(3; 2; 3)$, $(3; 2; 4)$, $(4; 2; 3)$, $(3; 2; 5)$ and $(5; 2; 3)$ respectively, on the sphere; note that $\mathcal{O} \cong \mathcal{D}_{(02)}(\mathcal{C})$ and $\mathcal{I} \cong \mathcal{D}_{(02)}(\mathcal{D})$. Together with the infinite families of hypermaps $\mathcal{D}_n$ with monodromy group D_n and $\mathcal{P}_n$ with monodromy group $D_n \times C_2$ ($n \in \mathbb{N}$), of types $(n; n; 1)$ and $(2; 2; n)$, respectively, they complete, up to duality and isomorphism, the uniform spherical hypermaps.



$\mathcal{D}_n$



$\mathcal{P}_n$

The last column of Table 1 displays the uniform spherical hypermaps (which are regular by last lemma) of type $(l;m;n)$ with $l \leq m \leq n$.

Lemma 7. *If $\mathcal{H}$ is a hypermap such that all hyperfaces have valency 1, then $\mathcal{H}$ is the "dihedral" hypermap $\mathcal{D}_n$, a regular hypermap on the sphere with n hyperfaces.*

Proof. Let H be a hypermap-subgroup of $\mathcal{H}$. All hyperfaces having valency 1 implies that $R_0R_1 \in H^d$ for all $d \in \Delta$ (i.e., R_0R_1 stabilises all the flags). Then $H\langle R_1, R_2\rangle = H\langle R_0, R_2\rangle = H\langle R_0, R_1, R_2\rangle = \Delta/_r H = \Omega$; that is, $\mathcal{H}$ has only one hypervertex and one hyperedge. Hence $\mathcal{H} \cong \mathcal{D}_n$, where n is the valency of the hyperedge and the hyperface of $\mathcal{H}$. $\square$

2 Constructing bipartite hypermaps

By the Reidemeister-Schreier rewriting process [16] it can be shown that

$$\Delta^{\hat{0}} \cong C_2 * C_2 * C_2 * C_2 = \langle R_1\rangle * \langle R_2\rangle * \langle {R_1}^{R_0}\rangle * \langle {R_2}^{R_0}\rangle\,.$$

As a consequence we have an epimorphism $\varphi : \Delta^{\hat{0}} \longrightarrow \Delta$.

Any such epimorphism φ induces a transformation (not an operation) of hypermaps, by transforming each hypermap $\mathcal{H} = (\Omega_{\mathcal{H}}; h_0, h_1, h_2)$ with hypermap subgroup H into a hypermap $\mathcal{H}^{\varphi^{-1}} = (\Omega; t_0, t_1, t_2)$ with hypermap subgroup $H\varphi^{-1}$.

$$\mathcal{H}^{\varphi^{-1}}\left\{\begin{tikzcd} \Delta \arrow[d, equal, "2"] & \\ \Delta^{\hat{0}} \arrow[r, "\varphi"] \arrow[d, dash] & \Delta \arrow[d, dash] \\ H\varphi^{-1} \arrow[r] & H \end{tikzcd}\right\}\mathcal{H}$$

Algebraically, $\mathcal{H}^{\varphi^{-1}} = (\Delta/_r H\varphi^{-1}; s_0, s_1, s_2)$ with $s_i = (H\varphi^{-1})_{\Delta} R_i$ acting on $\Omega = \Delta/_r H\varphi^{-1}$ by right multiplication. Here $(H\varphi^{-1})_{\Delta}$ denotes the core of $H\varphi^{-1}$ in Δ. In the following lemma we list three elementary, but useful, properties of this transformation φ.

Lemma 8. *Let $g \in \Delta$, $W = (H\varphi^{-1})_{\Delta} w \in \Delta/(H\varphi^{-1})_{\Delta} = \mathrm{Mon}(\mathcal{H}^{\varphi^{-1}})$ and $H\varphi^{-1}g \in \Omega$ be a flag of $\mathcal{H}^{\varphi^{-1}}$. Then,*

(1) *If $g \in \Delta^{\hat{0}}$, then $(H\varphi^{-1})^g = H^{g\varphi}\varphi^{-1}$. If $g \notin \Delta^{\hat{0}}$, then $(H\varphi^{-1})^g = \left(H^{(gR_0)\varphi}\varphi^{-1}\right)^{R_0}$.*

(2) $(H\varphi^{-1})_{\Delta^{\hat{0}}} = H_\Delta\varphi^{-1}$ *and* $(H\varphi^{-1})_\Delta = H_\Delta\varphi^{-1} \cap (H_\Delta\varphi^{-1})^{R_0}$.

(3) $W \in \mathrm{Stab}(H\varphi^{-1}g) \Leftrightarrow w \in (H\varphi^{-1})^g \Leftrightarrow \begin{cases} w\varphi \in H^{g\varphi}, & \text{if } g \in \Delta^{\hat{0}} \\ w^{R_0}\varphi \in H^{(gR_0)\varphi}, & \text{if } g \notin \Delta^{\hat{0}}. \end{cases}$ *Moreover,* $W \in \mathrm{Stab}(H\varphi^{-1}g)$ *implies that* $w \in \Delta^{\hat{0}}$.

Proof. (1) If $g \in \Delta^{\hat{0}}$, then $x \in H^{g\varphi}\varphi^{-1} \Leftrightarrow x\varphi \in H^{g\varphi} \Leftrightarrow (x\varphi)^{(g\varphi)^{-1}} = (x\varphi)^{g^{-1}\varphi} = x^{g^{-1}}\varphi \in H \Leftrightarrow x \in (H\varphi^{-1})^g$. If $g \notin \Delta^{\hat{0}}$, then $gR_0 \in \Delta^{\hat{0}}$ and so $(H\varphi^{-1})^g = \left((H\varphi^{-1})^{(gR_0)}\right)^{R_0} = \left(H^{(gR_0)\varphi}\varphi^{-1}\right)^{R_0}$.

(2) Since φ is onto, the above item translates into these two results.

(3) $W \in \mathrm{Stab}(H\varphi^{-1}g) = \mathrm{Stab}(H\varphi^{-1})^g \Leftrightarrow w \in (H\varphi^{-1})^g$. Since $H\varphi^{-1} \lhd \Delta^{\hat{0}}$, this implies that $w \in \Delta^{\hat{0}}$.

If $g \in \Delta^{\hat{0}}$, then $w \in (H\varphi^{-1})^g \overset{(1)}{=} H^{g\varphi}\varphi^{-1} \Leftrightarrow w\varphi \in H^{g\varphi}$.

If $g \notin \Delta^{\hat{0}}$, then $gR_0 \in \Delta^{\hat{0}}$ and so, by above, $w \in (H\varphi^{-1})^g \Leftrightarrow w^{R_0} \in (H\varphi^{-1})^{gR_0} \Leftrightarrow (w^{R_0})\varphi \in H^{(gR_0)\varphi}$. □

Remark: For simplicity we will not distinguish W from w, and so we will see W as a word on R_0, R_1 and R_2 in Δ instead of a coset word $(H\varphi^{-1})_\Delta w$.

Theorem 9. *If* $\mathcal{H} \cong \mathcal{G}^{\varphi^{-1}}$ *for some hypermap* $\mathcal{G}$*, then* $\Delta^{\hat{0}}$-$\mathrm{Mon}(\mathcal{H}) \cong \mathrm{Mon}(\mathcal{G})$.

Proof. By Lemma 8(2) we deduce that
$\Delta^{\hat{0}}$-$\mathrm{Mon}(\mathcal{H}) = \Delta^{\hat{0}}/H_{\Delta^{\hat{0}}} = \Delta^{\hat{0}}/(G\varphi^{-1})_{\Delta^{\hat{0}}} = \Delta^{\hat{0}}/G_\Delta\varphi^{-1} \cong \Delta/G_\Delta = \mathrm{Mon}(\mathcal{G})$. □

Among many possible canonical epimorphisms $\varphi : \Delta^{\hat{0}} \to \Delta$, there are two that induce transformations preserving the underlying surface, namely φ_W and φ_P defined by

$$R_1\varphi_W = R_1, \quad R_2\varphi_W = R_2, \quad R_1{}^{R_0}\varphi_W = R_0, \quad R_2{}^{R_0}\varphi_W = R_2,$$

$$R_1\varphi_P = R_1, \quad R_2\varphi_P = R_2, \quad R_1{}^{R_0}\varphi_P = R_0, \quad R_2{}^{R_0}\varphi_P = R_0.$$

Denote by $Wal(\mathcal{H})$ the hypermap $\mathcal{H}^{\varphi_W^{-1}}$ and by $Pin(\mathcal{H})$ the hypermap $\mathcal{H}^{\varphi_P^{-1}}$. $Wal(\mathcal{H})$ is a map; in fact, since $(R_0R_2)^2 = R_2{}^{R_0}R_2$ and $((R_0R_2)^2)^{R_0} = R_2R_2{}^{R_0}$ we have $(R_0R_2)^2\varphi_W = ((R_0R_2)^2)^{R_0}\varphi_W = 1$, and hence, by Lemma 8(3), for all $g \in \Delta$, $(R_0R_2)^2 \in \mathrm{Stab}(H\varphi_W^{-1}g)$.

Both hypermaps $Wal(\mathcal{H})$ and $Pin(\mathcal{H})$ have the same underlying surface as $\mathcal{H}$ but while $Wal(\mathcal{H})$ is a map (bipartite map since $H\varphi_W^{-1} \subseteq \Delta^{\hat{0}}$), the well known Walsh bipartite map of $\mathcal{H}$ [24, 4], $Pin(\mathcal{H})$ is not necessarily a map.

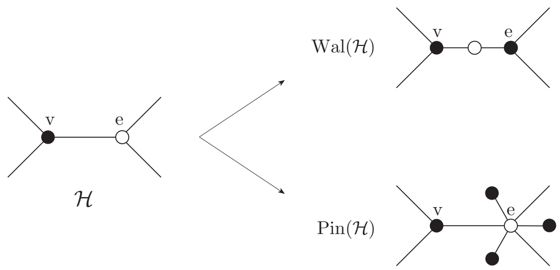


Figure 1: Topological construction of $Wal(\mathcal{H})$ and $Pin(\mathcal{H})$.

Theorem 10 (Properties of φ_W). *Let $\mathcal{H}$ be a hypermap. Then:*

1. *$\mathcal{H}$ is uniform of type $(l;m;n)$ if and only if $Wal(\mathcal{H})$ is bipartite-uniform of bipartite-type $(l,m;2;2n)$ if $l \leq m$ or $(m,l;2;2n)$ if $l \geq m$;*

2. *$\mathcal{H}$ is regular if and only if $Wal(\mathcal{H})$ is bipartite-regular.*

Proof. Let H be a hypermap subgroup of $\mathcal{H}$. Then $H\varphi_W^{-1}$ is a hypermap subgroup of $Wal(\mathcal{H})$.

(10.1) ($\Rightarrow$) Let us suppose that $\mathcal{H}$ is uniform of type $(l;m;n)$. Note first that

$$\begin{aligned} R_1R_2 &= (R_1R_2)\varphi_W\,, && (1)\\ R_0R_2 &= ({R_1}^{R_0}{R_2}^{R_0})\varphi_W = (R_1R_2)^{R_0}\varphi_W\,, && (2)\\ R_0R_1 &= ({R_1}^{R_0}R_1)\varphi_W = (R_0R_1)^2\varphi_W\,. && (3) \end{aligned}$$

Let W denote a word in R_0, R_1, R_2 and $\omega g \in \Omega_{Wal(\mathcal{H})}$ be any flag ($g \in \Delta$). We already know that the valency of the hyperedge containing ωg is 2 ($Wal(\mathcal{H})$ is a map) and that the valency of the hyperface contains ωg is even. Let l' and n' be the valencies of the hypervertex and the hyperface containing ωg, respectively.

(1) $g \in \Delta^{\hat{0}}$. From (1) and Lemma 8(1) we have $(R_1R_2)^k \in H^{g\varphi_W}$ if and only if $(R_1R_2)^k \in H^{g\varphi_W}\varphi_W^{-1} = (H\varphi_W^{-1})^g$; that is, according to Lemma 8(3),

$$(R_1R_2)^k \in \mathrm{Stab}(H(g\varphi_W)) \Leftrightarrow (R_1R_2)^k \in \mathrm{Stab}((H\varphi_W^{-1})g)\,. \qquad (4)$$

Analogously, from (3) we get $(R_0R_1)^k \in H^{g\varphi_W}$ if and only if $(R_0R_1)^{2k} \in H^{g\varphi_W}\varphi_W^{-1} = (H\varphi_W^{-1})^g$ that is, according to Lemma 8(3),

$$(R_0R_1)^k \in \mathrm{Stab}(H(g\varphi_W)) \Leftrightarrow (R_0R_1)^{2k} \in \mathrm{Stab}((H\varphi_W^{-1})g)\,. \qquad (5)$$

Now the uniformity of $\mathcal{H}$ implies $l' = l$ and $n' = 2n$.

(2) $g \notin \Delta^{\hat{0}}$. Since $gR_0 \in \Delta^{\hat{0}}$ we get from (2),

$$\begin{aligned} (R_0R_2)^k \in H^{(gR_0)\varphi_W} &\Leftrightarrow ((R_1R_2)^{R_0})^k \in H^{(gR_0)\varphi_W}\varphi_W^{-1} = (H\varphi_W^{-1})^{gR_0}\\ &\Leftrightarrow (R_1R_2)^k \in (H\varphi_W^{-1})^g\,; \end{aligned}$$

and from (3),

$$\begin{aligned}(R_0R_1)^k \in H^{(gR_0)\varphi_W} \quad &\Leftrightarrow \quad (R_0R_1)^{2k} \in H^{gR_0\varphi_W}\varphi_W^{-1} = (H\varphi_W^{-1})^{gR_0} \\ &\Leftrightarrow \quad (R_1R_0)^{2k} \in (H\varphi_W^{-1})^g\,.\end{aligned}$$

This implies that

$$(R_0R_2)^k \in \mathrm{Stab}(H(gR_0)\varphi_W) \Leftrightarrow (R_1R_2)^k \in \mathrm{Stab}(H\varphi_W^{-1}g), \tag{6}$$

$$(R_0R_1)^k \in \mathrm{Stab}(H(gR_0)\varphi_W) \Leftrightarrow (R_1R_0)^{2k} \in \mathrm{Stab}(H\varphi_W^{-1}g). \tag{7}$$

Likewise, the uniformity of $\mathcal{H}$ now implies that $l' = m$ and $n' = 2n$.

Combining (1) and (2) and assuming, without loss of generality, that $l \leq m$, we find that $Wal(\mathcal{H})$ is bipartite-uniform of bipartite-type $(l, m; 2; 2n)$.

($\Leftarrow$) Let us assume that $Wal(\mathcal{H})$ is bipartite-uniform of bipartite-type $(l, m; 2; 2n)$. Being bipartite, $Wal(\mathcal{H})$ has two orbits of vertices: the "black" vertices, all with valency l (say), and the "white" vertices, all with valency m. Without loss of generality, all the flags $H\varphi_W^{-1}g$, $g \in \Delta^{\hat{0}}$, are adjacent to "black" vertices while all the flags $H\varphi_W^{-1}gR_0$, $g \in \Delta^{\hat{0}}$, are adjacent to "white" vertices. As seen before, the equivalence (1) for $g \in \Delta^{\hat{0}}$ gives rise to the equivalence (4), which expresses the fact that all the hypervertices of $\mathcal{H}$ have the same valency l; the equivalence (2) for $g \notin \Delta^{\hat{0}}$ gives rise to the equivalence (6), which says that all the hyperedges of $\mathcal{H}$ have the same valency m; finally, the equivalence (3) gives rise to the equivalence (5) if $g \in \Delta^{\hat{0}}$ or the equivalence (7) if $g \notin \Delta^{\hat{0}}$, and they express the fact that all the hyperfaces of $\mathcal{H}$ have the same valency n. Hence $\mathcal{H}$ is uniform of type $(l; m; n)$ (or $(m; l; n)$ since the positional order of l and m in the bipartite-type of $Wal(\mathcal{H})$ is ordered by increasing value).

(10.2) $\mathcal{H}$ is regular $\Leftrightarrow H \lhd \Delta \Leftrightarrow H\varphi_W^{-1} \lhd \Delta^{\hat{0}} \Leftrightarrow Wal(\mathcal{H})$ is bipartite-regular since φ_W is an epimorphism. $\square$

Theorem 11. *$\mathcal{H}$ is a bipartite map if and only if $\mathcal{H} \cong Wal(\mathcal{G})$ for some hypermap $\mathcal{G}$.*

Proof. Only the necessary condition needs to be proved. If $\mathcal{H}$ is a bipartite map, then $H \subseteq \Delta^{\hat{0}}$. Since $\mathcal{H}$ is a map, $((R_0R_2)^2)^g \in H$ for all $g \in \Delta$; therefore $\ker\varphi_W = \langle (R_0R_2)^2 \rangle^{\Delta^{\hat{0}}} \subseteq H$. This implies that $H\varphi_W\varphi_W^{-1} = H\ker\varphi_W = H$ and hence $\mathcal{H} \cong Wal(\mathcal{G})$ where $\mathcal{G}$ is a hypermap with hypermap subgroup $G = H\varphi_W$. $\square$

Theorem 12 (Properties of φ_P). *Let $\mathcal{H}$ be a hypermap. Then,*

1. *$Pin(\mathcal{H})$ is a bipartite hypermap such that all hypervertices in one $\Delta^{\hat{0}}$-orbit have valency 1;*

2. *$\mathcal{H}$ is uniform of type $(l; m; n)$ if and only if $Pin(\mathcal{H})$ is bipartite-uniform of bipartite-type $(1, l; 2m; 2n)$;*

3. *$\mathcal{H}$ is regular if and only if $Pin(\mathcal{H})$ is bipartite-regular.*

Proof. Let H be a hypermap subgroup of $\mathcal{H}$. Then $H\varphi_{_P}^{-1}$ is a hypermap subgroup of $Pin(\mathcal{H})$.

(1) $Pin(\mathcal{H})$ is bipartite since $H\varphi_{_P}^{-1} \subseteq \Delta^{\hat{0}}$. We have $(R_1R_2)^{R_0}\varphi_{_P} = ({R_1}^{R_0}{R_2}^{R_0})\varphi_{_P} = 1$; therefore, by Lemma 2 (2), $R_1R_2 \in \mathrm{Stab}(H\varphi_{_P}^{-1}g)$ for all $g \notin \Delta^{\hat{0}}$, i.e, all hypervertices in the same $\Delta^{\hat{0}}$-orbit of the hypervertex containing the flag $H\varphi_{_P}^{-1}R_0$ have valency 1.

(2) Let us suppose that $\mathcal{H}$ is uniform of type $(l;m;n)$. We proceed similarly as for $\varphi_{_W}$, keeping in mind that all hypervertices of $Pin(\mathcal{H})$ adjacent to flags $H\varphi_{_P}^{-1}g$, for $g \notin \Delta^{\hat{0}}$, have valency 1. Starting from the equalities,

$$\begin{aligned}
R_1R_2 &= (R_1R_2)\varphi_{_P}\,, \\
R_0R_2 &= ({R_2}^{R_0}R_2)\varphi_{_P} = (R_0R_2)^2\varphi_{_P}\,, \\
R_0R_1 &= ({R_1}^{R_0}R_1)\varphi_{_P} = (R_0R_1)^2\varphi_{_P}\,.
\end{aligned}$$

one gets the following equivalences,

$$\begin{aligned}
&(R_1R_2)^k \in \mathrm{Stab}(Hg\varphi_{_P}) \Leftrightarrow (R_1R_2)^k \in \mathrm{Stab}(H\varphi_{_P}^{-1}g),\ \forall\, g \in \Delta^{\hat{0}}\,, \\
&(R_0R_2)^k \in \mathrm{Stab}(Hg\varphi_{_P}) \Leftrightarrow (R_0R_2)^{2k} \in \mathrm{Stab}(H\varphi_{_P}^{-1}g),\ \forall\, g \in \Delta^{\hat{0}}\,, \\
&(R_0R_2)^k \in \mathrm{Stab}(H(gR_0)\varphi_{_P}) \Leftrightarrow (R_2R_0)^{2k} \in \mathrm{Stab}(H\varphi_{_P}^{-1}g),\ \forall\, g \notin \Delta^{\hat{0}}\,, \\
&(R_0R_1)^k \in \mathrm{Stab}(Hg\varphi_{_P}) \Leftrightarrow (R_0R_1)^{2k} \in \mathrm{Stab}(H\varphi_{_P}^{-1}g),\ \forall\, g \in \Delta^{\hat{0}}\,, \\
&(R_0R_1)^k \in \mathrm{Stab}(H(gR_0)\varphi_{_P}) \Leftrightarrow (R_1R_0)^{2k} \in \mathrm{Stab}(H\varphi_{_P}^{-1}g),\ \forall\, g \notin \Delta^{\hat{0}}\,.
\end{aligned}$$

This clearly shows that $Pin(\mathcal{H})$ is bipartite-uniform of bipartite-type $(1,l;2m;2n)$. Reciprocally, if $Pin(\mathcal{H})$ is bipartite-uniform of bipartite-type $(1,l;2m;2n)$ then, reversing the above argument in a similar way as we did for $Wal(\mathcal{H})$ in the proof of Theorem 10, we easily conclude that $\mathcal{H}$ is uniform of type $(l;m;n)$.

(3) Since $\varphi_{_P}$ is an epimorphism, $\mathcal{H}$ is regular $\Leftrightarrow$ $H \lhd \Delta$ $\Leftrightarrow$ $H\varphi_{_P}^{-1} \lhd \Delta^{\hat{0}}$ $\Leftrightarrow$ $Pin(\mathcal{H})$ is bipartite-regular. $\square$

Theorem 13. *If $\mathcal{H}$ is a bipartite hypermap such that all hypervertices in one $\Delta^{\hat{0}}$-orbit have valency 1, then $\mathcal{H} \cong Pin(\mathcal{G})$ for some hypermap $\mathcal{G}$.*

Proof. As in Theorem 13, only the necessary condition needs to be proved. Let H be a hypermap subgroup of $\mathcal{H}$. By taking H^{R_0} instead of H if necessary, we may assume, without loss of generality, that all hypervertices in the $\Delta^{\hat{0}}$-orbit of the hypervertex that contains the flag HR_0 have valency 1, i.e, $R_1R_2 \in H^{R_0g}$ for all $g \in \Delta^{\hat{0}}$. Then $((R_1R_2)^{R_0})^h \in H$ for all $h \in \Delta^{\hat{0}}$; therefore $\ker\varphi_{_P} = \langle (R_1R_2)^{R_0}\rangle^{\Delta^{\hat{0}}} \subseteq H$. This implies that $H\varphi_{_P}\varphi_{_P}^{-1} = H\ker\varphi_{_P} = H$ and hence $\mathcal{H} \cong Pin(\mathcal{G})$, where $\mathcal{G}$ is the hypermap with hypermap subgroup $G = H\varphi_{_P}$. $\square$

Theorem 14. *$Wal(\mathcal{D}_{(0\,1)}(\mathcal{H})) \cong Wal(\mathcal{H})$.*

Proof. If H is a hypermap subgroup of $\mathcal{H}$, then $H{\varphi_{_W}}^{-1}$ and $H(0\,1)^{\circ}{\varphi_{_W}}^{-1}$ are hypermap subgroups of $Wal(\mathcal{H})$ and $Wal(\mathcal{D}_{(0\,1)}(\mathcal{H}))$, respectively. Since $g\varphi_{_W}\sigma = g\iota^{R_0}\varphi_{_W}$ for all

$g \in \Delta^{\hat{0}}$, where $\sigma = (0\,1)^{\circ}$ and ι^{R_0} is the automorphism given by conjugation by R_0, we have

$$H\sigma\varphi_W^{-1} = H\varphi_W^{-1}\iota^{R_0}, \tag{8}$$

that is, the hypermap subgroup $H(0\,1)^{\circ}\varphi_W^{-1}$ of $Wal(\mathcal{D}_{(0\,1)}(\mathcal{H}))$ is just a conjugate under R_0 of the hypermap subgroup of $Wal(\mathcal{H})$ and so they are isomorphic. $\square$

Theorem 15. *$Pin(\mathcal{D}_{(1\,2)}(\mathcal{H})) = \mathcal{D}_{(1\,2)}(Pin(\mathcal{H}))$.*

Proof. Let H be a hypermap subgroup of $\mathcal{H}$ and $\sigma = (1\,2)^{\circ}$. Then $H\sigma\varphi_P^{-1}$ and $H\varphi_P^{-1}\sigma$ are hypermap subgroups of $Pin(\mathcal{D}_{(1\,2)}(\mathcal{H}))$ and $\mathcal{D}_{(1\,2)}(Pin(\mathcal{H}))$, respectively. The equality $\sigma|_{\Delta^{\hat{0}}}\varphi_P = \varphi_P\sigma$ actually shows that

$$H\sigma\varphi_P^{-1} = H\varphi_P^{-1}\sigma\,; \tag{9}$$

so they represent the same hypermap. $\square$

Theorem 16. *If $Wal(\mathcal{H}) \cong Wal(\mathcal{G})$, then $\mathcal{H} \cong \mathcal{G}$ or $\mathcal{H} \cong \mathcal{D}_{(01)}(\mathcal{G})$.*

Proof. If $Wal(\mathcal{H}) \cong Wal(\mathcal{G})$ then $H\varphi_W^{-1} = (G\varphi_W^{-1})^g$ for some $g \in \Delta$.

(i) $g \in \Delta^{\hat{0}}$. Then $(G\varphi_W^{-1})^g = G^{g\varphi_W}\varphi_W^{-1}$, by Lemma 8(1), and then we have

$$H = H\varphi_W^{-1}\varphi_W = G^{g\varphi_W}\varphi_W^{-1}\varphi_W = G^{g\varphi_W}\,;$$

that is, $\mathcal{H} \cong \mathcal{G}$.

(ii) $g \notin \Delta^{\hat{0}}$. Then $gR_0 \in \Delta^{\hat{0}}$ and

$$(G\varphi_W^{-1})^g = \left((G\varphi_W^{-1})^{gR_0}\right)^{R_0} = \left(G^{(gR_0)\varphi_W}\varphi_W^{-1}\right)^{R_0} = G^{(gR_0)\varphi_W}\sigma\varphi_W^{-1}\,,$$

using (8), where $\lambda = \iota^{R_0}$ and $\sigma = (0\,1)^{\circ}$. Therefore

$$H = H\varphi_W^{-1}\varphi_W = G^{(gR_0)\varphi_W}\sigma\varphi_W^{-1}\varphi_W = G^{(gR_0)\varphi_W}\sigma,$$

which says that $\mathcal{H} \cong D_{\sigma}(\mathcal{G})$. $\square$

Theorem 17. *If $Pin(\mathcal{H}) \cong Pin(\mathcal{G})$, then $\mathcal{H} \cong \mathcal{G}$.*

Proof. As before, let H and G be hypermap-subgroups of $\mathcal{H}$ and $\mathcal{G}$. If $Pin(\mathcal{H}) \cong Pin(\mathcal{G})$ then $H\varphi_P^{-1} = (G\varphi_P^{-1})^g$ for some $g \in \Delta$.

(i) If $g \in \Delta^{\hat{0}}$ then, as before, $(G\varphi_P^{-1})^g = G^{g\varphi_P}\varphi_P^{-1}$ and then $H = G^{g\varphi_P}$, showing that $\mathcal{H} \cong \mathcal{G}$.

(ii) Suppose that $g \notin \Delta^{\hat{0}}$. As for $b \in \Delta^{\hat{0}}$, $(R_1R_2)^{R_0b}\varphi_P = 1 \in H \cap G$ so that $(R_1R_2)^{R_0b}$ belongs to both $H\varphi_P^{-1}$ and $G\varphi_P^{-1}$, for all $b \in \Delta^{\hat{0}}$. Then (1) $R_1R_2 \in (H\varphi_P^{-1})^{b^{-1}R_0}$ and (2) since $(R_1R_2)^{R_0bg} \in (G\varphi_P^{-1})^g = H\varphi_P^{-1}$, $R_1R_2 \in (H\varphi_P^{-1})^{g^{-1}b^{-1}R_0}$. Since $b^{-1}R_0$ runs all over $\Delta\backslash\Delta^{\hat{0}}$ and $g^{-1}b^{-1}R_0$ runs all over $\Delta^{\hat{0}}$, when $b \in \Delta^{\hat{0}}$, then $R_1R_2 \in (H\varphi_P^{-1})^d$, for all $d \in \Delta$. This implies that all the hypervertices of $Pin(\mathcal{H})$ have valency 1. By a dual version of Lemma 7, $Pin(\mathcal{H})$ is a "star"-like hypermap (see Figure 2);

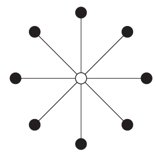


Figure 2: $Pin(\mathcal{H}) = D_{(0\,2)}(\mathcal{D}_n)$.

that is, $Pin(\mathcal{H}) = D_{(0\,2)}(\mathcal{D}_n)$. Hence $Pin(\mathcal{H})$ is a regular hypermap on the sphere with n (even) hypervertices. Thus $H\varphi_P^{-1}$, as well as $(G\varphi_P^{-1})^g$, is normal in Δ. Therefore, $H\varphi_P^{-1} = G\varphi_P^{-1}$ and hence $H = G$. $\square$

The proof of the above theorem reveals the following information,

Lemma 18. *If $Pin(\mathcal{H})$ is not isomorphic to $D_{(0\,2)}(\mathcal{D}_n)$ for any even n, then $Pin(\mathcal{H}) \cong Pin(\mathcal{G})$ implies that $H\varphi_P^{-1} = (G\varphi_P^{-1})^g$ for some $g \in \Delta^{\hat{0}}$.*

2.1 Euler formula for bipartite-uniform hypermaps

In this subsection we write the Euler characteristic of a bipartite-uniform hypermap in terms of its bipartite-type. Let $\mathcal{H} = (\Omega_{\mathcal{H}}; h_0, h_1, h_2)$ be a bipartite-uniform hypermap with Euler characteristic χ, let V, E and F be the numbers of hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, respectively, and let V_1 and $V_2 = V - V_1$ be the numbers of hypervertices of the two $\Delta^{\hat{0}}$-orbits in $\Omega_{\mathcal{H}}$. By Lemma 3, $\chi = V_1 + V_2 + E + F - \frac{|\Omega_{\mathcal{H}}|}{2}$. Let $(l_1, l_2; m; n)$ be the bipartite-type of $\mathcal{H}$. Then $V_1 = \frac{|\Omega_{\mathcal{H}}|}{4l_1}$, $V_2 = \frac{|\Omega_{\mathcal{H}}|}{4l_2}$, $E = \frac{|\Omega_{\mathcal{H}}|}{2m}$ and $F = \frac{|\Omega_{\mathcal{H}}|}{2n}$. Replacing these values in the above formula we get the following result:

Lemma 19 (Euler formula for bipartite-uniform hypermaps). *If $\mathcal{H}$ is a bipartite-uniform hypermap of bipartite-type $(l_1, l_2; m; n)$, then*

$$\chi = \frac{|\Omega_{\mathcal{H}}|}{2}\left(\frac{1}{2l_1} + \frac{1}{2l_2} + \frac{1}{m} + \frac{1}{n} - 1\right).$$

2.2 Spherical bipartite-uniform hypermaps

In this subsection we classify the bipartite-uniform hypermaps $\mathcal{K}$ on the sphere. The main results were already given before; all we need now is to apply them directly to the sphere ($\chi = 2$).

Let $\mathcal{K}$ be a bipartite-uniform hypermap of bipartite-type $(l_1, l_2; m; n)$ on the sphere. Then $\chi = 2 > 0$ and $\frac{1}{2l_1} + \frac{1}{2l_2} + \frac{1}{m} + \frac{1}{n} > 1$. Suppose, without loss of generality, that $l_1 \le l_2$ and $m \le n$. Then

$$\begin{aligned}
\frac{1}{l_1} + \frac{2}{m} \ge \frac{1}{2l_1} + \frac{1}{2l_2} + \frac{1}{m} + \frac{1}{n} > 1 \quad &\Rightarrow \quad \frac{1}{l_1} > \frac{1}{2} \quad \text{or} \quad \frac{2}{m} > \frac{1}{2} \\
&\Leftrightarrow \quad l_1 < 2 \quad \text{or} \quad m < 4 \\
&\Leftrightarrow \quad l_1 = 1 \quad \text{or} \quad m = 2 \\
&\qquad \text{(since } m \text{ is even)}
\end{aligned}$$

From this result and Theorems 11 and 13, we deduce the following theorem.

Theorem 20. *If $\mathcal{K}$ is a spherical bipartite-uniform hypermap, then $\mathcal{K} \cong Wal(\mathcal{R})$ or $\mathcal{K} \cong Pin(\mathcal{R})$ for some spherical uniform hypermap $\mathcal{R}$, unique up to isomorphism. Moreover, as $\mathcal{K}$ is bipartite-regular if and only if $\mathcal{R}$ is regular, and on the sphere all uniform hypermaps are regular, then all bipartite-uniform hypermaps on the sphere are bipartite-regular.*

#	l_1	l_2	m	n	V_1	V_2	E	F	$\|\Omega\|$	$\mathcal{K}$
1	1	1	$2n$	$2n$	n	n	1	1	$4n$	$Pin(\mathcal{D}_{(02)}(\mathcal{D}_n))$
2	1	2	4	$2n$	$2n$	n	n	2	$8n$	$Pin(\mathcal{P}_n)$
3	1	2	6	6	12	6	4	4	48	$Pin(\mathcal{D}_{(01)}(\mathcal{T}))$
4	1	2	6	8	24	12	8	6	96	$Pin(\mathcal{D}_{(01)}(\mathcal{C}))$
5	1	2	6	10	60	30	20	12	240	$Pin(\mathcal{D}_{(01)}(\mathcal{D}))$
6	1	3	4	6	12	4	6	4	48	$Pin(\mathcal{T})$
7	1	3	4	8	24	8	12	6	96	$Pin(\mathcal{C})$
8	1	3	4	10	60	20	30	12	240	$Pin(\mathcal{D})$
9	1	4	4	6	24	6	12	8	96	$Pin(\mathcal{D}_{(02)}(\mathcal{C}))$
10	1	5	4	6	60	12	30	20	240	$Pin(\mathcal{D}_{(02)}(\mathcal{D}))$
11	1	n	2	$2n$	n	1	n	1	$4n$	$Pin(\mathcal{D}_{(12)}(\mathcal{D}_n))$
12	1	n	4	4	$2n$	2	n	n	$8n$	$Pin(\mathcal{D}_{(02)}(\mathcal{P}_n))$
13	2	2	2	$2n$	n	n	$2n$	2	$8n$	$Wal(\mathcal{P}_n)$
14	2	3	2	6	6	4	12	4	48	$Wal(\mathcal{T})$
15	2	3	2	8	12	8	24	6	96	$Wal(\mathcal{C})$
16	2	3	2	10	30	20	60	12	240	$Wal(\mathcal{D})$
17	2	4	2	6	12	6	24	8	96	$Wal(\mathcal{D}_{(02)}(\mathcal{C}))$
18	2	5	2	6	30	12	60	20	240	$Wal(\mathcal{D}_{(02)}(\mathcal{D}))$
19	2	n	2	4	n	2	$2n$	n	$8n$	$Wal(\mathcal{D}_{(02)}(\mathcal{P}_n))$
20	3	3	2	4	4	4	12	6	48	$Wal(\mathcal{D}_{(12)}(\mathcal{T}))$
21	3	4	2	4	8	6	24	12	96	$Wal(\mathcal{D}_{(12)}(\mathcal{C}))$
22	3	5	2	4	20	12	60	30	240	$Wal(\mathcal{D}_{(12)}(\mathcal{D}))$
23	n	n	2	2	1	1	n	n	$4n$	$Wal(\mathcal{D}_n)$

Table 2: The bipartite-regular hypermaps on the sphere.

Based on the knowledge of regular hypermaps on the sphere, we display in Table 2 all the possible values (up to duality) for the bipartite-type of the bipartite-regular hypermaps on the sphere and the unique hypermap (up to isomorphism) with such a bipartite-type. Notice that the map of bipartite-type $(1, n; 2; 2n)$ can be constructed from $\mathcal{D}_n$ either via a Wal transformation $Wal(\mathcal{D}_{(02)}(\mathcal{D}_n))$ or via a Pin transformation $Pin(\mathcal{D}_{(12)}(\mathcal{D}_n))$. Since $Wal(\mathcal{D}_{(02)}(\mathcal{D}_n)) \cong Wal(\mathcal{D}_{(12)}(\mathcal{D}_n))$ these two constructions (Wal and Pin) can actually be carried forward on the same hypermap $\mathcal{D}_{(12)}(\mathcal{D}_n)$. The Tetrahedron $\mathcal{R} = \mathcal{T}$, which is self-dual, gives rise to $Wal(\mathcal{D}_{(0\,1)}(\mathcal{T})) = Wal(\mathcal{T}) = Wal(\mathcal{D}_{(02)}(\mathcal{T}))$.

3 Irregularity and chirality

We follow the same terminology and notations used in [3]. Let $\mathcal{K}$ be a bipartite (that is, $\Delta^{\hat{0}}$-conservative) hypermap with hypermap-subgroup $K < \Delta^{\hat{0}}$. If $\mathcal{K}$ is not regular (that is, not Δ-regular), then its closure cover $\mathcal{K}^{\Delta}$ is the largest regular hypermap covered by $\mathcal{K}$ and its covering core $\mathcal{K}_{\Delta}$ is the smallest regular hypermap covering $\mathcal{K}$. Hence we have two normal subgroups in Δ, the normal closure K^{Δ} containing K, and the core K_{Δ} contained in K. Since $K_{\Delta} \lhd K$, although K may not be normal in K^{Δ}, we have a group

$$\Upsilon_{\Delta}(\mathcal{K}) = K/K_{\Delta}$$

called the *lower-irregularity group* of $\mathcal{K}$. Its size is the *lower-irregularity index* and is denoted by $\iota_{\Delta}(\mathcal{K})$. The *upper-irregularity index*, denoted by $\iota^{\Delta}(\mathcal{K})$, is the index $|K^{\Delta} : K|$. If $\mathcal{K}$ is bipartite-regular, then $K \lhd \Delta^{\hat{0}}$, and since K^{Δ} is a subgroup of $\Delta^{\hat{0}}$, $K \lhd K^{\Delta}$ and we have another group, the *upper-irregularity group*

$$\Upsilon^{\Delta}(\mathcal{K}) = K^{\Delta}/K.$$

Since the index of $\Delta^{\hat{0}}$ in Δ is 2, the upper- and lower-irregularity groups are isomorphic; so their upper- and lower-irregularity indices are equal ($\mathcal{K}$ is irregularity balanced). The common group $\Upsilon^{\Delta}(\mathcal{K}) \cong \Upsilon_{\Delta}(\mathcal{K}) = \Upsilon$ is the *irregularity group* of the bipartite-regular hypermap $\mathcal{K}$ and the common value $\iota^{\Delta}(\mathcal{K}) = \iota_{\Delta}(\mathcal{K}) = \iota$ is its *irregularity index*. This has value 1 if and only if $\mathcal{K}$ is regular. Being bipartite-regular, $\mathcal{K}$ is isomorphic to a regular $\Delta^{\hat{0}}$-marked hypermap (see [1])

$$\mathcal{Q} = (G, a, b, c, d) \cong (\Delta^{\hat{0}}/K, KA, KB, KC, KD)\,,$$

where $\Delta^{\hat{0}} = \langle A, B, C, D\rangle \cong C_2 * C_2 * C_2 * C_2$ and K is the $\Delta^{\hat{0}}$-hypermap subgroup of $\mathcal{Q}$ (and the hypermap subgroup of $\mathcal{K}$). Here G is the group generated by a, b, c, d. To compute the irregularity group of $\mathcal{K}$ we use:

Lemma 21. *If G has presentation $\langle a, b, c, d \mid R = 1\rangle$, where $R = \{R_1, \dots, R_k\}$ is a set of relators $R_i = R_i(a, b, c, d)$ then $\Upsilon^{\Delta}(\mathcal{K}) = \langle R^{R_0}\rangle^G$.*

See [3] for the proof.

The definition of chirality given in [2] is slightly different from that used in [6, 7, 8, 9]. If $\mathcal{K}$ is bipartite ($K < \Delta^{\hat{0}}$), not necessarily bipartite-regular, then $\mathcal{K}$ is $\Delta^{\hat{0}}$-*chiral*, or bipartite-chiral, if the normaliser $N_{\Delta}(K)$ of K in Δ is a subgroup of $\Delta^{\hat{0}}$. In other words, $\mathcal{K}$ is $\Delta^{\hat{0}}$-chiral if the group of automorphisms $Aut(\mathcal{K}) \cong N_{\Delta}(K)/K$ contains no "symmetry" besides $\Delta^{\hat{0}}$.

Let $\mathcal{K}$ be a $\Delta^{\hat{0}}$-chiral hypermap. If $\mathcal{K}$ is bipartite-regular ($\Delta^{\hat{0}}$-regular), then $K \lhd \Delta^{\hat{0}}$ and so we have $N_{\Delta}(K) = \Delta^{\hat{0}}$. Thus $\mathcal{K}$ is $\Delta^{\hat{0}}$-chiral if and only if K is not normal in Δ; that is, if and only if $\mathcal{K}$ is irregular. As $\Delta^{\hat{0}}$ has index 2 in Δ, with transversal $\{1, R_0\}$, we have $K = K^{\langle R_0\rangle} = KK^{R_0} = K^{\Delta}$ if and only if $R_0 \in N_{\Delta}(K)$; that is, if and only if $KR_0 \in Aut(\mathcal{K})$. Hence the upper-irregularity index ι^{Δ} gives a "measure" of "how close" $\mathcal{K}$ is to having the "symmetry" KR_0 outside $\Delta^{\hat{0}}$. For this reason we also call the upper-irregularity index (which coincides with the lower-irregularity index) the $\Delta^{\hat{0}}$-*chirality index* of the bipartite-regular $\mathcal{K}$. This expresses how "close" $\mathcal{K}$ is to getting a "symmetry" outside $\Delta^{\hat{0}}$, or in other words, how close it is to losing $\Delta^{\hat{0}}$-chirality.

The same happens to any normal subgroup Θ with index two in Δ. In particular, for $\Theta = \Delta^+$, the upper irregularity index (or simply the irregularity index) of a Δ^+-regular (that is, orientably regular) hypermap coincides with the Δ^+-chirality index. This explains the use of chirality index in place of irregularity index (of orientably regular hypermaps) in the papers [6, 7, 8, 9]. For more information and a general definition of chirality group see [2].

$\mathcal{K}$	$\mathcal{K}^\Delta$	l,m,n	$\lvert\Omega\rvert$	$\mathcal{K}_\Delta$	l,m,n	$\lvert\Omega\rvert$	genus	ι	Υ
$Pin(\mathcal{D}_{(02)}(\mathcal{D}_n))$	$\mathcal{D}_{(02)}(\mathcal{D}_{2n})$	$1,2n,2n$	$4n$	$\mathcal{D}_{(02)}(\mathcal{D}_{2n})$	$1,2n,2n$	$4n$	0	1	1
$Pin(\mathcal{P}_n)$	$\mathcal{D}_{(02)}(\mathcal{D}_4)$	$1,4,4$	8	$\mathcal{K}_{\Delta 2}$	$2,4,2n$	$8n^2$	$\frac{(n-1)^2+1}{2}$	n	$D_{\frac{n}{2}}$, n even
	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 3}$	$2,4,2n$	$16n^2$	$(n-1)^2$	$2n$	D_n, n odd
$Pin(\mathcal{D}_{(01)}(\mathcal{T}))$	$\mathcal{D}_{(02)}(\mathcal{D}_6)$	$1,6,6$	12	$\mathcal{K}_{\Delta 4}$	$2,6,6$	192	9	4	V_4
$Pin(\mathcal{D}_{(01)}(\mathcal{C}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 5}$	$2,6,8$	2304	121	24	S_4
$Pin(\mathcal{D}_{(01)}(\mathcal{D}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 6}$	$2,6,10$	14400	841	60	A_5
$Pin(\mathcal{T})$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 7}$	$3,4,6$	576	37	12	A_4
$Pin(\mathcal{C})$	$\mathcal{D}_{(02)}(\mathcal{D}_4)$	$1,4,4$	8	$\mathcal{K}_{\Delta 8}$	$3,4,8$	1152	85	12	A_4
$Pin(\mathcal{D})$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 9}$	$3,4,10$	14400	1141	60	A_5
$Pin(\mathcal{D}_{(02)}(\mathcal{C}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 10}$	$4,4,6$	2304	193	24	S_4
$Pin(\mathcal{D}_{(02)}(\mathcal{D}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 11}$	$5,4,6$	14400	1381	60	A_5
$Pin(\mathcal{D}_{(12)}(\mathcal{D}_n))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 12}$	$n,2,2n$	$4n^2$	$\frac{(n-1)(n-2)}{2}$	n	C_n
$Pin(\mathcal{D}_{(02)}(\mathcal{P}_n))$	$\mathcal{D}_{(02)}(\mathcal{D}_4)$	$1,4,4$	8	$\mathcal{K}_{\Delta 13}$	$n,4,4$	$8n^2$	$(n-1)^2$	n	C_n
$Wal(\mathcal{P}_n)$	$\mathcal{P}_{2n}$	$2,2,2n$	$8n$	$\mathcal{P}_{2n}$	$2,2,2n$	$8n$	0	1	1
$Wal(\mathcal{T})$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 15}$	$6,2,6$	576	25	12	A_4
$Wal(\mathcal{C})$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 16}$	$6,2,8$	2304	121	24	S_4
$Wal(\mathcal{D})$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 17}$	$6,2,10$	14400	841	60	A_5
$Wal(\mathcal{D}_{(02)}(\mathcal{C}))$	$\mathcal{P}_6$	$2,2,6$	24	$\mathcal{K}_{\Delta 18}$	$4,2,6$	384	9	4	V_4
$Wal(\mathcal{D}_{(02)}(\mathcal{D}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 19}$	$10,2,6$	14400	841	60	A_5
$Wal(\mathcal{D}_{(02)}(\mathcal{P}_n))$	$\mathcal{P}_4$	$2,2,4$	16	$\mathcal{K}_{\Delta 20}$	$n,2,4$	$4n^2$	$\frac{(n-2)^2}{4}$	$\frac{n}{2}$	$C_{\frac{n}{2}}$, n even
	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 21}$	$2n,2,4$	$16n^2$	$(n-1)^2$	$2n$	D_n, n odd
$Wal(\mathcal{D}_{(12)}(\mathcal{T}))$	$\mathcal{C}$	$3,2,4$	48	$\mathcal{C}$	$3,2,4$	48	0	1	1
$Wal(\mathcal{D}_{(12)}(\mathcal{C}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 23}$	$12,2,4$	2304	97	24	S_4
$Wal(\mathcal{D}_{(12)}(\mathcal{D}))$	$\mathcal{D}_{(02)}(\mathcal{D}_2)$	$1,2,2$	4	$\mathcal{K}_{\Delta 24}$	$15,2,4$	14400	661	60	A_5
$Wal(\mathcal{D}_n)$	$\mathcal{D}_{(02)}(\mathcal{P}_n)$	$n,2,2$	$4n$	$\mathcal{D}_{(02)}(\mathcal{P}_n)$	$n,2,2$	$4n$	0	1	1

Table 3: $\mathcal{K}$, $\mathcal{K}^\Delta$ and $\mathcal{K}_\Delta$.

Computing the irregularity group Υ.

Let $\mathcal{K} = \mathcal{H}^{\varphi^{-1}} = Wal(\mathcal{H})$ or $Pin(\mathcal{H})$ conform $\varphi = \varphi_W$ or φ_P, respectively, and let H be the hypermap subgroup of a regular hypermap $\mathcal{H}$ of type $(l;m;n)$. The inverse image $K = H\varphi^{-1}$ is the hypermap subgroup of $\mathcal{K}$. The lower-irregularity index of $\mathcal{K}$, $\Upsilon_\Delta(\mathcal{K}) = K/K_\Delta$, is isomorphic to its upper-irregularity group $\Upsilon^\Delta(\mathcal{K}) = K^\Delta/K$, a subgroup of the $\Delta^{\hat{0}}$-monodromy group $G = \Delta^{\hat{0}}/K$ of $\mathcal{K}$. This common group, the irregularity group Υ, can be computed in the following way. According to Theorem 9, the group $G \cong \mathrm{Mon}(\mathcal{H})$ is a known group (see Table 1). Being G the $\Delta^{\hat{0}}$-monodromy group of a bipartite-regular hypermap, using φ we can rewrite G in the following form

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, R = 1 \rangle,$$

such that $a^{R_0} = c$, $b^{R_0} = d$, $c^{R_0} = a$ and $d^{R_0} = b$; R stands for a set of relators on a, b, c, d. By Lemma 21,

$$\Upsilon = \langle R^{R_0} \rangle^G$$

is the closure subgroup of R^{R_0} in G. This calculation is easily performed and the results for $\Upsilon(\mathcal{K})$ can be seen in the last column of Table 3. For an example of how this calculation is carried out see Theorem 23.

However, since $\varphi = \varphi_W$ or φ_P sends generators of $\Delta^{\hat{0}}$ of odd length in Δ to generators of Δ of odd length in Δ, we have necessarily $\Delta^+\varphi^{-1} = \Delta^{+\hat{0}}$, where $\Delta^{+\hat{0}} = \Delta^+ \cap \Delta^{\hat{0}}$. Since $H \lhd \Delta^+$, then $K^{\Delta}/K \lhd \Delta^{+\hat{0}}/K = \Delta^+\varphi^{-1}/H\varphi^{-1} \cong \Delta^+/H = Aut^+(\mathcal{H})$; that is, $\Upsilon = K^{\Delta}/K$ is a normal subgroup of $Aut^+(\mathcal{H})$.

Let $A = R_1$, $B = R_2$, $C = R_1^{R_0}$ and $D = R_2^{R_0}$. Then $\Delta^{\hat{0}} = \langle A, B, C, D\rangle$.

(1) If $\mathcal{K} = Wal(\mathcal{H})$ then $K = \langle BD, (AB)^l, (DC)^m, (CA)^n\rangle^{\Delta^{\hat{0}}}$ so $K^{\Delta} = \langle BD, (AB)^l, (DC)^m, (CA)^n\rangle^{\Delta}$. Let $d = \gcd(l, m)$. Since $(AB)^m = ((DC)^{-m})^{R_0}$ and $(DC)^l = ((AB)^{-l})^{R_0}$ then $(AB)^d$ and $(DC)^d$ also belong to K^{Δ}. Hence if $d = 1$, then AB and DC belong to K^{Δ} and so $K^{\Delta} = \Delta^{+\hat{0}}$. Therefore $\Upsilon = Aut^+(\mathcal{H})$ when $d = 1$.

(2) If $\mathcal{K} = Pin(\mathcal{H})$, then $K = \langle CD, (AB)^l, (BD)^m, (CA)^n\rangle^{\Delta^{\hat{0}}}$; so $K^{\Delta} = \langle CD, (AB)^l, (BD)^m, (CA)^n\rangle^{\Delta}$. Let $d = \gcd(m, n)$. Since $K^{\Delta}D = K^{\Delta}C$, $K^{\Delta}(CA)^m = K^{\Delta}(DA)^m = (K^{\Delta}(BC)^m)^{R_0} = (K^{\Delta}(BD)^m)^{R_0} = K^{\Delta}$ and so $(CA)^m \in K^{\Delta}$. Similarly, $(BD)^n \in K^{\Delta}$. Hence if $d = 1$, then $K^{\Delta} = \Delta^{+\hat{0}}$ and consequently $\Upsilon = Aut^+(\mathcal{H})$.

Therefore the general calculations mentioned above only need to be carried out for the cases where $d \neq 1$, namely the cases 2 (for n even), 3, 7, 12, 17 and 19 (for n even).

Computing the closure cover $\mathcal{K}^{\Delta}$.

Once the irregularity index is calculated, it is an easy task to compute the closure cover $\mathcal{K}^{\Delta}$ of $\mathcal{K} = \mathcal{H}^{\varphi^{-1}}$, simply because the genus of the closure cover is zero and in the sphere the type determines uniquely a uniform (or regular) hypermap. Let $(l; m; n)$ be the type of the closure cover $\mathcal{K}^{\Delta}$ and let $(r, s; u; v)$ be the bipartite-type of the spherical bipartite-regular hypermap $\mathcal{K}$. The number of flags $|\Omega_{\mathcal{K}^{\Delta}}|$ of the closure cover must divide the number of flags $|\Omega_{\mathcal{K}}|$ of $\mathcal{K}$. Also l divides $gcd(r, s)$, m divides u and n divides v. The greatest possible values for l, m and n are $gcd(r, s)$, u and v, respectively. Moreover, when $gcd(r, s) = 1$ we must have $l = 1$ in which case $m = n$ and the greatest possible values are achieved for $m = n = gcd(u, v)$. Since $\mathcal{K}^{\Delta}$ is a regular hypermap on the sphere and is determined by l, m and n, we must check if in each case the above choice of l, m and n give rise to a spherical type (cf. Table 1). If not we choose the second greatest, the third greatest and so forth. For each bipartite-regular hypermap $\mathcal{K}$ in Table 2, where $(l_1, l_2; m; n)$ is our $(r, s; u; v)$, taking the greatest values for the triple (l, m, n) we get a spherical type. To check if such triple determines a hypermap covered by $\mathcal{K}$ we take a half-turn in the middle of each hyperedge of $\mathcal{K}$; these half-turns determine a covering $\mathcal{K} \mapsto \mathcal{K}^{\Delta}$. The results can be seen in Table 3.

Computing the covering core $\mathcal{K}_{\Delta}$.

The covering core is already computed since we know its monodromy group

$$Mon(\mathcal{K}_{\Delta}) = Mon(\mathcal{K})$$

and their canonical generators. Feeding these parameters in GAP [12], for example, we get the rest of the information shown in the Table 3. In this table we observe two isolated maps (not in families) with less then 100 edges, the map $D_{(1\,2)}(\mathcal{K}_{\Delta 4})$ with 48 edges and Petrie path of length 4, and $\mathcal{K}_{\Delta 18}$ with 96 edges and Petrie path of length 6. In [25], where we can find a good list of regular maps up to 100 edges (although the list is guaranteed to be complete only up to 49), these maps are $P(70)$ and $DP(190)$ on pages 144 and 181 respectively. These can be consulted in the recently created Census of orientably-regular maps [27].

4 Final comments

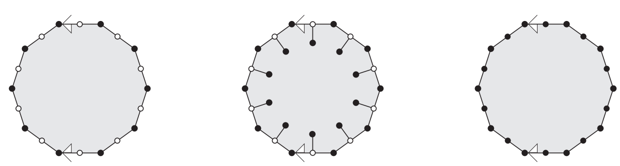
By examining Table 3 we observe the following extra result,

Theorem 22. *The irregularity (or chirality) index of a bipartite-regular hypermap can be any positive integer number. Moreover, cyclic groups and dihedral groups are irregularity groups of bipartite-regular hypermaps.*

Using the Pin and Wal transformations we can say a little more.

Theorem 23. *On each orientable surface of genus g there are $\Delta^{\hat{0}}$-chiral hypermaps (that is, irregular bipartite-regular hypermaps) with irregularity indices $2g+1$, $4g+2$ and $4g$.*

Proof. Just take the $Pin(\mathcal{M}_k)$ and the $Wal(\mathcal{M}_k)$ constructions over the one-face regular map $\mathcal{M}_k$ formed from a single $2k$-gon by identifying opposite edges orientably. The map $\mathcal{M}_k$ has type $(k;2;2k)$ or $(2k;2;2k)$ according as k is odd or even. The monodromy group of $\mathcal{M}_k$ is the dihedral group D_{2k} generated by the involutions r_0, r_1 and r_2 subject to the relations $(r_0r_1)^{2k} = 1$ and $r_2 = r_0(r_1r_0)^k$. The genus of $\mathcal{M}_k$ is $\frac{k-1}{2}$ if k is odd and $\frac{k}{2}$ otherwise. Hence each orientable surface of genus g supports two maps $\mathcal{M}_k$, one for k odd and another for k even. Note that $\mathcal{M}_k$ has 1 or 2 vertices according as k is even or odd.



(a) (b) (c)

Figure 3: (a) The $\mathcal{M}_k$ map (opposite edges identified orientably). (b) $Pin(\mathcal{M}_k)$. (c) $Wal(\mathcal{M}_k)$.

The bipartite-regular hypermap $Pin(\mathcal{M}_k)$ has bipartite-type $(1,k;4;4k)$ if k is odd and $(1,2k;4;4k)$ otherwise. The bipartite-regular map $Wal(\mathcal{M}_k)$ has type $(2,k;2;4k)$ or $(2,2k;2;4k)$ according as k is odd or even. Let H be the hypermap subgroup of $\mathcal{M}_k$ and $K = H\varphi^{-1}$, where $\varphi = \varphi_P$ or φ_W.

(1) *The hypermap $Pin(\mathcal{M}_k)$.* The epimorphism φ_P induces an isomorphism $G = \Delta^{\hat{0}}/K \to \Delta/H$, mapping $a \mapsto r_1$, $b \mapsto r_2$, $c \mapsto r_0$ and $d \mapsto r_0$. That is, $c = d$ in the $\Delta^{\hat{0}}$-monodromy

group of $Pin(\mathcal{M}_k)$. With the help of φ_P we rewrite $Mon(\mathcal{M}_k)$ in function of a, b, c and d to get the $\Delta^{\hat{0}}$-monodromy group

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, c = d, (ca)^{2k} = 1, b = c(ac)^k \rangle .$$

In this case $R = \{cd^{-1}, (ac)^{2k}, c(ac)^k b^{-1}\}$ and the irregularity group of $Pin(\mathcal{M}_k)$ is the normal closure of R^{R_0} in G; thus

$$\Upsilon = \langle ab^{-1}, (ca)^{2k}, a(ca)^k d^{-1} \rangle^G = \langle ab \rangle^G = \langle ab \rangle.$$

Since $ab = (ac)^{k+1}$, this group has size k if k is odd and size $2k$ otherwise. Hence $Pin(\mathcal{M}_k)$ has irregularity index $\iota = k = 2g+1$, for k odd, and $\iota = 2k = 4g$ for k even.

(2) *The map* $Wal(\mathcal{M}_k)$. Proceeding similarly we obtain

$$\begin{aligned} G &= \Delta^{\hat{0}} - Mon(Wal(\mathcal{M}_k)) \\ &= \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, b = d, (ca)^{2k} = 1, b = c(ac)^k \rangle \end{aligned}$$

and irregularity group $\Upsilon = \langle ac \rangle = C_{2k}$ cyclic, giving rise to irregularity indices $\iota = 2k = 4g+2$ when k is odd and $\iota = 2k = 4g$ when k is even. $\square$

For non-orientable surfaces we cannot answer affirmatively since to obtain $\Delta^{\hat{0}}$-chiral hypermaps the *Pin* and *Wal* constructions need regular hypermaps and we know that there are none on the non-orientable surfaces with negative characteristic 0, 1, 16, 22, 25, 37, and 46 [5, 28].

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