# On Mixed Codes with Covering Radius 1 and Minimum Distance 2

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**Abstract**

Let $R$, $S$ and $T$ be finite sets with $|R| = r$, $|S| = s$ and $|T| = t$. A code $C \subset R \times S \times T$ with covering radius 1 and minimum distance 2 is closely connected to a certain generalized partial Latin rectangle. We present various constructions of such codes and some lower bounds on their minimal cardinality $K(r, s, t; 2)$. These bounds turn out to be best possible in many instances. Focussing on the special case $t = s$ we determine $K(r, s, s; 2)$ when $r$ divides $s$, when $r = s - 1$, when $s$ is large, relative to $r$, when $r$ is large, relative to $s$, as well as $K(3r, 2r, 2r; 2)$. Some open problems are posed. Finally, a table with bounds on $K(r, s, s; 2)$ is given.

## 1 Introduction

Let $Q$ denote a finite alphabet with $|Q| = q \geq 2$. The Hamming distance $d(y, y')$ between $y, y' \in Q^n$ denotes the number of coordinates in which $y$ and $y'$ differ. For $y \in Q^n$ and $C \subset Q^n$ with $C \neq \emptyset$ we set $d(y, C) = \min_{x \in C} d(y, x)$. We say that $y$ is $R$-covered by $C$ if $d(y, C) \leq R$ and that $C' \subset Q^n$ is $R$-covered by $C$, if every $y \in C'$ is $R$-covered by $C$. A code $C \subset Q^n$ of length $n$ has covering radius (at most) $R$, if $Q^n$ is $R$-covered by $C$. $C$ has minimum distance (at least) $d$, when any two distinct codewords have Hamming distance at least $d$. Combinatorial coding theory deals with $A_q(n, d)$, the maximal cardinality of a code $C \subset Q^n$ with minimum distance $d$, and $K_q(n, R)$, the minimal cardinality of a code $C \subset Q^n$ with covering radius $R$, see [2].

$q$-ary codes with covering radius (at most) 1 and minimum distance (at least) 2 as well as the corresponding non-extendable partial multiquasigroups have been studied in [9, 7, 8, 1, 6]. Equivalent objects are pairwise non-attacking rooks which cover all cells of a generalized chessboard and non-extendable partial Latin hypercubes. Denote by $K_q(n, 1, 2)$ the minimal cardinality of a code $C \subset Q^n$ with covering radius 1 and minimum distance 2. Well-known results are $K_q(2, 1, 2) = q$ and $K_q(3, 1, 2) = \lceil q^2/2 \rceil$ as well as $K_2(4, 1, 2) = 4$, $K_2(5, 1, 2) = 8$, $K_2(6, 1, 2) = 12$, $K_3(4, 1, 2) = 9$, $K_4(4, 1, 2) = 28$ and $K_q(n + 1, 1, 2) \leq q \cdot K_q(n, 1, 2)$, see [4, 3, 8, 6].

A natural generalization is to consider mixed codes with covering radius 1 and minimum distance 2. In the whole paper $r$, $s$, $t$ denote positive integers and $R = \{1, 2, ..., r\}$, $S = \{1, 2, ..., s\}$ as well as $T = \{1, 2, ..., t\}$. The minimal cardinality $K(r, s, t)$ of a code $C \subset R \times S \times T$ with covering radius 1 was studied by Numata [5], see also [2, Section 3.7]. Let $K(r, s; 2)$ and $K(r, s, t; 2)$ denote the minimal cardinality of a code $C \subset R \times S$ and of a code $C \subset R \times S \times T$, both with covering radius 1 and minimum distance 2, respectively. In case of codes of length 2, $K(r, s; 2) = \min\{r, s\}$ is obvious. The present paper deals with codes of length 3. Note that $K(r, s, t; 2)$ as well as $K(r, s, t)$ are invariant under permutation of the parameters $r$, $s$ and $t$.

There is an interesting connection between $K(r, s, t; 2)$ and certain generalized partial Latin rectangles. A Latin square of order $r$ is an $r \times r$ matrix with entries from a $r$-set $R$ such that every element of $R$ appears exactly once in every row and every column.

**Definition 1.** *A generalized partial Latin rectangle of order $s \times t$ and size $m$ with entries from a $r$-set $R$ is an $s \times t$ matrix with $m$ filled and $st - m$ empty cells such that every element of $R$ appears at most once in every row and every column. In case of $s = t$, we call it generalized partial Latin square.*

Clearly, such an object corresponds to a code $C \subset R \times S \times T$ with minimum distance 2, and vice versa.

**Definition 2.** *A generalized partial Latin rectangle with entries from $R$ is called non-extendable if for each empty cell every element of $R$ appears in the row or the column of that cell.*

Clearly, a non-extendable generalized partial Latin rectangle of order $s \times t$ and size $m$ with entries from $R$ corresponds to a code $C \subset R \times S \times T$ of cardinality $m$ with covering radius 1 and minimum distance 2, and vice versa. Hence, the existence of such an object yields $K(r, s, t; 2) \leq m$.

Figure I. A non-extendable generalized partial Latin square of order 3 and size 7 with entries from $\{1, 2, 3, 4\}$ and the corresponding code.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | |
| 3 | | 4 |

$$\begin{array}{lll} \{(1,1,1), & (2,1,2), & (3,1,3), \\ \ (2,2,1), & (1,2,2), & \\ \ (3,3,1), & & (4,3,3)\} \end{array}$$

The paper is organized as follows: In Section 2 we give upper bounds for $K(r,s,t;2)$ by presenting various constructions of such codes (or the corresponding partial Latin rectangles). Our focus will be on the special case $t = s$. In Section 3 we give lower bounds for $K(r,s,t;2)$, which will be used in Section 4, to prove the optimality of some of the constructions of Section 2. In this way we determine $K(r,s,s;2)$ when $r$ divides $s$ (Theorem 22), when $r = s-1$ (Theorem 27), when $s \geq r^2$ (Theorem 23), when $r \geq 2s-2$ (Theorem 26) as well as $K(3r,2r,2r;2)$ (Theorem 24). In Section 5 open problems are posed. Finally, a table with bounds on $K(r,s,s;2)$ is given.

For technical reasons we set $K(a,b,c;2) = 0$ if at least one of the variables equals zero.

# 2 Constructions

We often deal with the special case $t = s$. A trivial upper bound is

$$K(r,s,s;2) \leq s \cdot \min\{r,s\}. \tag{1}$$

We start with our basic construction:

**Theorem 3.** *Let $n, s_1, ..., s_n$ be positive integers satisfying $s = \sum_{i=1}^{n} s_i$. Let $R_i$ be an $s_i$-subset of $R$ for every $i \in \{1,...,n\}$ such that $R_i \cup R_j \cup R_k = R$ for all $i \neq j \neq k \neq i$. Set $R_{ij} = R \setminus (R_i \cup R_j)$ and $r_{ij} = |R_{ij}|$ for all $i \neq j$. Then*

$$K(r,s,s;2) \leq \sum_{i=1}^{n} s_i^2 + 2 \sum_{1 \leq i < j \leq n} K(r_{ij}, s_i, s_j; 2).$$

*Proof.* Let $A_{ii}$ be a Latin square of order $s_i$ with entries from $R_i =: R_{ii}$. If $i < j$ then let $A_{ij}$ be a non-extendable generalized partial Latin rectangle of order $s_i \times s_j$ and size $K(r_{ij}, s_i, s_j; 2)$ with entries from $R_{ij}$. Set $A_{ji} = A_{ij}^T$. Since $R_{ij} \cap R_{ik} = R_{ji} \cap R_{ki} = \emptyset$ if $j \neq k$, the matrix

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \tag{2}$$

is the desired non-extendable generalized partial Latin square of order $s$ and size $\sum_{i=1}^{n} s_i^2 + 2\sum_{i<j} K(r_{ij}, s_i, s_j; 2)$ with entries from $R$. □

The following corollary is a generalization of Kalbfleisch and Stanton's [4] construction which proved $K_q(3,1,2) = K(q,q,q;2) \leq \lceil q/2 \rceil^2 + \lfloor q/2 \rfloor^2 = \lceil q^2/2 \rceil$.

**Corollary 4.** *Let $n, s_1, ..., s_n$ be positive integers satisfying $s = \sum_{i=1}^{n} s_i$. Let $R_i$ be an $s_i$-subset of $R$ for every $i \in \{1,...,n\}$ such that $R_i \cup R_j = R$ for all $i \neq j$. Then*

$$K(r,s,s;2) \leq \sum_{i=1}^{n} s_i^2. \tag{3}$$

*Proof.* Apply Theorem 3. Since $R_{ij} = \emptyset$ if $i \neq j$, all matrices $A_{ij}$ are empty in that case. $\square$

**Corollary 5.** *Assume $r$ divides $s$. Set $n = s/r + 1$ and write $s = (n-1)r = qn + c$ with $0 \leq c < n$. Then $K(r, s, s; 2) \leq sq + c(q + 1)$.*

*Proof.* If $r = 1$ then $q = 0$ and $c = s$. Hence, the desired bound follows by (1). Let $r \geq 2$, implying $q > 0$. For $i \in \{1, ..., n\}$ we set

$$s_i = \lfloor (s + i - 1)/n \rfloor = \begin{cases} q & \text{if} \quad i \leq n - c \\ q + 1 & \text{if} \quad n - c < i. \end{cases} \tag{4}$$

Then $\sum_{i=1}^{n} s_i = q(n-c) + (q+1)c = qn + c = s$ implying $\sum_{i=1}^{n}(r - s_i) = rn - s = r$. Since $1 \leq q \leq s_i \leq q + 1 \leq r$ we thus may partition $R = \bigcup_{i=1}^{n} R'_i$ into pairwise disjoint subsets of cardinality $|R'_i| = r - s_i$. Then $R_i = R \setminus R'_i$ is an $s_i$-subset of $R$ for every $i \in \{1, ..., n\}$ such that $R_i \cup R_j = R$ for all $i \neq j$. By (3) and (4) we now get

$$K(r, s, s; 2) \leq \sum_{i=1}^{n} s_i^2 = q^2(n - c) + (q + 1)^2 c = sq + c(q + 1).$$

$\square$

**Corollary 6.** *$K(r, s, s; 2) \leq rs - r^2 + r$ if $s \geq r^2$.*

*Proof.* If $r = 1$ then the bound follows from (1), so assume $r \geq 2$. We modify Corollary 4 (with $n = r + 1$): Let $A_{ii}$ be a Latin square of order $r - 1$ with entries from $R \setminus \{i\}$ if $1 \leq i \leq r$. Let $A_{r+1,r+1}$ be a non-extendable generalized partial Latin square of order $s - r(r-1) \geq r$ and size $r(s - r(r-1))$ with entries from $R$, which is easy to construct from a Latin square of the same order. Then a matrix of type (2), where all matrices $A_{ij}$ are empty if $i \neq j$, is the desired object and, hence, $K(r, s, s; 2) \leq (r-1)^2 r + r(s - r(r-1)) = rs - r^2 + r$. $\square$

**Corollary 7.** *$K(r + s + t, s + t, s + t; 2) \leq s^2 + t^2 + 2K(r, s, t; 2)$.*

*Proof.* Apply Theorem 3 with $(r, s)$ replaced by $(r + s + t, s + t)$, $R$ replaced by $\{1, ..., r + s + t\}$, $n = 2$ and $R_1 = \{1, ..., s\}$ as well as $R_2 = \{s + 1, ..., s + t\}$. $\square$

**Corollary 8.** *$K(r + s, r + 2s, r + 2s; 2) \leq r^2 + 2s^2 + 2K(r, s, s; 2)$ holds true. Especially $K(2r, 3r, 3r; 2) \leq 3r^2 + 2\lceil r^2/2 \rceil$.*

*Proof.* Apply Theorem 3 with $(r, s)$ replaced by $(r + s, r + 2s)$, $R$ replaced by $\{1, ..., r + s\}$, $n = 3$ and $R_1 = \{1, ..., r\}$ as well as $R_2 = R_3 = \{r + 1, ..., r + s\}$. $\square$

The following technical theorem has important consequences.

**Theorem 9.** *Assume $R' \subset R$, $S' \subset S$ and $T' \subset T$. Let $C \subset R \times S \times T$ be a code with covering radius 1 and minimum distance 2. Then there is a code $C' \subset R' \times S' \times T'$ of cardinality*

$$|C'| \leq |\{x \in C \mid d(x, R' \times S' \times T') \leq 1\}| \leq |C|$$

*with covering radius 1 and minimum distance 2.*

*Proof.* Set $\mathcal{W} = R' \times S' \times T'$ and $n = |C \setminus \mathcal{W}|$. We recursively define a sequence $C_0, ..., C_n \subset R \times S \times T$ of codes by the following procedure. Let $C_0 = C$. Assume $i \in \{1, ..., n\}$ and fix $x \in C_{i-1} \setminus \mathcal{W}$. If there exists a $y \in \mathcal{W}$ with $d(x, y) = 1$, which is not covered by $C_{i-1} \cap \mathcal{W}$ then set $C_i = \{y\} \cup C_{i-1} \setminus \{x\}$, otherwise set $C_i = C_{i-1} \setminus \{x\}$ (the latter surely is the case if $d(x, \mathcal{W}) > 1$). Clearly, we have $C_n \subset \mathcal{W}$. Moreover by induction one easily sees, that $C_i \cap \mathcal{W}$ has minimum distance 2 for all $i$ and $\mathcal{W}$ is covered by $C_i$. Hence $C' = C_n$ is the desired code. □

**Corollary 10.** *$r' \leq r$, $s' \leq s$ and $t' \leq t$ imply $K(r', s', t'; 2) \leq K(r, s, t; 2)$.*

**Corollary 11.** *$K(r_1, s_1, t_1; 2) + K(r_2, s_2, t_2; 2) \leq K(r_1 + r_2, s_1 + s_2, t_1 + t_2; 2)$.*

*Proof.* Let $R_1 \cup R_2$, $S_1 \cup S_2$ and $T_1 \cup T_2$ be decompositions of $R$, $S$ and $T$ respectively with $|R_i| = r_i$, $|S_i| = s_i$ and $|T_i| = t_i$ for $i \in \{1, 2\}$. Let $C \subset R \times S \times T$ be a code with covering radius 1 and minimum distance 2 satisfying $|C| = K(r_1 + r_2, s_1 + s_2, t_1 + t_2; 2)$. Then the codes $C^{(1)}$ and $C^{(2)}$ defined by $C^{(i)} = \{x \in C \mid d(x, R_i \times S_i \times T_i) \leq 1\}$ are disjoint. Now Theorem 9 guarantees the existence of codes $C_i \subset R_i \times S_i \times T_i$ with covering radius 1, minimum distance 2 and $|C_i| \leq |C^{(i)}|$ for both $i$. This implies $K(r_1, s_1, t_1; 2) + K(r_2, s_2, t_2; 2) \leq |C_1| + |C_2| \leq |C^{(1)}| + |C^{(2)}| \leq |C| = K(r_1 + r_2, s_1 + s_2, t_1 + t_2; 2)$. □

We now give a construction, which combines Theorem 3 with Theorem 9.

**Theorem 12.** *Let $n, a, s_1, ..., s_n$ be positive integers satisfying $a < s = \sum_{i=1}^{n} s_i$ and $a \leq s_1$. Let $R_i$ be an $s_i$-subset of $R$ for every $i \in \{1, ..., n\}$ such that $R_i \cup R_j \cup R_k = R$ for all $i \neq j \neq k \neq i$. Set $R_{ij} = R \setminus (R_i \cup R_j)$ and $r_{ij} = |R_{ij}|$ for all $i \neq j$. Then*

$$K(r, s - a, s - a; 2) \leq \left( \sum_{i=1}^{n} s_i^2 + 2 \sum_{1 \leq i < j \leq n} K(r_{ij}, s_i, s_j; 2) \right) - a^2.$$

*Proof.* Let $C \subset R \times S \times S$ be a code of cardinality $m = \sum_{i=1}^{n} s_i^2 + 2 \sum_{1 \leq i < j \leq n} K(r_{ij}, s_i, s_j; 2)$ with covering radius 1 and minimum distance 2 according to the proof of Theorem 3. Set $S' = S \setminus \{1, ..., a\}$. Since $|C \cap (R \times (S \setminus S') \times (S \setminus S'))| = a^2$, Theorem 9 guarantees the existence of a code $C' \subset R \times S' \times S'$ of cardinality $|C'| \leq m - a^2$ with covering radius 1 and minimum distance 2. □

The application of Theorem 12 to $K(r, r, r; 2) = \lceil r^2/2 \rceil$ shows

**Corollary 13.** *If $a \leq \lceil r/2 \rceil$ then $K(r, r - a, r - a; 2) \leq \lceil r^2/2 \rceil - a^2$.*

We give another variant of Corollary 4.

**Theorem 14.** *Let $n \geq 3, s_1, ..., s_n$ be positive integers satisfying $s = \sum_{i=1}^{n} s_i$. Let $R_i$ be an $s_i$-subset of $R$ for every $i \in \{1, ..., n\}$ such that $R_i \cup R_j = R$ for all $i \neq j$. For $i \leq \lfloor n/2 \rfloor$ set $t_i = \max\{s_i, s_{n+1-i}\}$. If $r' \leq t_i$ for all $i \leq \lfloor n/2 \rfloor$ then*

$$K(r + r', s, s; 2) \leq \sum_{i=1}^{n} s_i^2 + 2r' \sum_{i=1}^{\lfloor n/2 \rfloor} t_i.$$

*Proof.* Set $I = \{1, ..., \lfloor n/2 \rfloor\}$ and $R' := \{r+1, ..., r+r'\}$. W.l.o.g. let $s_i \geq s_{n+1-i}$ for all $i \in I$, implying $t_i = s_i$. Let $A_{ii}$ be a Latin square of order $s_i$ with entries from $R_i$ for all $i \in \{1, ..., n\}$. For every $i \in I$ let $A_{i,n+1-i}$ be a partial Latin rectangle of order $s_i \times s_{n+1-i}$ and size $r' \cdot s_{n+1-i}$ with entries from $R'$, which exists since $r' \leq s_i$. Clearly, every element of $R'$ appears exactly once in every column and exactly once in $s_{n+1-i}$ of the $s_i$ rows, while it does not appear in the remaining $s_i - s_{n+1-i}$ rows of $A_{i,n+1-i}$. Set $A_{n+1-i,i} = A^T_{i,n+1-i}$. Let $A_{ij}$ be an empty $s_i \times s_j$ matrix if $i \neq j \neq n+1-i$. Let $A^{(0)}$ be the matrix of type (2). By construction, it is a generalized partial Latin square of order $s$ and size $\sum_{i=1}^n s_i^2 + 2r' \sum_{i=1}^{\lfloor n/2 \rfloor} s_{n+1-i}$ with entries from $R \cup R'$. Set

$$M^{(0)} = \{(x, i, i') \in R' \times I \times \mathbb{N} \mid \text{ there is no } x \text{ in row } i' \leq s_i \text{ of } A_{i,n+1-i}\}$$

and $m = |M^{(0)}| = r' \sum_{i=1}^{\lfloor n/2 \rfloor}(s_i - s_{n+1-i})$. We recursively define a sequence $A^{(1)}$, ..., $A^{(m)}$ of generalized partial Latin squares of order $s$ and a sequence $M^{(1)}$, ..., $M^{(m)}$ of sets by the following procedure. Assume $k \in \{1, ..., m\}$ and choose $(x, i, i') \in M^{(k-1)}$. Denote by $a$ the row of $A^{(k-1)}$ corresponding to row $i'$ of $A_{i,n+1-i}$, i.e. $a = i' + \sum_{j=1}^{i-1} s_j$. If there is an empty cell $(a, b)$ in $A^{(k-1)}$ with $a < b \leq n$, such that neither the row $a$ nor the column $b$ has entry $x$, then construct $A^{(k)}$ from $A^{(k-1)}$ by adding entry $x$ in both, position $(a, b)$ and position $(b, a)$. Otherwise let $A^{(k)} = A^{(k-1)}$. In any case set $M^{(k)} = M^{(k-1)} \setminus \{(x, i, i')\}$. By induction, one easily sees that $|M^{(k)}| = |M^{(0)}| - k$ and $A^{(m)}$ is the desired non-extendable generalized partial Latin square. $\square$

Theorem 14 as well as Theorem 12 are often applied in combination with Corollary 5, where the numbers $s_i$ are defined by (4), see for instance the tables in Section 5. It is possible to modify Theorem 14 by using Theorem 9, similar to the proof of Theorem 12.

The next theorem is a modification of [7, Theorem 4].

**Theorem 15.** *$K(tr, ts, ts; 2) \leq t^2 K(r, s, s; 2)$.*

*Proof.* Let $B$ denote a non-extendable generalized partial Latin square of order $s$ and size $K(r, s, s; 2)$ with entries from $R$. Replace every entry $x$ in $B$ by a Latin square of order $t$ with entries from $T \times \{x\}$ and every empty cell in $B$ by an empty $t \times t$ matrix. The resulting matrix is a non-extendable generalized partial Latin square of order $ts$ and size $t^2 K(r, s, s; 2)$ with entries from the $tr$-set $T \times R$. $\square$

**Theorem 16.** *If $s \geq 2$ then*

$$K(2s-2, s, s; 2) \leq \begin{cases} s(s-1) & \textit{if } s \textit{ is even} \\ s(s-1)+1 & \textit{if } s \textit{ is odd.} \end{cases}$$

*Proof.* First assume that $s \geq 3$ is odd. We have to construct a non-extendable generalized partial Latin square of order $s$ and size $s(s-1)+1$ with entries from $R^* = \{1, ..., 2s-2\}$.

For $i,\, j \in S$ set

$$a_{ij} = \begin{cases} i+j-1 & \text{if } i \le j \text{ and } i+j \le s+1 \\ i+j-s-1 & \text{if } i \le j < s \le i+j-2 \\ 2i-2 & \text{if } j = s \text{ and } 1 < i \le (s+1)/2 \\ 2i-s-2 & \text{if } j = s \text{ and } (s+1)/2 < i \\ i+j+s-3 & \text{if } j+1 < i \text{ and } i+j \le s+1 \\ i+j-1 & \text{if } j+1 < i < s < i+j-1 \\ 2j+s-2 & \text{if } i = s \text{ and } 1 < j \le (s-1)/2 \\ 2j & \text{if } i = s \text{ and } (s-1)/2 < j \le s-2 \end{cases}$$

and let $a_{ij}$ be empty if $i = j+1$. It is easy to see that $A = (a_{ij})$ is a generalized partial Latin square of the desired order and size with entries from $R^*$. For $j \in S \setminus \{s\}$ consider the empty cell in position $(j+1, j)$. Every element of $R^*$ appears exactly once in the union of row $j+1$ and column $j$. Hence, $A$ is non-extendable.

Now assume that $s \ge 2$ is even. Set

$$a_{ij} = \begin{cases} i+j-2 & \text{if } i < j \text{ and } i+j \le s+1 \\ i+j-s-1 & \text{if } i < j < s \le i+j-2 \\ 2i-2 & \text{if } j = s \text{ and } 1 < i \le s/2 \\ 2i-s-1 & \text{if } j = s \text{ and } s/2 < i < s \\ i+j+s-3 & \text{if } j < i \text{ and } i+j \le s+1 \\ i+j-2 & \text{if } j < i < s < i+j-1 \\ 2j+s-3 & \text{if } i = s \text{ and } 1 < j \le s/2 \\ 2j-2 & \text{if } i = s \text{ and } s/2 < j < s \end{cases}$$

and let $a_{ii}$ be empty. Analogously, $A = (a_{ij})$ is the desired non-extendable generalized partial Latin square. □

**Corollary 17.** *If $s \le r < 2s$ with even $r$ and $2s - r$ divides $s$, then $K(r, s, s; 2) \le rs/2$.*

*Proof.* Set $m = s/(2s-r)$ and $t = (2s-r)/2$. From Theorem 15 and Theorem 16 follows $K(r, s, s; 2) = K(t(4m-2), 2tm, 2tm; 2) \le t^2 K(4m-2, 2m, 2m) \le t^2 2m(2m-1) = rs/2$. □

# 3 Lower bounds

For technical reasons we define the minimal cardinality $K'(r, s, t; 2)$ of a code $C \subset R \times S \times T$ with covering radius 1 and minimum distance 2 which satisfies $|C \cap (\{1\} \times S \times T)| = \min\{s, t\}$. Again let $K'(a, b, c; 2)$ equal zero, if one of the variables equals zero.

The following lemma is from [7].

**Lemma 18.** *Let $x_i, y_i$ with $i \in \{1, ..., n\}$ be integers satisfying $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $|y_i - y_j| \le 1$ for all $i, j$. Then $\sum_{i=1}^n x_i^2 \ge \sum_{i=1}^n y_i^2$.*

Now, we generalize [7, Theorem 1].

**Theorem 19.** *Let $B$ be an $s\times t$ matrix with entries from $\{0,1\}$. Assume for every 0-entry in $B$ the number of 1's together in the row and the column of that entry is at least $r$. Let $m$ denote the total number of 1's in $B$. If $m = a_1 s + b_1 = a_2 t + b_2$ with $0 \le b_1 < s$ and $0 \le b_2 < t$, then*

$$(r+s+t)m \ge rst + sa_1^2 + (2a_1+1)b_1 + ta_2^2 + (2a_2+1)b_2. \tag{5}$$

*Proof.* Let $B=(b_{ij})$ and denote by $D=\{(i,j)\in S\times T \mid b_{ij}=1\}$ the set of all positions of 1-entries. Clearly, $|D|=m$. For $i\in S$, $j\in T$ let $\varphi(j)=|\{k\in S \mid (k,j)\in D\}|$ and $\psi(i)=|\{k\in T \mid (i,k)\in D\}|$. It is easy to see that $\sum_{j\in T}\varphi(j)=\sum_{i\in S}\psi(i)=m$. Let $E=\{(i,j,k)\in S\times T\times S \mid (k,j)\in D\}$ and $F=\{(i,j,k)\in S\times T\times T \mid (i,k)\in D\}$. Clearly, $|E|=\sum_{(i,j)\in S\times T}\varphi(j)=sm$ and $|F|=\sum_{(i,j)\in S\times T}\psi(i)=tm$. Let $E'=\{(i,j,k)\in E \mid (i,j)\in D\}$ and $F'=\{(i,j,k)\in F \mid (i,j)\in D\}$. Lemma 18 implies

$$|E'| = \sum_{(i,j)\in D}\varphi(j) = \sum_{j\in T}\varphi^2(j) \ge b_2(a_2+1)^2 + (t-b_2)a_2^2$$

and, analogously,

$$|F'| = \sum_{(i,j)\in D}\psi(i) = \sum_{i\in S}\psi^2(i) \ge b_1(a_1+1)^2 + (s-b_1)a_1^2.$$

Combining these results shows

$$\begin{aligned}(st-m)r &\le \sum_{(i,j)\in(S\times T)\setminus D}(\varphi(j)+\psi(i)) = |E|-|E'|+|F|-|F'| \\ &\le (s+t)m - b_2(a_2+1)^2 - (t-b_2)a_2^2 - b_1(a_1+1)^2 - (s-b_1)a_1^2,\end{aligned}$$

and (5) follows. □

From this we deduce

**Theorem 20.** *Let $m \le st$ be an integer satisfying $m = a_1 s + b_1 = a_2 t + b_2$ with $0 \le b_1 < s$ and $0 \le b_2 < t$. If*

$$(r+s+t)m < rst + sa_1^2 + (2a_1+1)b_1 + ta_2^2 + (2a_2+1)b_2 \tag{6}$$

*holds, then $K(r,s,t;2) > m$ as well as $K'(r-1,s,t;2) > m$ (when $r \ge 2$).*

*Proof.* Assume to the contrary, that $C \subset R\times S\times T$ is a code with covering radius 1, minimum distance 2 and $|C| = K(r,s,t;2) = m' \le m$. Let $A=(a_{ij})$ be the corresponding non-extendable generalized partial Latin rectangle of size $m'$, where we assume $a_{ij}=0$ if the corresponding cell is empty. Let $B'=(b_{ij})$ be the matrix of the same order with $b_{ij}=\min\{a_{ij},1\}\in\{0,1\}$. If necessary, replace some 0's in $B'$ by 1's until the total number of 1's in the new matrix $B$ is $m$. Since $A$ is non-extendable, every element of $R$ appears in the row and the column of an 0-entry in $A$. Thus for every 0-entry in $B$ the number of 1's together in the row and the column of that entry is at least $r$. Therefore the propositions of Theorem 19 are satisfied and (5) holds, contradicting (6). Hence $K(r,s,t;2) > m$. An easy modification yields the bound $K'(r-1,s,t;2) > m$ for $r \geq 2$. Use the fact, that if $C \subset ((R \setminus \{r\}) \times S \times T)$ additionally satisfies $|C \cap (\{1\} \times S \times T)| = \min\{s,t\}$, then the entry 1 in the partial Latin rectangle $A$ occurs twice in the row and the column corresponding to a 0-entry in $A$. $\square$

**Theorem 21.** *Let $a, b$ be nonnegative integers. Let $C \subset R \times S \times T$ be a code with $|C| = K(r,s,t;2)$, covering radius 1 and minimum distance 2, such that $|C \cap (\{1\} \times S \times T)| = a \leq b \leq \min\{s,t\}$. Then*

$$K(r,s,t;2) \geq K(r,b,b;2) + (s-b)(t-b)$$

*and*

$$K(r,s,t;2) \geq K'(r,a,a;2) + (s-a)(t-a) \tag{7}$$

*hold true.*

*Proof.* There is a $(s-b)$-set $S^* \subset S$ and a $(t-b)$-set $T^* \subset T$ such that $C \cap ((\{1\} \times S^* \times T) \cup (\{1\} \times S \times T^*)) = \emptyset$ holds. Thus for $v^* \in S^*$, $w^* \in T^*$ the word $(1, v^*, w^*)$ can only be covered by a codeword $(u, v^*, w^*) \in C$ with a suitable $u \in R$. Hence, $|C \cap (R \times S^* \times T^*)| = |S^*| \cdot |T^*| = (s-b)(t-b)$. An application of Theorem 9 with $R' = R$, $S' = S \setminus S^*$ and $T' = T \setminus T^*$ yields the existence of a code $C' \subset R \times S' \times T'$ with covering radius 1, minimum distance 2 and cardinality $|C'| \leq |C| - |C \cap (R \times S^* \times T^*)|$, proving the first inequality. If $a$ is used instead of $b$, the obtained code $C'$ satisfies $|C' \cap (\{1\} \times S' \times T')| = a = \min\{|S'|, |T'|\}$ (as can be seen by the proof of Theorem 9) and (7) follows. $\square$

Theorem 21 is usually used in combination with Theorem 20. Use (7) to lower-bound $K(r,s,t;2)$ and use Theorem 20 to lower-bound the occurring expression $K'(r,a,a;2)$, see for instance the proof of Theorem 27 or the table in Section 5.

# 4 Some optimal codes

In this section we use the lower bounds of Section 3 to prove the optimality of some codes constructed in Section 2.

**Theorem 22.** *Assume $r$ divides $s$. Set $n = s/r + 1$. We write*

$$(n-1)r = qn + c \qquad \text{with} \quad 0 \leq c < n. \tag{8}$$

*Then $K(r,s,s;2) = sq + c(q+1)$.*

*Proof.* The upper bound $K(r,s,s;2) \leq sq + c(q+1)$ is stated in Corollary 5. Concerning the lower bound we apply Theorem 20 with $(r,s,t)$ replaced by $(s,r,s)$ and $m = sq + c(q+1) - 1$. We set $u = n - 1 = s/r$ and distinguish between two cases.

Case I. $\quad c > 0$.

By $0 \le q = \lfloor ur/(u+1) \rfloor < r$ and $1 \le c \le u$ we have

$$0 \le c(q+1) - 1 < cr \le ur = s. \tag{9}$$

Moreover $1 \le c \le u$ implies $(c-u-1)r \le -1 \le (c-1)(u-c)-1$, which is equivalent to

$$(c-1)r \le c\left(\frac{ur-c}{u+1}+1\right) - 1 = c(q+1)-1. \tag{10}$$

We set

$$\begin{aligned} a_1 &= uq + c - 1, \\ b_1 &= c(q+1) - (c-1)r - 1, \\ a_2 &= q, \\ b_2 &= c(q+1) - 1. \end{aligned}$$

From (9) and (10) it follows that $m = a_1 r + b_1 = a_2 s + b_2$ with $0 \le b_1 < r$ and $0 \le b_2 < s$. Making frequent use of (8) we now get

$$\begin{aligned} & (r+2s)m + 2ur + r \\ = \; & r(1+2u)(urq + c(q+1) - 1) + 2ur + r \\ = \; & r((c-1)(2u(q+1)-c) + 2u(q+1) + (1+2u)urq + c(c+q)) \\ = \; & r((c-1)(2u(q+1)-c) + 2u(q+1) + (1+2u)urq + cu(r-q)) \\ = \; & r((c-1)(2u(q+1)-c) + 2u(q+1) + ur(q(u+1)+c) + uq(ur-c)) \\ = \; & r((c-1)(2u(q+1)-c) + 2u(q+1) + u^2r^2 + u(u+1)q^2) \\ = \; & r(u^2r^2 + (uq+c-1)^2 + 2uc(q+1) - (2uq+2c-1)(c-1) + uq^2) \\ = \; & u^2r^3 + r(uq+c-1)^2 + 2(q(u+1)+c)c(q+1) - (2uq+2c-1)(c-1)r + urq^2 \\ = \; & u^2r^3 + r(uq+c-1)^2 + (2uq+2c-1)(c(q+1)-(c-1)r-1) \\ & +urq^2 + (2q+1)(c(q+1)-1) + 2(q(u+1)+c) \\ = \; & rs^2 + ra_1^2 + (2a_1+1)b_1 + sa_2^2 + (2a_2+1)b_2 + 2ur. \end{aligned}$$

Therefore (6) is satisfied (remember that $(r,s,t)$ is replaced by $(s,r,s)$). Moreover $m \le rs$ by (9) and $q < r$. An application of Theorem 20 now yields the bound $K(s,r,s;2) = K(r,s,s;2) \ge m+1 = sq + c(q+1)$.

Case II. $\quad c = 0$.

In this case $m = sq - 1 = a_1 r + b_1 = a_2 s + b_2$ $(0 \le b_1 < r,\ 0 \le b_2 < s)$ with

$$\begin{aligned} a_1 &= uq - 1, \\ b_1 &= r - 1, \\ a_2 &= q - 1, \\ b_2 &= s - 1. \end{aligned}$$

Making use of (8) with $c = 0$ for eliminating $q$ we get

$$\begin{aligned}(r+2s)m &= u^2r^3 - r + u^3r^3/(u+1) - 2ur \\ &= rs^2 + ra_1^2 + (2a_1+1)b_1 + sa_2^2 + (2a_2+1)b_2 - r - 2 \\ &< rs^2 + ra_1^2 + (2a_1+1)b_1 + sa_2^2 + (2a_2+1)b_2.\end{aligned}$$

Like in Case I we get the bound $K(r, s, s; 2) \geq sq + c(q + 1)$. □

**Theorem 23.** *If $s \geq r^2 - r + 1$ then*

$$K(r, s, s; 2) \geq rs - r^2 + r. \tag{11}$$

*If $s \geq r^2$ then equality holds in* (11).

*Proof.* We apply Theorem 20 with $(r, s, t)$ replaced by $(s, r, s)$ and $m = rs - r^2 + r - 1 \leq rs$. By $s \geq r^2 - r + 1$ we have $m = a_1r + b_1 = a_2s + b_2$ $(0 \leq b_1 < r,\ 0 \leq b_2 < s)$ with

$$\begin{aligned}a_1 &= s - r, \\ b_1 &= r - 1, \\ a_2 &= r - 1, \\ b_2 &= s - r^2 + r - 1.\end{aligned}$$

We get

$$\begin{aligned}(r+2s)m &= (r+2s)(rs - r^2 + r - 1) \\ &= rs^2 + ra_1^2 + (2a_1+1)b_1 + sa_2^2 + (2a_2+1)b_2 - r \\ &< rs^2 + ra_1^2 + (2a_1+1)b_1 + sa_2^2 + (2a_2+1)b_2.\end{aligned}$$

Thus (6) is satisfied. Now (11) follows by an application of Theorem 20.

If $s \geq r^2$ then $K(r, s, s; 2) = rs - r^2 + r$ now follows by Corollary 6. □

**Theorem 24.**

$$K(3r, 2r, 2r; 2) = 2r^2 + 2\lceil r^2/2 \rceil = \begin{cases} 3r^2 & \text{if } r \text{ is even} \\ 3r^2 + 1 & \text{if } r \text{ is odd.} \end{cases} \tag{12}$$

*Proof.* For the upper bound, use Corollary 7 with $t = s = r$. For the lower bound we apply Theorem 20 with $(r, s, t)$ replaced by $(3r, 2r, 2r)$ and distinguish between two cases. If $r$ is even then set $m = 3r^2 - 1 \leq 2r \cdot 2r$. We have $m = a_1 \cdot 2r + b_1 = a_2 \cdot 2r + b_2$ $(0 \leq b_1, b_2 < 2r)$ with

$$a_1 = a_2 = 3r/2 - 1,$$

$$b_1 = b_2 = 2r - 1.$$

We now get

$$7r(3r^2 - 1) = 21r^3 - 7r = 12r^3 + 2 \cdot 2ra_1^2 + 2(2a_1 + 1)b_1 - (r + 2).$$

If $r$ is odd then set $m = 3r^2 \leq 2r \cdot 2r$. We have $m = a_1 \cdot 2r + b_1 = a_2 \cdot 2r + b_2$ $(0 \leq b_1, b_2 < 2r)$ with

$$a_1 = a_2 = r + \lfloor r/2 \rfloor,$$
$$b_1 = b_2 = r.$$

We now get

$$7r \cdot 3r^2 = 21r^3 = 12r^3 + 2 \cdot 2ra_1^2 + 2(2a_1 + 1)b_1 - r.$$

Therefore inequality (6) is satisfied in both cases and the lower bound of (12) follows. □

We now show that Corollary 17 yields another infinite family of exact values for $K(r, s, s; 2)$.

**Theorem 25.** *If $s \leq r < 2s$ with even $r$, then*

$$K(r, s, s; 2) \geq rs/2. \tag{13}$$

*If additionally $2s - r$ divides $s$ then equality holds in* (13).

*Proof.* We apply Theorem 20 with $t = s$. Set $m = rs/2 - 1 \leq s^2$. We have $m = a_1 s + b_1 = a_2 s + b_2$ $(0 \leq b_1, b_2 < s)$ with

$$a_1 = a_2 = r/2 - 1,$$
$$b_1 = b_2 = s - 1.$$

We now get

$$\begin{aligned}(r + 2s)(rs/2 - 1) + (2s + 2 - r) &= rs^2 + 2s(r/2 - 1)^2 + 2(r - 1)(s - 1) \\ &= rs^2 + 2 \cdot sa_1^2 + 2 \cdot (2a_1 + 1)b_1.\end{aligned}$$

Therefore inequality (6) is satisfied and (13) follows by an application of Theorem 20. Corollary 17 completes the proof. □

In the next theorem we determine $K(r, s, s; 2)$ for $r \geq 2s - 2$.

**Theorem 26.** $K(r, s, s; 2) = s^2$ *holds true, when $r \geq 2s - 1$. Moreover if $s \geq 2$ then*

$$K(2s - 2, s, s; 2) = \begin{cases} s(s-1) & \text{if } s \text{ is even} \\ s(s-1) + 1 & \text{if } s \text{ is odd.} \end{cases}$$

*Proof.* First assume, $|R| = r \geq 2s - 1$. $K(r, s, s; 2) \leq s^2$ follows by (1). Equality holds, since it is easily seen, that a generalized partial Latin square of order $s$ and size $m < s^2$ with entries from $R$ cannot be non-extendable.

Now assume $r = 2s - 2$. The upper bound holds by Theorem 16. The bound $K(2s - 2, s, s; 2) \geq s(s - 1)$ follows from (13). Assume $K(2s - 2, s, s; 2) = s(s - 1)$. It suffices to show that $s$ must be even. By our assumption there exists a non-extendable generalized partial Latin square $A$ of order $s$ and size $s(s - 1)$ with entries from $\{1, ..., 2s - 2\}$. Especially there are exactly $s$ empty cells in $A$. Since $A$ is non-extendable, every row and every column of $A$ contains exactly one empty cell. Thus every cell in $A$ with entry 1 corresponds to exactly two distinct empty cells in $A$ lying in the row and column of that cell with entry 1. In the other direction, every empty cell corresponds to exactly one cell in $A$ with entry 1 either in the row or in the column of that empty cell, since $A$ is non-extendable. Now it is clear, that the number of empty cells, i.e. $s$ must be even. $\square$

A slight modification of the first statement of Theorem 26 gives

$$K'(r,s,s;2) = s^2 \tag{14}$$

if $r \geq 2s-2$.

**Theorem 27.** *If $s \geq 2$ then $K(s-1,s,s;2) = K(s,s,s;2)$.*

*Proof.* Corollary 10 proves $K(s-1,s,s;2) \leq K(s,s,s;2) = \lceil s^2/2 \rceil$. Suppose there is a code $C \subset (S \setminus \{s\}) \times S \times S$ of cardinality $|C| < \lceil s^2/2 \rceil$ with covering radius 1 and minimum distance 2. There exists $x \in S \setminus \{s\}$ such that

$$a := |C \cap (\{x\} \times S \times S)| \leq \lfloor |C|/(s-1) \rfloor \leq \lceil s/2 \rceil.$$

W.l.o.g. let $x = 1$. If $s$ is even, then Theorem 26 and Theorem 21 (with $r = s-1$, $t = s$ and $b = s/2$) imply the contradiction $(s/2)^2 = K(s-1, s/2, s/2; 2) \leq |C| - (s/2)^2 < (s/2)^2$. If $s$ is odd and $a = (s+1)/2$, equation (14) and (7) imply the contradiction $a^2 = K'(s-1,a,a;2) \leq |C| - ((s-1)/2)^2 < a^2$. If $s$ is odd and $a \leq b := (s-1)/2$, Theorem 26 and Theorem 21 imply the contradiction $b^2 = K(s-1,b,b;2) \leq |C| - ((s+1)/2)^2 < b^2$. $\square$

# 5 Open problems and a table

In general, very little is known about the mathematical differences of the quantities $K_q(n,1)$ and $K_q(n,1,2)$, so it might be interesting to compare the properties of $K(r,s,t;2)$ considered in this paper with the properties of $K(r,s,t)$ considered by Numata [5]. Both quantities appear to have their own flavor. In some cases the first quantity is known while the second is not, and vice versa. For instance, when $s < r$ then Numata determined $K(r,s,s)$ (see also [2], Theorem 3.7.4), whereas most of the values of $K(r,s,s;2)$ remain an open problem in the case $s < r < 2s-2$. On the other hand, if $r < s$ and $r$ divides $s$, then $K(r,s,s;2)$ is known, whereas $K(r,s,s)$ is unknown!

Trivially

$$K(r,s,s) \leq K(r,s,s;2) \tag{15}$$

is valid, but we do not know in general, when equality holds. This surely is the case if $r = s$ or $r \geq 3s-2$, since then both values equal $\lceil s^2/2 \rceil$ (see Kalbfleisch and Stanton [4]) and $s^2$ (see Theorem 26 and Numata [5]), respectively. The inequality is strict however, when $2s-1 \leq r \leq 3s-3$ (see Numata [5]). Let us have a look on the case $r < s$. Here we propose the following conjecture:

**Conjecture 28.** *If $r$ divides $s$, then equality holds in* (15).

In many cases however inequality (15) is strict. So Numata showed

$$K(r,s,s) \leq rs - \lfloor r^2/2 \rfloor \tag{16}$$

whenever $r < s$. Especially

$$K(s-1,s,s) \leq \begin{cases} K(s,s,s) & \text{if } s \text{ is even} \\ K(s,s,s)-1 & \text{if } s \text{ is odd.} \end{cases} \tag{17}$$

So by Theorem 27 we have $K(s-1,s,s) < K(s-1,s,s;2)$ whenever $s$ is odd.

Numata conjectured, that equality always holds in (16). This conjecture in general is incorrect however. So for instance by (15) and Corollary 5 we get $K(3,6,6) \leq 12$, whereas (16) only yields $K(3,6,6) \leq 14$. In the case $r = s-1$ however we support Numata's conjecture:

**Conjecture 29.** *In* (17) *equality always holds.*

We conjecture that Theorem 27 can be generalized as follows.

**Conjecture 30.** *For every nonnegative integer $a$ there exists an integer $s(a)$, such that $K(s-a,s,s;2) = K(s,s,s;2)$ holds whenever $s > s(a)$.*

As a counterpart to Theorem 27 we have $K(s,s-1,s-1;2) \leq K(s,s,s;2)-1$ by Corollary 13. We do not know however, whether equality always holds. Another question is, whether the code constructed in Corollary 8 is optimal for $s = r$, i.e. if $K(2r,3r,3r;2) = 3r^2 + 2\lceil r^2/2 \rceil$ holds.

Finally, we give a table with exact values of and bounds on $K(r,s,s;2)$ in case of $r,s \leq 16$. Upper bounds: unmarked upper bounds refer to (1), $A$ to Corollary 5, $B$ to Corollary 6, $C$ to Corollary 7, $D$ to Corollary 8, $E$ to Corollary 10, $F$ to Theorem 12, $G$ to Theorem 14 and $H$ to Theorem 16. Lower bounds: unmarked lower bounds refer to Theorem 20, $a$ refers to a combination of Theorem 20 and 21 (see the end of Section 3), and $b$ to Theorem 26.

Table I. Bounds on $K(r,s,s;2)$ for $r \leq 16$ and $s \leq 8$.

| | $s=1$ | $s=2$ | $s=3$ | $s=4$ | $s=5$ | $s=6$ | $s=7$ | $s=8$ |
|---|---|---|---|---|---|---|---|---|
| $r=1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $r=2$ | 1 | $2A$ | $a5F$ | $6B$ | $8B$ | $10B$ | $12B$ | $14B$ |
| $r=3$ | 1 | 4 | $5A$ | $8F$ | $10-11F$ | $12A$ | $15-17F$ | $18-20F$ |
| $r=4$ | 1 | 4 | $b7H$ | $8A$ | $a13F$ | $a16D$ | $a20-21F$ | $22A$ |
| $r=5$ | 1 | 4 | 9 | $11-12C$ | $13A$ | $a18F$ | $21-25F$ | $25-28G$ |
| $r=6$ | 1 | 4 | 9 | $12C$ | $15-17F$ | $18A$ | $a25E$ | $a30-32F$ |
| $r=7$ | 1 | 4 | 9 | 16 | $19-21C$ | $22-24C$ | $25A$ | $a32E$ |
| $r=8$ | 1 | 4 | 9 | 16 | $b21H$ | $24-28C$ | $28-31F$ | $32A$ |
| $r=9$ | 1 | 4 | 9 | 16 | 25 | $28C$ | $33-37F$ | $37-40C$ |
| $r=10$ | 1 | 4 | 9 | 16 | 25 | $30H$ | $35-39C$ | $40-44C$ |
| $r=11$ | 1 | 4 | 9 | 16 | 25 | 36 | $40-41C$ | $45-48C$ |
| $r=12$ | 1 | 4 | 9 | 16 | 25 | 36 | $b43H$ | $48C$ |
| $r=13$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | $54-56C$ |
| $r=14$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | $56C$ |
| $r=15$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 |
| $r=16$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 |

Table II. Bounds on $K(r,s,s;2)$ for $r \leq 16$ and $9 \leq s \leq 16$.

| | $s=9$ | $s=10$ | $s=11$ | $s=12$ | $s=13$ | $s=14$ | $s=15$ | $s=16$ |
|---|---|---|---|---|---|---|---|---|
| $r=1$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ |
| $r=2$ | $16B$ | $18B$ | $20B$ | $22B$ | $24B$ | $26B$ | $28B$ | $30B$ |
| $r=3$ | $21B$ | $24B$ | $27B$ | $30B$ | $33B$ | $36B$ | $39B$ | $42B$ |
| $r=4$ | $26-27F$ | $29-32F$ | $33-35F$ | $36A$ | $40-43F$ | $44-48F$ | $48-51F$ | $52B$ |
| $r=5$ | $30-33F$ | $34A$ | $39-41F$ | $44-48F$ | $48-53F$ | $53-56F$ | $57A$ | $62-64F$ |
| $r=6$ | $a35-37D$ | $38-40G$ | $43-47F$ | $48A$ | $54-57F$ | $60-66F$ | $65-73F$ | $71-78F$ |
| $r=7$ | $36-41F$ | $42-46G$ | $a49-56E$ | $54-56G$ | $60-65F$ | $66A$ | $73-75F$ | $79-86F$ |
| $r=8$ | $a41E$ | $a48-50F$ | $a53-61F$ | $58-64D$ | $65-76E$ | $72-76G$ | $79-85F$ | $86A$ |
| $r=9$ | $41F$ | $a50E$ | $55-61E$ | $a63-72F$ | $a71-81D$ | $78-86G$ | $85-96E$ | $93-96G$ |
| $r=10$ | $45-49F$ | $50F$ | $a61E$ | $a70-72E$ | $a76-85F$ | $82-92D$ | $90-101D$ | $a100-106G$ |
| $r=11$ | $51-57F$ | $56-60C$ | $61F$ | $a72E$ | $78-85E$ | $a88-98F$ | $a97-113F$ | $105-116G$ |
| $r=12$ | $54-63F$ | $60-66C$ | $66-71F$ | $72F$ | $a85E$ | $a96-98E$ | $a103-113E$ | $110-126G$ |
| $r=13$ | $60-65C$ | $66-72C$ | $73-81F$ | $79-84C$ | $85F$ | $a98E$ | $105-113E$ | $a117-128E$ |
| $r=14$ | $63-67C$ | $70-76C$ | $77-89F$ | $84-92C$ | $91-97F$ | $98F$ | $a113E$ | $a126-128E$ |
| $r=15$ | $69-73E$ | $76C$ | $84-95E$ | $91-96C$ | $99-109F$ | $106-112C$ | $113F$ | $a128E$ |
| $r=16$ | $b73H$ | $80-84C$ | $88-95C$ | $96-104C$ | $104-119F$ | $112-122C$ | $120-127F$ | $128F$ |

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