Two Color Off-diagonal Rado-type Numbers

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Abstract

We show that for any two linear homogeneous equations $\mathcal{E}_0, \mathcal{E}_1$, each with at least three variables and coefficients not all the same sign, any 2-coloring of $\mathbb{Z}^+$ admits monochromatic solutions of color 0 to $\mathcal{E}_0$ or monochromatic solutions of color 1 to $\mathcal{E}_1$. We define the 2-color off-diagonal Rado number $RR(\mathcal{E}_0, \mathcal{E}_1)$ to be the smallest N such that $[1, N]$ must admit such solutions. We determine a lower bound for $RR(\mathcal{E}_0, \mathcal{E}_1)$ in certain cases when each $\mathcal{E}_i$ is of the form $a_1x_1 + \ldots + a_nx_n = z$ as well as find the exact value of $RR(\mathcal{E}_0, \mathcal{E}_1)$ when each is of the form $x_1 + a_2x_2 + \ldots + a_nx_n = z$. We then present a Maple package that determines upper bounds for off-diagonal Rado numbers of a few particular types, and use it to quickly prove two previous results for diagonal Rado numbers.

0 Introduction

For $r \geq 2$, an r-coloring of the positive integers $\mathbb{Z}^+$ is an assignment $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$. Given a diophantine equation $\mathcal{E}$ in the variables $x_1, \ldots, x_n$, we say a solution $\{\bar{x}_i\}_{i=1}^n$ is monochromatic if $\chi(\bar{x}_i) = \chi(\bar{x}_j)$ for every i, j pair. A well-known theorem of Rado states that, for any $r \geq 2$, a linear homogeneous equation $c_1x_1 + \ldots + c_nx_n = 0$ with each $c_i \in \mathbb{Z}$ admits a monochromatic solution in $\mathbb{Z}^+$ under any r-coloring of $\mathbb{Z}^+$ if and only if some nonempty subset of $\{c_i\}_{i=1}^n$ sums to zero. The smallest N such that any r-coloring of $\{1, 2, \ldots, N\} = [1, N]$ satisfies this condition is called the r-color Rado number for the equation $\mathcal{E}$. However, Rado also proved the following, much lesser known, result.

Theorem 0.1 (*Rado [6]*) Let $\mathcal{E}$ be a linear homogeneous equation with integer coefficients. Assume that $\mathcal{E}$ has at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits a monochromatic solution to $\mathcal{E}$.

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Remark. Theorem 0.1 cannot be extended to more than 2 colors, without restriction on the equation. For example, Fox and Radoičić [2] have shown, in particular, that there exists a 3-coloring of $\mathbb{Z}^+$ that admits no monochromatic solution to $x + 2y = 4z$. For more information about equations that have finite colorings of $\mathbb{Z}^+$ with no monochromatic solution see [1] and [2].

In [4], the 2-color Rado numbers are determined for equations of the form $a_1x_1 + \ldots + a_nx_n = z$ where one of the a_i's is 1. The case when $\min(a_1, \ldots, a_n) = 2$ is done in [5], while the general case is settled in [3].

In this article, we investigate the "off-diagonal" situation. To this end, for $r \in \mathbb{Z}^+$ define an off-diagonal Rado number for the equations $\mathcal{E}_i$, $0 \leq i \leq r-1$, to be the least integer N (if it exists) for which any r-coloring of $[1, N]$ must admit a monochromatic solution to $\mathcal{E}_i$ of color i for some $i \in [0, r-1]$. In this paper, when $r = 2$ we will prove the existence of such numbers and determine particular values and lower bounds in several specific cases when the two equations are of the form $a_1x_1 + \ldots + a_nx_n = z$.

1 Existence

The authors were unable to find an English translation of the proof of Theorem 0.1. For the sake of completeness, we offer a simplified version of Rado's original proof.

Proof of Theorem 0.1 (due to Rado [6]) Let $\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{\ell} \beta_i y_i$ be our equation, where $k \geq 2$, $\ell \geq 1$, $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$, and $\beta_i \in \mathbb{Z}^+$ for $1 \leq i \leq \ell$. By setting $x = x_1 = x_2 = \cdots = x_{k-1}$, $y = x_k$, and $z = y_1 = y_2 = \cdots = y_\ell$, we may consider solutions to

$$ax + by = cz,$$

where $a = \sum_{i=1}^{k-1} \alpha_i$, $b = c_k$, and $c = \sum_{i=1}^{\ell} \beta_i$. We will denote $ax + by = cz$ by $\mathcal{E}$.

Let $m = \mathrm{lcm}\left(\frac{a}{\gcd(a,b)}, \frac{c}{\gcd(b,c)}\right)$. Let (x_0, y_0, z_0) be the solution to $\mathcal{E}$ with $\max(x, y, z)$ a minimum, where the maximum is taken over all solutions of positive integers to $\mathcal{E}$. Let $A = \max(x_0, y_0, z_0)$.

Assume, for a contradiction, that there exists a 2-coloring of $\mathbb{Z}^+$ with no monochromatic solution to $\mathcal{E}$. First, note that for any $n \in \mathbb{Z}^+$, the set $\{in : i = 1, 2, \ldots, A\}$ cannot be monochromatic, for otherwise $x = x_0 n$, $y = y_0 n$, and $z = z_0 n$ is a monochromatic solution, a contradiction.

Let $x = m$ so that $\frac{bx}{a}, \frac{bx}{c} \in \mathbb{Z}^+$. Letting red and blue be our two colors, we may assume, without loss of generality, that x is red. Let y be the smallest number in $\{im : i = 1, 2, \ldots, A\}$ that is blue. Say $y = \ell m$ so that $2 \leq \ell \leq A$.

For some $n \in \mathbb{Z}^+$, we have that $z = \frac{b}{a}(y-x)n$ is blue, otherwise $\{i\frac{b}{a}(y-x) : i = 1, 2, \ldots\}$ would be red, admitting a monochromatic solution to $\mathcal{E}$. Then $w = \frac{a}{c}z + \frac{b}{c}y$ must be red, for otherwise $az + by = cw$ and z, y, and w are all blue, a contradiction. Since x and w are both red, we have that $q = \frac{c}{a}w - \frac{b}{a}x = \frac{b}{a}(y-x)(n+1)$ must be blue, for otherwise x, w, and q give a red solution to $\mathcal{E}$. As a consequence, we see that $\left\{i\frac{b}{a}(y-x) : i = n, n+1, \ldots\right\}$

is monochromatic. This gives us that $\left\{i\frac{b}{a}(y-x)n : i = 1, 2, \ldots, A\right\}$ is monochromatic, a contradiction. $\square$

Using the above result, we offer an “off-diagonal” consequence.

Theorem 1.1 Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be linear homogeneous equations with integer coefficients. Assume that $\mathcal{E}_0$ and $\mathcal{E}_1$ each have at least 3 variables with both positive and negative coefficients. Then any 2-coloring of $\mathbb{Z}^+$ admits either a solution to $\mathcal{E}_0$ of the first color or a solution to $\mathcal{E}_1$ of the second color.

Proof. Let $a_0, a_1, b_0, b_1, c \in \mathbb{Z}^+$ and denote by $\mathcal{G}_i$ the equation $a_i x + b_i y = cz$ for $i = 0, 1$. Via the same argument given in the proof to Theorem 0.1, we may consider solutions to $\mathcal{G}_0$ and $\mathcal{G}_1$. (The coefficients of z may be taken to be the same in both equations by finding the lcm of the original coefficients of z and adjusting the other coefficients accordingly.)

Let the colors be red and blue. We want to show that any 2-coloring admits either a red solution to $\mathcal{G}_0$ or a blue solution to $\mathcal{G}_1$. From Theorem 0.1, we have monochromatic solutions to each of these equations. Hence, we assume, for a contradiction, that any monochromatic solution to $\mathcal{G}_0$ is blue and that any monochromatic solution to $\mathcal{G}_1$ is red. This gives us that for any $i \in \mathbb{Z}^+$, if ci is blue, then $(a_1 + b_1)i$ is red (else we have a blue solution to $\mathcal{G}_1$).

Now consider monochromatic solutions in $c\mathbb{Z}^+$. Via the obvious bijection between colorings of $c\mathbb{Z}^+$ and $\mathbb{Z}^+$ and the fact that linear homogeneous equations are unaffected by dilation, Theorem 0.1 gives us the existence of monochromatic solutions in $c\mathbb{Z}^+$. If cx, cy, cz solve $\mathcal{G}_0$ and are the same color, then they must be blue. Hence, $\hat{x} = (a_1 + b_1)x$, $\hat{y} = (a_1 + b_1)y$, and $\hat{z} = (a_1 + b_1)z$ are all red. But, $\hat{x}, \hat{y}, \hat{z}$ solve $\mathcal{G}_0$. Thus, we have a red solution to $\mathcal{G}_0$, a contradiction. $\square$

2 Two Lower Bounds

Given the results in the previous section, we make a definition, which uses the following notation.

Notation For $n \in \mathbb{Z}^+$ and $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, denote by $\mathcal{E}_n(\vec{a})$ the linear homogeneous equation $\sum_{i=1}^{n} a_i x_i = 0$.

Definition For $k, \ell \geq 3, \vec{b} \in \mathbb{Z}^k$, and $\vec{c} \in \mathbb{Z}^\ell$, we let $RR(\mathcal{E}_k(\vec{b}), \mathcal{E}_\ell(\vec{c}))$ be the minimum integer N, if it exists, such that any 2-coloring of $[1, N]$ admits either a solution to $\mathcal{E}_k(\vec{b})$ of the first color or a solution to $\mathcal{E}_\ell(\vec{c})$ of the second color.

We now develop a general lower bound for certain types of those numbers guaranteed to exist by Theorem 1.1.

Theorem 2.1 For $k, \ell \geq 2$, let $b_1, b_2, \ldots, b_{k-1}, c_1, c_2, \ldots, c_{\ell-1} \in \mathbb{Z}^+$. Consider $\mathcal{E}_k = \mathcal{E}_k(b_1, b_2, \ldots, b_{k-1}, -1)$ and $\mathcal{E}_\ell = \mathcal{E}_\ell(c_1, c_2, \ldots, c_{\ell-1}, -1)$, written so that $b_1 = \min(b_1, b_2, \ldots, b_{k-1})$ and $c_1 = \min(c_1, c_2, \ldots, c_{\ell-1})$. Assume that $t = b_1 = c_1$. Let $q = \sum_{i=2}^{k-1} b_i$ and

$s = \sum_{i=2}^{\ell-1} c_i$. Let (without loss of generality) $q \geq s$. Then

$$RR(\mathcal{E}_k, \mathcal{E}_\ell) \geq t(t+q)(t+s) + s.$$

Proof. Let $N = t(t+q)(t+s) + s$ and consider the 2-coloring of $[1, N-1]$ defined by coloring $[s+t, (q+t)(s+t)-1]$ red and its complement blue. We will show that this coloring avoids red solutions to $\mathcal{E}_k$ and blue solutions to $\mathcal{E}_\ell$.

We first consider any possible red solution to $\mathcal{E}_k$. The value of x_k would have to be at least $t(s+t) + q(s+t) = (q+t)(s+t)$. Thus, there is no suitable red solution. Next, we consider $\mathcal{E}_\ell$. If $\{x_1, x_2, \ldots, x_{\ell-1}\} \subseteq [1, s+t-1]$, then $x_\ell < (q+t)(s+t)$. Hence, the smallest possible blue solution to $\mathcal{E}_\ell$ has $x_i \in [(q+t)(s+t), N-1]$ for some $i \in [1, \ell-1]$. However, this gives $x_\ell \geq t(q+t)(s+t) + s > N-1$. Thus, there is no suitable blue solution. □

The case when $k = \ell = 2$ in Theorem 2.1 can be improved somewhat in certain cases, depending upon the relationship between t, q, and s. This result is presented below.

Theorem 2.2 Let $t, j \in \mathbb{Z}^+$. Let $\mathcal{F}_j^t$ represent the equation $tx + jy = z$. Let $q, s \in \mathbb{Z}^+$ with $q \geq s \geq t$. Define $m = \frac{\gcd(t,q)}{\gcd(t,q,s)}$. Then

$$RR(\mathcal{F}_q^t, \mathcal{F}_s^t) \geq t(t+q)(t+s) + ms.$$

Proof. Let $N = t(t+q)(t+s) + ms$ and consider the 2-coloring χ of $[1, N-1]$ defined by coloring

$$R = [s+t, (q+t)(s+t)-1] \cup \{t(t+q)(t+s) + is : 1 \leq i \leq m-1\}$$

red and $B = [1, N-1] \setminus R$ blue. We will show that this coloring avoids red solutions to $\mathcal{F}_q^t$ and blue solutions to $\mathcal{F}_s^t$.

We first consider any possible red solution to $\mathcal{F}_q^t$. The value of z would have to be at least $t(s+t) + q(s+t) = (q+t)(s+t)$ and congruent to 0 modulo m. Since $t(t+q)(t+s) \equiv 0 \pmod{m}$ but $is \not\equiv 0 \pmod{m}$ for $1 \leq i \leq m-1$, there is no suitable red solution. Next, we consider $\mathcal{F}_s^t$. If $\{x, y\} \subseteq [1, s+t-1]$, then $s+t \leq z < (q+t)(s+t)$. Hence, the smallest possible blue solution to $\mathcal{F}_s^t$ has x or y in $[(q+t)(s+t), N-1]$. However, this gives $z \geq t(q+t)(s+t) + s > N-1$. By the definition of the coloring, z must be red. Thus, there is no suitable blue solution to $\mathcal{F}_s^t$. □

3 Some Exact Numbers

In this section, we will determine some of the values of $RR_1(q,s) = RR(x+qy = z, x+sy = z)$, where $1 \leq s \leq q$. The subscript 1 is present to emphasize the fact that we are using $t = 1$ as defined in Theorem 2.1. In this section we will let $RR_t(q,s) = RR(tx + qy = z, tx + sy = z)$ and we will denote the equation $tx + jy = z$ by $\mathcal{F}_j^t$.

Theorem 3.1 Let $1 \leq s \leq q$. Then

$$RR_1(q,s) = \begin{cases} 2q + 2\left\lfloor \frac{q+1}{2} \right\rfloor + 1 & \text{for } s = 1 \\ (q+1)(s+1)+s & \text{for } s \geq 2. \end{cases}$$

Proof. We start with the case $s = 1$. Let $N = 2q + 2\left\lfloor \frac{q+1}{2} \right\rfloor + 1$. We first improve the lower bound given by Theorem 2.1 for this case.

Let γ be the 2-coloring of $[1, N-1]$ defined as follows. The first $2\lfloor \frac{q+1}{2} \rfloor - 1$ integers alternate colors with the color of 1 being blue. We then color $\left[2\lfloor \frac{q+1}{2} \rfloor, 2q+1\right]$ red. We color the last $2\lfloor \frac{q+1}{2} \rfloor - 1$ integers with alternating colors, where the color of $2q+2$ is blue.

First consider possible blue solutions to $x + y = z$. If $x, y \leq 2\lfloor \frac{q+1}{2} \rfloor - 1$, then $z \leq 2q$. Under γ, such a z must be red. Now, if exactly one of x and y is greater than $2q+1$, then z is odd and greater than $2q+1$. Again, such a z must be red. Finally, if both x and y are greater than $2q+1$, then z is too big. Hence, γ admits no blue solution to $x + y = z$.

Next, we consider possible red solutions to $x + qy = z$. If $x, y \leq \lfloor \frac{q+1}{2} \rfloor - 1$, then z must be even. Also, since x and y must both be at least 2 under γ, we see that $z \geq 2q+2$. Under γ, such a z must be blue. If one (or both) of x or y is greater than $\lfloor \frac{q+1}{2} \rfloor - 1$, then $z \geq N - 1$, with equality possible. However, with equality, the color of z is blue. Hence, γ admits no red solution to $x + qy = z$.

We move onto the upper bound. Let χ be a 2-coloring of $[1, N]$ using the colors red and blue. Assume, for a contradiction, that there is no red solution to $\mathcal{F}_q^1$ and no blue solution to $\mathcal{F}_1^1$. We break the argument into 3 cases.

`Case 1.` 1 is red. Then $q+1$ must be blue since otherwise $(x, y, z) = (1, 1, q+1)$ would be a red solution to $\mathcal{F}_q^1$. Since $(q+1, q+1, 2q+2)$ satisfies $\mathcal{F}_1^1$, we have that $2q+2$ must be red. Now, since $(q+2, 1, 2q+2)$ satisfies $\mathcal{F}_q^1$, we see that $q+2$ must be blue. Since $(2, q+2, q+4)$ satisfies $\mathcal{F}_1^1$ we have that $q+4$ must be red. This implies that 4 must be blue since $(4, 1, q+4)$ satisfies $\mathcal{F}_q^1$. But then $(2, 2, 4)$ is a blue solution to $\mathcal{F}_1^1$, a contradiction.

`Case 2.` 1 is blue and q is odd. Note that in this case we have $N = 3q+2$. Since 1 is blue, 2 must be red, which, in turn, implies that $2q+2$ must be blue. Since $(q+1, q+1, 2q+2)$ solves $\mathcal{F}_1^1$, we see that $q+1$ must be red. Now, since $(j, 2q+2, 2q+j+2)$ solves $\mathcal{F}_1^1$ and $(j+2, 2, 2q+j+2)$ solves $\mathcal{F}_q^1$, we have that for any $j \in \{1, 3, 5, \ldots, q\}$, the color of j is blue. With 2 and q both red, we have that $3q$ is blue, which implies that $3q+1$ must be red. Since $(q+1, 2, 3q+1)$ solves $\mathcal{F}_q^1$, we see that $q+1$ must be blue, and hence $q+2$ is red. Considering $(q+2, 2, 3q+2)$, which solves $\mathcal{F}_q^1$, and $(q, 2q+2, 3q+2)$, which solves $\mathcal{F}_1^1$, we have an undesired monochromatic solution, a contradiction.

`Case 3.` 1 is blue and q is even. Note that in this case we have $N = 3q+1$. As in Case 2, we argue that for any $j \in \{1, 3, 5, \ldots, q-1\}$, the color of j is blue. As in Case 2, both 2 and $q+1$ must be red, so that $3q+1$ must be blue. But $(q-1, 2q+2, 3q+1)$ is then a blue solution to $\mathcal{F}_1^1$, a contradiction.

Next, consider the cases when $s \geq 2$. From Theorem 2.1, we have $RR_1(q,s) \geq (q+1)(s+1)+s$. We proceed by showing that $RR_1(q,s) \leq (q+1)(s+1)+s$.

In the case when $s=1$ we used an obvious "forcing" argument. As such, we have automated the process in the Maple package `SCHAAL` [8]. The package is detailed in the next subsection, but first we finish the proof. Using `SCHAAL` we find the following (where we use the fact that $s \geq 2$):

1) If 1 is red, then the elements in $\{s, q+s+1, qs+q+s+1\}$ must be both red and blue, a contradiction.

2) If 1 is blue and $s-1$ is red, then the elements in $\{1, 2, 2q-1, 2s+1, 2q+1, 2q+2s-1, 2q+2s+1\}$ must be both red and blue, a contradiction.

3) If 1 and $s-1$ are both blue, the analysis is a bit more involved. First, by assuming $s \geq 2$ we find that 2 must be red and s must be blue. Hence, we cannot have $s=2$ or $s=3$, since if $s=2$ then 2 is both red and blue, and if $s=3$ then since $s-1$ is blue, we again have that 2 is both red and blue. Thus, we may assume that $s \geq 4$. Using `SCHAAL` with $s \geq 4$ now produces the result that the elements in $\{4, s+1, q+1, 2s-1, 2s, q+2s+1, 3s+1, 5q+1, 4q+s+1, 4q+2s-1, 4q+2s, 4q+3s+1, 5q+2s+1, qs-3q+1, qs-3q+2s+1, qs-3q+s-1, qs+q+1, qs+q+s-1, qs+q+2s+1\}$ must be both red and blue, a contradiction.

This completes the proof of the theorem. $\square$

Using the above theorem, we offer the following corollary.

Corollary 3.2 For $k, \ell \in \mathbb{Z}^+$, let $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in \mathbb{Z}^+$. Assume $\sum_{i=1}^k a_i \geq \sum_{i=1}^\ell b_i$. Then $RR_1 = RR_1\left(x + \sum_{i=1}^k a_i y_i = z, x + \sum_{i=1}^\ell b_i y_i = z\right)$ is

$$RR_1 = \begin{cases} 2\displaystyle\sum_{i=1}^k a_i + 2\left\lfloor \frac{\sum_{i=1}^k a_i + 1}{2} \right\rfloor + 1 & \text{for } \displaystyle\sum_{i=1}^\ell b_i = 1 \\ \left(\displaystyle\sum_{i=1}^k a_i + 1\right)\left(\displaystyle\sum_{i=1}^\ell b_i + 1\right) + \displaystyle\sum_{i=1}^\ell b_i & \text{for } \displaystyle\sum_{i=1}^\ell b_i \geq 2. \end{cases}$$

Proof. We start by proving that the coloring given in the proof of Theorem 3.1 which provides the lower bound for the case $s=1$ also provides (with a slight modification) a lower bound for the case when $\sum_{i=1}^\ell b_i = 1$. In this situation, we must show that the coloring where the first $2\left\lfloor \frac{\sum_{i=1}^k a_i+1}{2} \right\rfloor - 1$ integers alternate colors with the color of 1 being blue. We then color $\left[2\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\rfloor, 2\sum_{i=1}^k a_i + 1\right]$ red. We color the last $2\left\lfloor \frac{\sum_{i=1}^k a_i+1}{2} \right\rfloor - 1$ integers with alternating colors, where the color of $2\sum_{i=1}^k a_i + 2$ is blue. An obvious parity argument shows that there is no blue solution to $x+y=z$ (this is the case when $\sum_{i=1}^\ell b_i = 1$) exists, so it remains to show that no red solution to $x + \sum_{i=1}^k a_i y_i = z$

exists under this coloring. Now, if x and all the y_i's are less than $2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor$, then z would be even and have value at least $2\sum_{i=1}^k a_i+2$. This is not possible, so at least one of $x,y_1,\ldots,y_k$ must have value at least $2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor$. If $x\geq 2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor$, then $z\geq 2\sum_{i=1}^k a_i+2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor$. Hence, either z is blue or too big. So, assume, without loss of generality, that $y_1\geq 2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor$. If $a_1=1$, then $z=x+y_1+\sum_{i=2}^k a_iy_i\geq 2+2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor+2\sum_{i=2}^k a_i=2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor+2\sum_{i=1}^k a_i$ and again either z is blue or too big. If $a_1\geq 2$ (and we may assume that $k\geq 2$ so that $\sum_{i=1}^k a_i+1\geq 4$), then $z=x+a_1y_1+\sum_{i=2}^k a_iy_i>a_1\cdot 2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor+2\sum_{i=2}^k a_iy_i\geq 2(a_1+\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor)+2\sum_{i=2}^k a_iy_i=2\left\lfloor\frac{\sum_{i=1}^k a_i+1}{2}\right\rfloor)+2\sum_{i=1}^k a_iy_i$ and z is too big.

Next, by coupling the above lower bound with Theorem 2.1 (using $t=1$), it remains to prove that the righthand sides of the theorem's equations serve as upper bounds for $N=RR_1(x+\sum_{i=1}^k a_iy_i=z,x+\sum_{i=1}^\ell b_iy_i=z)$. Letting $q=\sum_{i=1}^k a_i$ and $s=\sum_{i=1}^\ell b_i$, any solution to $x+qy=z$ (resp., $x+sy=z$) is a solution to $x+\sum_{i=1}^k a_iy_i$ (resp., $x+\sum_{i=1}^\ell b_iy_i=z$) by letting all y_i's equal y. Hence, $N\leq RR_1(q,s)$ and we are done. $\square$

Remark. When $a_i=1$ for $1\leq i\leq k$, $\ell=1$, and $b_1=1$ the numbers in Corollary 3.2 are called the off-diagonal generalized Schur numbers. In this case, the values of the numbers have been determined [7].

3.1 About the Maple Package SCHAAL

This package is used to try to automatically provide an upper bound for the off-diagonal Rado-type numbers $RR_t(q,s)$. The package employs a set of rules to follow, while the overall approach is an implementation of the above "forcing" argument.

Let $t\geq 2$ be given, keep $q\geq s$ as parameters, and define $N=tqs+t^2q+(t^2+1)s+t^3$. We let $\mathcal{R}$ and $\mathcal{B}$ be the set of red, respectively blue, elements in $[1,N]$. The package SCHAAL uses the following rules.

For $x,y\in\mathcal{R}$,
- R1) if $q|(y-tx)$ and $y-tx>0$, then $\frac{y-tx}{q}\in\mathcal{B}$;
- R2) if $t|(y-qx)$ and $y-qx>0$, then $\frac{y-qx}{t}\in\mathcal{B}$;
- R3) if $(q+t)|x$ then $\frac{x}{q+t}\in\mathcal{B}$.

For $x,y\in\mathcal{B}$,
- B1) if $s|(y-tx)$ and $y-tx>0$, then $\frac{y-tx}{s}\in\mathcal{R}$;
- B2) if $t|(y-sx)$ and $y-sx>0$, then $\frac{y-sx}{t}\in\mathcal{R}$;
- B3) if $(s+t)|x$ then $\frac{x}{s+t}\in\mathcal{R}$.

We must, of course, make sure that the elements whose colors are implied by the above rules are in $[1,N]$. This is done by making sure that the coefficients of qs, q, and s, as well as the constant term are nonnegative and at most equal to the corresponding coefficients in $tqs + t^2q + (t^2+1)s + t^3$ (hence the need for t to be an integer and not a parameter). See the Maple code for more details.

The main program of `SCHAAL` is `dan`. The program `dan` runs until $\mathcal{R} \cap \mathcal{B} \neq \emptyset$ or until none of the above rules produce a color for a new element.

3.2 Some Diagonal Results Using `SCHAAL`

Included in the package `SCHAAL` is the program `diagdan`, which is a cleaned-up version of `dan` in the case when $q = s$. Using `diagdan` we are able to reprove the main results found in [4] and [5]. However, our program is not designed to reproduce the results in [3], which keeps t as a parameter and confirms the conjecture of Hopkins and Schaal [4] that $R_t(q,q) = tq^2 + (2t^2+1)q + t^3$.

Theorem 3.3 (*Jones and Schaal [5]*) $R_1(q,q) = q^2 + 3q + 1$

Proof. By running `diagdan`$(\{1\}, \{\}, 1, q)$ we find immediately that the elements in $\{1, 2, q, 2q+1, q^2+2q+1\}$ must be both red and blue, a contradiction. □

Theorem 3.4 (*Hopkins and Schaal [4]*) $R_2(q,q) = 2q^2 + 9q + 8$

Proof. By running `diagdan`$(\{1\}, \{q\}, 2, q)$ we find immediately that the elements in $\{q + 2, 2q^2+5q, \frac{1}{2}(q^2+3q)\}$ must be both red and blue. We then run `diagdan`$(\{1,q\}, \{\}, 2, q)$ and find that the elements in $\{2, q+2, 2q, 6q, q^2+6q\}$ must be both red and blue. The program ran for about 10 seconds to obtain this proof. □

3.3 Some Values of $RR_t(q,s)$

We end this paper with some values of $RR_t(q,s)$ for small values of t, q and s.

t	q	s	Value		t	q	s	Value
2	3	2	43		3	5	4	172
2	4	2	50		3	6	4	201
2	5	2	58		3	7	4	214
2	6	2	66		3	8	4	235
2	7	2	74		3	9	4	264
2	8	2	82		3	10	4	277
2	9	2	90		3	6	5	231
2	10	2	98		3	7	5	245
2	4	3	66		3	8	5	269
2	5	3	73		3	9	5	303

Table 1: Small Values of $RR_t(q,s)$

t	q	s	Value		t	q	s	Value
2	6	3	86		3	10	5	317
2	7	3	93		3	7	6	276
2	8	3	106		3	8	6	303
2	9	3	112		3	9	6	330
2	10	3	126		3	10	6	357
2	5	4	88		3	8	7	337
2	6	4	100		3	9	7	381
2	7	4	112		3	10	7	397
2	8	4	124		3	9	8	420
2	9	4	136		3	10	8	437
2	10	4	148		3	10	9	477
2	6	5	122		4	5	4	292
2	7	5	131		4	6	4	324
2	8	5	150		4	7	4	356
2	9	5	159		4	8	4	388
2	10	5	178		4	9	4	432*
2	7	6	150		4	10	4	452
2	8	6	166		4	6	5	370
2	9	6	182		4	7	5	401
2	10	6	198		4	8	5	452
2	8	7	194		4	9	5	473
2	9	7	205		4	10	5	514
2	10	7	230		4	7	6	446
2	9	8	228		4	8	6	492
2	10	8	248		4	9	6	526
2	10	9	282		4	10	6	566
3	4	3	129		4	8	7	556
3	5	3	147		4	9	7	579
3	6	3	165		4	10	7	630
3	7	3	192*		4	9	8	632
3	8	3	201		4	10	8	680
3	9	3	219		4	10	9	746
3	10	3	237		5	11	5	820*

Table 1 cont'd: Small Values of $RR_t(q, s)$

These values were calculated by matching Theorem 2.2's lower bound with the Maple package `SCHAAL`'s upper bound. We use `SCHAAL` by letting 1 be red and then letting 1 be blue. In many cases this is sufficient, however in many of the remaining cases, we must consider subcases depending upon whether 2 is red or blue. If this is still not sufficient, we consider subsubcases depending upon whether the value in Table 1 (in the value column), the integer 3, the integer 4, or the integer 5, is red or blue. This is sufficient for all values in Table 1, expect for those marked with an *. This is because, except for those three values marked with an *, all values agree with the lower bound given by Theorem 2.2. For these three exceptional values, we can increase the lower bound given in Theorem 2.2.

Theorem 3.3 Let $t \geq 3$. Then $R_t(2t+1,t) \geq 6t^3 + 2t^2 + 4t$.

Proof. It is easy to check that the 2-coloring of $[1, 6t^3 + 2t^2 + 4t - 1]$ defined by coloring $\{1, 2, 6t\} \cup \{6t + 3, \ldots, 6t^2 + 2t - 1\} \cup \{6t^2 + 2t \leq i \leq 12t^2 + 4t : i \equiv 0 \,(\mathrm{mod}\ t)\}$ red and its complement blue avoids red solutions to $tx + (2t+1)y = z$ and blue solutions to $tx + ty = z$. (We use $t > 2$ so that $6t$ is the minimal red element that is congruent to 0 modulo t.) $\square$

Remark. The lower bound in the above theorem is not tight. For example, when $t = 6$, the 2-coloring of $[1, 1392]$ given by coloring $\{1, 2, 3, 37, 39, 40, 41, 43, 46, 47, 48, 49, 50, 52, 56\} \cup [58, 228] \cup \{234 \leq i \leq 558 : i \equiv 0 \,(\mathrm{mod}\ 6)\} \cup \{570, 576, 594, 606, 612, 648, 684\}$ red and its complement blue avoids red solutions to $6x + 13y = z$ and blue solutions to $6x + 6y = z$. Hence, $RR_t(2t + 1, t) > 6t^3 + 2t^2 + 4t$ for $t = 6$.

We are unable to explain why $(b, c) = (2t + 1, t)$ produces these "anomalous" values while others, e.g., $(b, c) = (2t - 1, t)$, appear not to do so.

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