On self-complementary cyclic packing of forests *

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Abstract

A graph is *self-complementary* if it is isomorphic to its complement. In this paper we prove that every forest of order $4p$ and size less than $3p$ is a subgraph of a self-complementary graph of order $4p$ with a cyclic self-complementary permutation. We also discuss some generalization of the main result.

Keywords: self-complementary graphs, permutation (structure), forest.

1 Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$. All graphs will be assumed to have neither loops nor multiple edges. For $W \subset V(G)$ we will denote by $G - W$ the graph obtained from G by removing vertices of W. A graph G is *self-complementary* (briefly, s-c) if it is isomorphic to its complement (cf. [4], [5], or [2]). It is clear that an s-c graph has $n \equiv 0, 1 \pmod 4$ vertices. A *self-complementary* permutation is a permutation which transforms one copy of a self-complementary graph into another. Ringel ([4]) and Sachs ([5]), independently, proved that a self-complementary permutation consists of cycles of lengths that are multiples of 4, except for one cycle of length one when $n \equiv 1 \pmod 4$. The following has been observed in [5].

Remark 1 *If σ is a self-complementary permutation of G then every odd power of σ is a self complementary permutation of G (while every even power of σ is an automorphism of G).*

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A sufficient condition for a graph to be a subgraph of a self-complementary graph was proved in [1].

Lemma 2 *Let $H = (V(H), E(H))$ be a graph of order $n \in \{4p, 4p+1\}$ and let σ be a permutation of its vertex set, such that every orbit of σ has a multiple of four vertices except, possibly, of one fix vertex in odd case. If for every edge $xy \in E$ we have $\sigma^{2k+1}(x)\sigma^{2k+1}(y) \notin E$ for every $k = 0, 1, ..., 2p-1$, then H is a subgraph of a self-complementary graph with self-complementary permutation σ.* ■

An *embedding* of G (in its complement $\overline{G}$) is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. In others words, an embedding is an (edge-disjoint) *placement* (or *packing*) of two copies of G (of order n) into the complete graph K_n. It is evident that subgraphs of self-complementary graphs of the same order are embeddable. The relationship between the property "to be embeddable" and the property "to be a subgraph of a self-complementary graph of the same order" was discussed in [1], [8], [9]. The structure of packing permutations was also studied in [3], [7] and [10].

We consider the special structure of self-complementary permutations. By Theorem 6, the expectation that a graph is a subgraph of a self-complementary graph H of the same order rises with the number of cycles of s-c permutation of H.

2 Main result

We think that the following conjecture may be true.

Conjecture 3 *Every graph G of order at most $n = 4p$ and size less then $\frac{3}{4}n = 3p$ is a subgraph of a self-complementary graph of order n with cyclic self-complementary permutation.*

We shall prove a result which gives some support to Conjecture 3.

Theorem 4 *Let $n = 4p$ and let F be a forest of order at most $n = 4p$ and size less then $\frac{3}{4}n = 3p$. Then F is a subgraph of a self-complementary graph H of order n with a cyclic self-complementary permutation.*

By Lemma 2, we obtain that if the star $K_{1,k}$ is a subgraph of a self-complementary graph of order $4p$ with a cyclic self-complementary permutation then $k \leq 3p-1$. Thus the star $K_{1,3p}$ is not a subgraph of any self-complementary graph of order $4p$ with cyclic permutation, and Theorem 4 is sharp, in a sense. Note also that an s-c graph may have two different s-c permutations. For example, (112345678) and (1278)(3456) are s-c permutations of the graph depicted in Fig. 1.

In fact, we shall prove that, for every $n = 4p$ there is a universal s-c graph of order n containing every forest of order at most $4p$ and size less then $3p$.

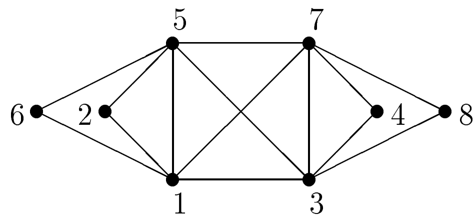

Figure 1: s-c graph with s-c permutations (12345678) and (1278)(3456)

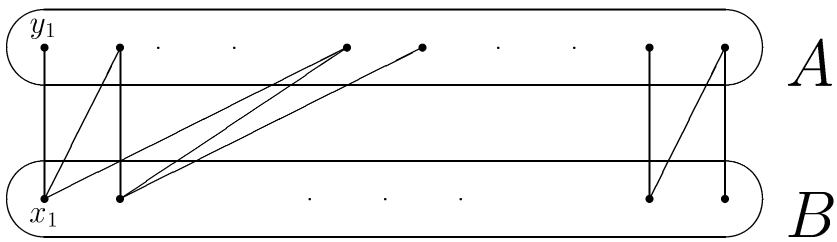

Figure 2: Graph F_{4p}

We need some additional definitions. Let H_{4p} be the graph defined in the following way. The vertices of H_{4p} are the numbers from 1 to $4p$. The even vertices form a clique. Additionally, each even vertex x is joined by an edge to p odd vertices $x + 1 \bmod 4p, x + 3 \bmod 4p, \ldots, x + 2p - 1 \bmod 4p$.

It is easy to see that H_{4p} is a self-complementary graph and that the corresponding packing permutation is cyclic (namely: $\sigma = (1234\ldots 4p)$). Let F_{4p} be the graph with the vertex set $A \cup B$ where $A = A(F_{4p}) = \{y_1, y_2, \ldots, y_{2p}\}$, $B = B(F_{4p}) = \{x_1, x_2, \ldots, x_{2p}\}$ drawn as in Fig. 2. The *left-hand side* L of the graph F_{4p} is the set of vertices $L = \{x_1, ..., x_p, y_1, ..., y_p\}$ and the *right-hand side* is $R = \{x_{p+1}, ..., x_{2p}, y_{p+1}, ..., y_{2p}\}$. The edges are defined as follows. The set $B = B(F_{4p})$ is a clique. Moreover, each vertex x_i is connected to the vertex y_i as well as to the vertices y_{i+k} for $1 \leq k < p$ if $i + k \leq 2p$. In particular, the vertex y_{2p} is the only neighbour of x_{2p} in $A = A(F_{4p})$.

It is immediate that F_{4p} is a subgraph of H_{4p}.

Theorem 5 *Let $n = 4p$ and let F be a forest of order at most $n = 4p$ and size less then $\frac{3}{4}n = 3p$. Then F is a subgraph of F_{4p}.*

3 Proof

The proof is by induction on p. It is not difficult to check that the theorem is true for $p = 1$ and $p = 2$. Assume that our theorem is true for a fixed $p \geq 2$. We shall show that it holds also for $n = 4(p + 1)$. Let F be a forest having $4(p + 1)$ vertices and at most $3(p + 1) - 1 = 3p + 2$ edges. We shall consider several cases. In each case we shall remove from F four vertices and at least three edges. Denote by F' the obtained forest. By induction, we can consider it as a subgraph of the graph F_{4p}. The graph F_{4p+4} will be drawn as a graph F_{4p} with four additional vertices f_1, f_2, f_3, f_4 (f_1, $f_4 \in A(F_{4p+4})$, f_2, $f_3 \in B(F_{4p+4})$) placed in the proper way. It is sufficient now to determine where the four removed from F vertices may be put.

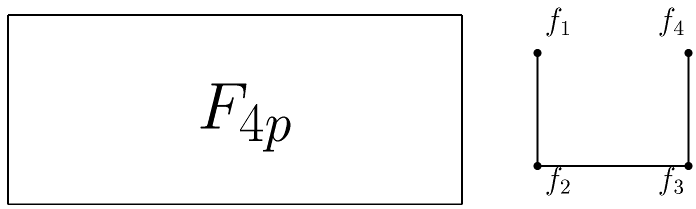

Figure 3: Case **1**:$T = P_4$

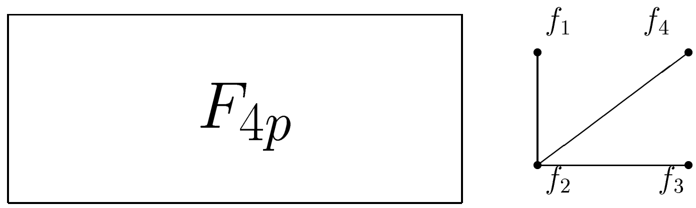

Figure 4: Case **1**:$T = K_{1,3}$

Case 1. The forest F has a component T of order 4.

Set $F' = F - V(T)$. Observe that T is the path of order 4 or the star $K_{1,3}$. Details of putting vertices of T on vertices f_1,...,f_4 are given in Fig. 3 and 4.

Case 2. The forest F has a component T of order 3.

It is obvious that if F consists of the tree T and isolated vertices then F is a subgraph of F_{4p+4}. Hence we may assume that there is a vertex $l \in V(F) - V(T)$ such that $d(l) = 1$ and the vertex $v \in V(F)$ such that v is the only neighbour of l. Let $F' = F - V(T) - \{l\}$.

Let us first suppose that the vertex v is in the set $A(F_{4p})$. If v is in the set $L \cap A(F_{4p})$ then we can put it on the vertex f_3 and we can put vertices of T on vertices f_1, f_2, f_4 (Fig. 5). Let us suppose that v is in the set $R \cap A(F_{4p})$. Details of addition vertices f_1,...,f_4 in this subcase are given in Fig. 6. Since $p - 2 \geq 1$, we can form F_{4p+4} in this way. Then we put the vertex l on f_3 and vertices of T on f_1, f_2, f_4.

If the vertex v is in the set $B(F_{4p})$ then the vertex l can be put on the vertex f_3 and vertices of T can be put on vertices f_1, f_2, f_4 (Fig. 7).

Case 3. The forest F has a component of order 2.

By Cases 1 and 2 we may assume that no tree of order 4 or 3 is a component of F. It is obvious that if every tree of F is an isolated vertex or a tree of order 2 (an isolated edge) then F is a subgraph of F_{4p+4}. Thus we may assume that there is the tree $\tilde{T}$ of order at least 5 as a component of F.

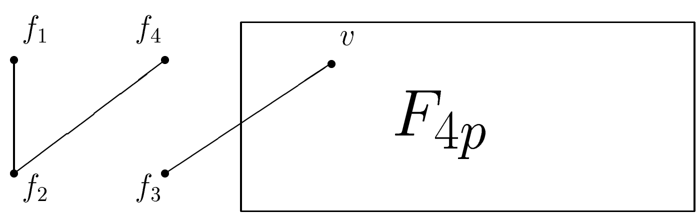

Figure 5: Case **2**: $v \in L \cap A(F_{4p})$

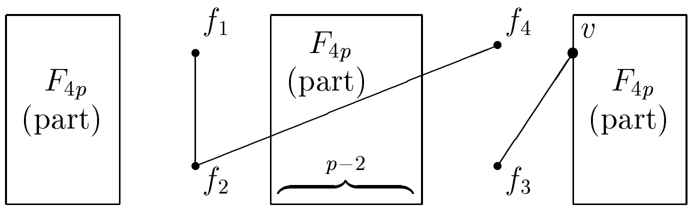

Figure 6: Case **2**:$v \in R \cap A(F_{4p})$

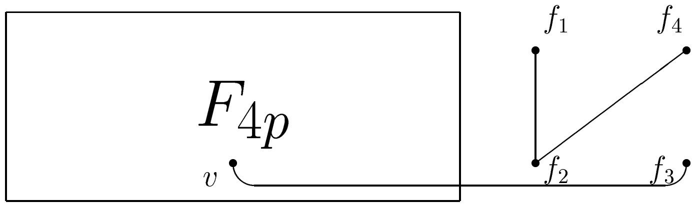

Figure 7: Case **2**:$v \in B(F_{4p})$

Subcase 3.1. There is a vertex of $\tilde{T}$ with at least two leaves as neighbours.

Set l_1, l_2, $v \in V(\tilde{T})$ such that $d(l_1) = d(l_2) = 1$ and v is the only neighbour of l_1 and l_2. In this subcase set $F' = F - V(T) - \{l_1, l_2\}$.

Let us first suppose that v is in the set $A(F_{4p})$. Observe that every neighbour of v in $\tilde{T} - \{l_1, l_2\}$ is in the set $B(F_{4p})$. Thus we can change the place of v by putting it on f_2. Then we put l_1, l_2 on f_1 and the place just left by v. We put vertices of T on f_3 and f_4 (Fig. 8).

Hence we may assume that v is in the set $B(F_{4p})$. Then we form F_{4p+4} by adding vertices f_1,...,f_4 as in Fig. 9. We put the leaves l_1, l_2 on f_1, f_2 and vertices of T on f_3, f_4.

Subcase 3.2. No two leaves of $\tilde{T}$ have a common neighbour.

Observe that there are vertices l, $v \in V(\tilde{T})$ such that $d(l) = 1$, $d(v) = 2$ and v is the only neighbour of l. Let w be the second neighbour of v. In this subcase let $F' = F - V(T) - \{l, v\}$. We put the vertices of T on f_3 and f_4 and the vertices l and v on f_1 and f_2, respectively. Details are given in Fig. 10 (the vertex w is in $A(F_{4p})$) and Fig. 11 (the vertex w is in $B(F_{4p})$).

Case 4. Every tree of F is either an isolated vertex or has at least 5 vertices.

Observe that F has at least $p + 1$ components. Since F has $4p$ vertices, there is an isolated vertex $u \in V(F)$. If F consists of isolated vertices then F is a subgraph of F_{4p+4}. Thus we may assume that there is a component T of order at least 5. Then there are five

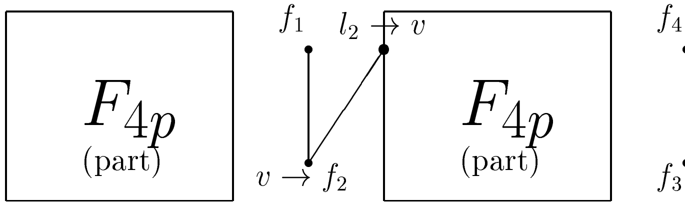

Figure 8: Subcase **3.1**: $v \in A(F_{4p})$

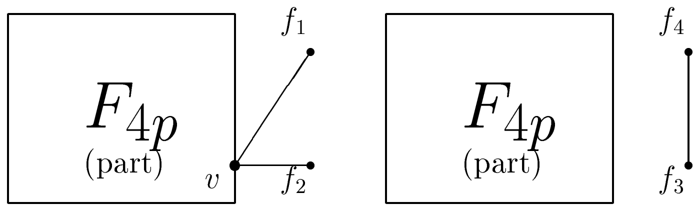

Figure 9: Subcase **3.1**: $v \in B(F_{4p})$

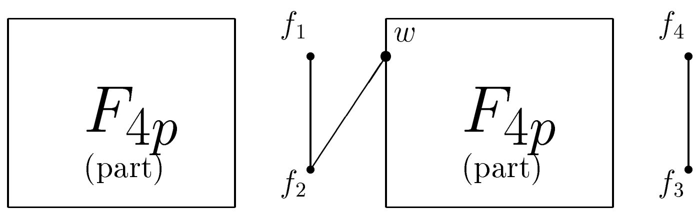

Figure 10: Subcase **3.2**: $w \in A(F_{4p})$

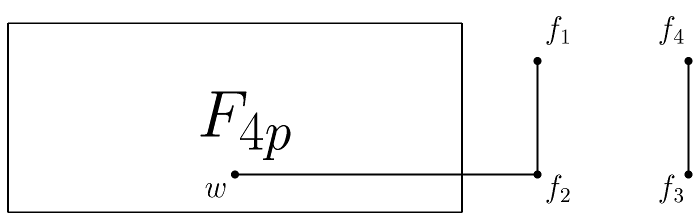

Figure 11: Subcase **3.2**:$w \in B(F_{4p})$

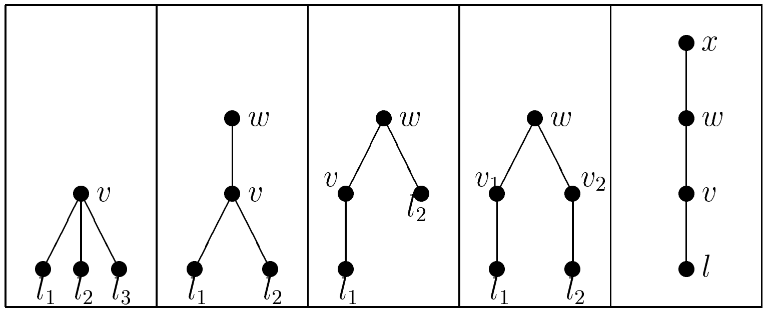

Figure 12: Five cases.

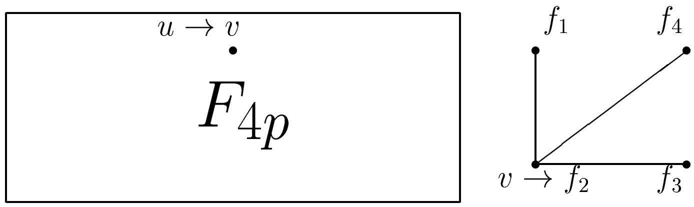

Figure 13: Subcase **4.1**:$v \in A(F_{4p})$

subcases given in Fig. 12.

Subcase 4.1. There is a vertex of T with at least three leaves as neighbours.

Set l_1, l_2, l_3, $v \in V(T)$ such that $d(l_1) = d(l_2) = d(l_3) = 1$ and v is the only neighbour of l_1, l_2 and l_3. In this subcase set $F' = F - \{l_1, l_2, l_3, u\}$.

Let us suppose that v is in the set $A(F_{4p})$. Then every neighbour of v in $T - \{l_1, l_2, l_3\}$ is in the set $B(F_{4p})$. Thus we can change the place of v by putting it on f_2. Then we put vertices l_1, l_2 and l_3 on vertices f_1, f_3 and f_4. The vertex u is put on the place just left by v (Fig. 13).

Hence we can assume that v is in the set $B(F_{4p})$. We put the vertices l_1, l_2 and l_3 on f_1, f_2 and f_3. The vertex u is put on f_4 (Fig. 14).

Subcase 4.2. There is a vertex of degree 3 in T with exactly two leaves as neighbours.

Set l_1, l_2, $v \in V(T)$ such that $d(l_1) = d(l_2) = 1$ and v is the only neighbour of l_1 and l_2. Then $d(v) = 3$. Let w denote the third neigbour of v. In this subcase set $F' = F - \{l_1, l_2, v, u\}$. If w is in the set $A(F_{4p})$ then we can change the place of w by putting it on f_3 and then the vertex v is put on f_2, vertices l_1, l_2 are put on f_1, f_4 and the vertex u is put on the place just left by w (Fig. 15). When w is in the set $B(F_{4p})$

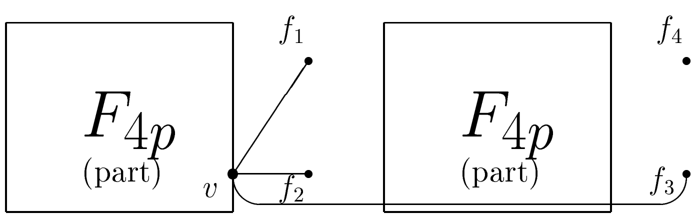

Figure 14: Subcase **4.1**: $v \in B(F_{4p})$

Figure 15: Subcase **4.2**:$w \in A(F_{4p})$

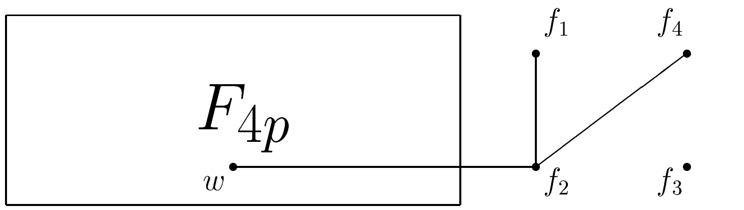

Figure 16: Subcase **4.2**:$w \in B(F_{4p})$

details of putting vertices l_1, l_2, v, u are given in Fig. 16.

Subcase 4.3. There are vertices $l_1, l_2, v, w \in V(T)$ such that $d(l_1) = d(l_2) = 1$, $d(v) = 2$, the vertex v is the only neighbour of l_1, the vertex w is a common neighbour of l_2 and v.

Set $F' = F - \{l_1, l_2, v, u\}$. Details of putting vertices l_1, l_2, v, u are given in Fig. 17 (when w is in $A(F_{4p})$) and in Fig. 18 (when w is in $B(F_{4p})$). Observe that when w is in the set $A(F_{4p})$ we can change the place of w by putting it on f_3 and put the vertex u on the place just left by w.

Subcase 4.4. There are vertices l_1, l_2, v_1, v_2, $w \in V(T)$ such that $d(l_1) = d(l_2) = 1$, $d(v_1) = d(v_2) = 2$, v_i is the only neighbour of l_i for $i = 1, 2$ and w is a common neighbour of v_1, v_2.

Set $F' = F - \{l_1, l_2, v_1, u\}$.

Let us suppose that w is in the set $A(F_{4p})$. Then every neighbour of w in $T - \{l_1, l_2, v_1\}$ is in the set $B(F_{4p})$. In particular $v \in B(F_{4p})$. We can change the place of w by putting it on f_3 and then we can put l_2 on the place just left by w (Fig. 19).

Thus we may assume that w is in the set $B(F_{4p})$. We can change the place of v_2 by putting it on f_3 and then we can put u on the place just left by v_2. We put vertices l_1, v_1, l_2 on vertices f_1, f_2 and f_4, respectively (Fig. 20).

Subcase 4.5. There are vertices l, v, w, $x \in V(T)$ such that $d(l) = 1$, $d(v) = d(w) = 2$, the vertex v is the only neighbour of l, the vertex w is a common neighbour of vertices v

Figure 17: Subcase **4.3**:$w \in A(F_{4p})$

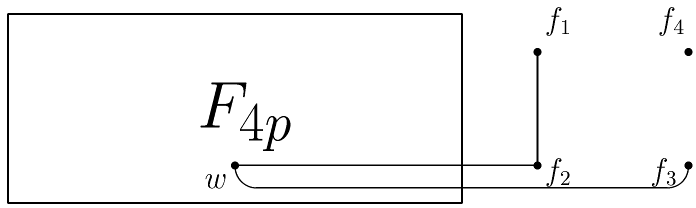

Figure 18: Subcase **4.3**:$w \in B(F_{4p})$

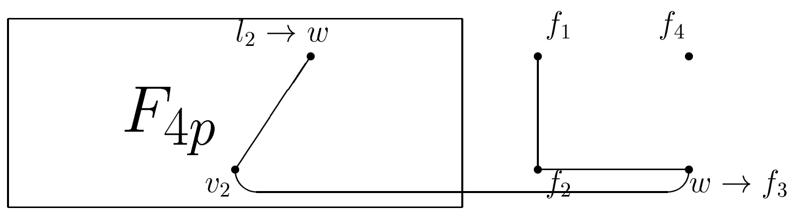

Figure 19: Subcase **4.4**:$w \in A(F_{4p})$

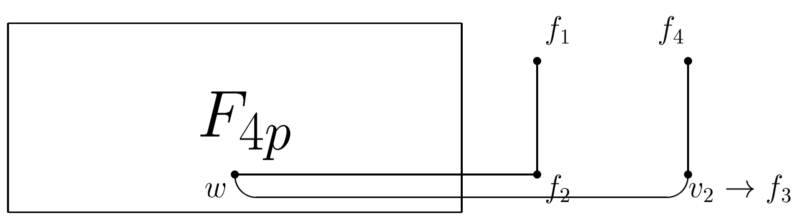

Figure 20: Subcase **4.4**:$w \in B(F_{4p})$

Figure 21: Subcase **4.5**:$x \in A(F_{4p})$

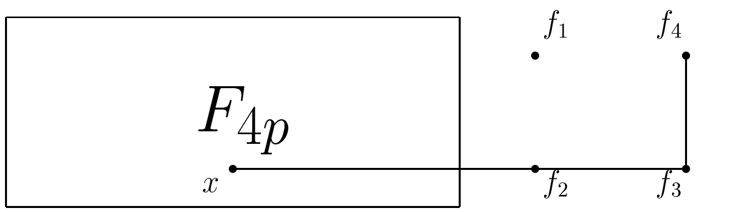

Figure 22: Subcase **4.5**:$x \in B(F_{4p})$

and x.

Set $F' = F - \{l, v, w, u\}$. Details of putting vertices l, v, w, u are given in Fig. 21 (when x is in $A(F_{4p})$) and Fig. 22 (when x is in $B(F_{4p})$). Observe that when x is in $A(F_{4p})$ we can change the place of x by putting it on f_3 and then put the vertex u on the place just left by x.

■

4 Generalizations

Theorem 6 *Let p and q be integers such that $p \geq 1$, $q \geq 0$ and let F be a forest of order at most $4p + 4q$ and size less then $3p + 4q$. Then F is a subgraph of a self-complementary graph H of order $n = 4(p+q)$, such that a self-complementary permutation of H has $q+1$ cycles, q of which having length four.*

Proof. The proof is by induction on q. For $q = 0$ Theorem 6 is exactly Theorem 4. Assuming that the theorem holds for an integer $q \geq 0$ we shall prove it for $q + 1$.
Let F be a forest of order $4p + 4(q + 1)$ and size at most $3p + 4(q + 1) - 1$. It is obvious that we can assume that F does not consists of only isolated vertices.

Let us first suppose that at least one of cases holds:
I. F has a component T of order at least 2 which is neither a star nor a path on 4 vertices.
II. Two components T_1, T_2 of F are trees of order at least 2 such that $T_1 \cup T_2$ is not the union of an isolated edge and a path (including an isolated edge).

In both cases there are four vertices: either l_1, v_1, l_2, $v_2 \in V(T)$ or l_1, $v_1 \in V(T_1)$, l_2, $v_2 \in V(T_2)$, respectively, such that $d(l_1) = d(l_2) = 1$, v_i is the only neighbour of l_i, $i = 1$, 2 and vertices v_1, v_2 cover at least four edges. Set $F' = F - \{l_1, l_2, v_1, v_2\}$. By induction hypothesis F' is contained in an s-c graph of order $4(p + q)$ with an s-c permutation σ'

having $q+1$ cycles, q of which of length four. By Lemma 2 and Remark 1 the permutation $\sigma = \sigma' \circ (v_1 l_1 v_2 l_2)$ is an s-c permutation with required properties.

We may assume that none of cases I, II holds. Then we obtain five possibilities:
i) F is the union of a star (including an isolated edge) and isolated vertices,
ii) F is the union of the path of order 4 and isolated vertices,
iii) F is the union of at least two isolated edges and possibly isolated vertices,
iv) F is the union of $K_{1,2}$, at least one isolated edge and possibly isolated vertices,
v) F is the union of the path of order 4, at least one isolated edge and possibly isolated vertices.

It is easily seen, by Lemma 2 and Remark 1, that in possibility i) and ii) the forest F is a subgraph of required s-c graph of order $4(p+q+1)$.

Let us consider possibilities iii), iv) and v). Since $|V(F)| = 4p + 4(q+1)$, $|E(F)| \leq 2p + 2q + 2$ in iii), iv) and $|E(F)| \leq 2p + 2q + 3$ in v). Let T_1, T_2 be two trees of F such that T_1, T_2 are isolated edges in iii) and $T_1 = K_{1,2}$, T_2 is an isolated edge in iv) and T_1 is the path of order 4, T_2 is an isolated edge in v). There are four vertices l_1, $v_1 \in V(T_1)$, l_2, $v_2 \in V(T_2)$ such that $d(l_1) = d(l_2) = d(v_2) = 1$, v_1 is the only neighbour of l_1. Set $F' = F - \{l_1, v_1, l_2, v_2\}$. Then $|E(F')| \leq 2p + 2q$ in iii), $|E(F')| \leq 2p + 2q - 1$ in iv), v). Thus in every possibility iii), iv), v) F' verifies the assumptions of the theorem for q. By induction hypothesis F' is contained by s-c graph of order 4(p+q) with an s-c permutation σ' having $q+1$ cycles, q of which of length four. Then $\sigma = \sigma' \circ (v_1 l_1 v_2 l_2)$ is an s-c permutation with required properties. ■

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