Using Lovász Local Lemma in the space of random injections

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Abstract

The Lovász Local Lemma is known to have an extension for cases where independence is missing but negative dependencies are under control. We show that this is often the case for random injections, and we provide easy-to-check conditions for the non-trivial task of verifying a negative dependency graph for random injections. As an application, we prove existence results for hypergraph packing and Turán type extremal problems. A more surprising application is that tight asymptotic lower bounds can be obtained for asymptotic enumeration problems using the Lovász Local Lemma.

1 Introduction

The Lovász Local Lemma is perhaps one of the most powerful probabilistic tools in combinatorics, which has numerous applications, in addition to combinatorics, in number theory and computer science.

When dependencies of the events are rare, the Lovász Local Lemma provides a general way of proving that with a positive (though tiny) probability, none of the events occur. In some cases an efficient algorithm has been found for finding elements of this tiny event [4]. The main contribution of this paper is to use the Lovász Local Lemma in a space with rich dependencies, in the set of random injections between two sets.

Let $A_1, A_2, \ldots, A_n$ be events in a probability space Ω. A graph G on vertices $[n]$ is called a *dependency graph* of the events A_i's if A_i is mutually independent of all A_j with $ij \notin E(G)$.

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Lemma 1 Lovász Local Lemma (first version) [9] *For each* $1 \leq i \leq n$*, suppose the event* A_i *satisfies* $\Pr(A_i) \leq p$*, and assume a dependency graph* G *is associated with these events. Assume that* d *is an upper bound for the degrees in* G*. If* $e(d+1)p < 1$*, then* $\Pr(\wedge_{i=1}^{n}\overline{A_i}) > 0$*. is positive.*

Here is a more general second version, Lemma 2, which implies Lemma 1 by setting $x_i = \frac{1}{d+1}$:

Lemma 2 Lovász Local Lemma (second version) [2] p. 64. *Let* $A_1, \ldots, A_n$ *be events with dependency graph* G*. If there exist numbers* $x_1, \ldots, x_n \in [0,1)$ *such that*

$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j).$$

for all i*, then*

$$\Pr(\wedge_{i=1}^{n}\overline{A_i}) \geq \prod_{i=1}^{n}(1 - x_i) > 0.$$

Going to further generality, a *negative dependency graph* for $A_1, \ldots, A_n$ is a simple graph on $[n]$ satisfying

$$\Pr(A_i | \wedge_{j \in S} \overline{A_j}) \leq \Pr(A_i), \tag{1}$$

for any index i and any subset $S \subseteq \{j \mid ij \notin E(G)\}$, if the conditional probability $\Pr(\wedge_{j \in S}\overline{A_j})$ is well-defined, i.e. > 0 (in [10], the terminology was *lopsidependency graph*). We will make use of the fact that Equation (1) trivially holds when $\Pr(A_i) = 0$, otherwise the following equation is equivalent to Equation (1):

$$\Pr(\wedge_{j \in S}\overline{A_j} \mid A_i) \leq \Pr(\wedge_{j \in S}\overline{A_j}). \tag{2}$$

Note that if A_i is mutually independent of A_j for $j \in S$, then we have

$$\Pr(A_i | \wedge_{j \in S} \overline{A_j}) = \Pr(A_i).$$

Thus, the dependency graphs always can be considered as negative dependency graphs.

Lemma 3 Lovász Local Lemma (third version) [10], or [2] p. 65. *Let* $A_1, \ldots, A_n$ *be events with a negative dependency graph* G*. If there exist* $x_1, \cdots, x_n \in [0,1)$ *with*

$$\Pr(A_i) \leq x_i \prod_{ij \in E(G)} (1 - x_j) \tag{3}$$

for all i*, then*

$$\Pr(\wedge_{i=1}^{n}\overline{A_i}) \geq \prod_{i=1}^{n}(1 - x_i) > 0. \tag{4}$$

Note that one easily obtains a version of Lemma 1 for the case of the negative dependency graph from Lemma 3 by setting $x_i = \frac{1}{d+1}$. For historical accuracy, [10] described the proof of Lemma 3 in this special setting. The manuscript [14] available on the web gives a detailed proof to Lemma 3. A variant of Lemma 3 has been proved in [1].

The main obstacle for using Lemma 3 is the difficulty to define a useful negative dependency graph other than a dependency graph. In this paper, we will consider the probability space over random injections. Let U and V be two finite sets with $|U| \leq |V|$. Consider the probability space $\Omega = I(U, V)$ of all injections from U to V equipped with a uniform distribution. We are going to provide a criterion for defining the negative dependency graph. We give applications of this criterion in permutation enumeration, hypergraph packing, and Turán type extremal problems.

We do not prove any *new* result on permutation enumeration, our point is that we are not aware of *any* previous application of the Lovász Local Lemma in this direction. Our proofs suggest that this possibility is there.

Both for hypergraph packing problems and Turán type extremal problems, the literature mostly focuses on best estimates for particular hypergraphs. Here we give *very general* bounds that are close to optimal in their general setting, and at the same time, are not very far from the best estimates for particular hypergraphs, when we apply the general setting for them.

For example, for any fixed bipartite graph G on s vertices and any graph H on n vertices, Alon and Yuster [3] proved that for sufficiently large n, H can be covered by vertex-disjoint copies of G if the minimum degree of H is at least $(\frac{1}{2} + \epsilon)n$ and s divides n. We obtain a general (but weaker) result (Theorem 3) on perfect packing problem for any hypergraph G.

For Turán type extremal problems, likewise, the literature focuses on particular excluded sub-hypergraphs, like $K_{r+1}^{(r)}$. The few general results available are about the number of edges [16], [12]. Our general results are about excluded sub-hypergraphs, in which every edge meets few other edges, and as before, our estimate is near tight when applied to the well-studied $K_{r+1}^{(r)}$.

The paper is organized as follows. In section 2, we prove our main theorem. We extend the Lovász Local Lemma to the space of random injections by establishing a simple criterion for defining the negative dependency graph. In section 3, we apply our main theorem to asymptotic permutation enumeration. We study the packing problem for any two hypergraphs in section 4 and the perfect packing problem in section 5. The last application on Turán type extremal problems will be given in section 6.

In a follow-up paper under preparation, we will show that Lemma 3 applies to the uniform probability space of perfect matchings of K_{2n} with a proper definition of the negative dependency graph; and we will also show how many of our asymptotically tight lower bounds can be turned actually into an asymptotic formula.

2 Main result

To state our result, we will use the following notations. Every injection from U to V can be viewed as a saturated matching of complete bipartite graph with partite sets U and V. In this sense, we define a matching to be a triple (S, T, f) satisfying

1. S is the subset of U and T is a subset of V.

2. The map $f : S \to T$ is a bijection.

We denote the set of all such matchings by $M(U, V)$. Note that the elements of $M(U, V)$ are partial functions from U to V that are injections, and $I(U, V) \subseteq M(U, V)$.

For any permutation ρ of V we define the map $\pi_\rho : M(U, V) \to M(U, V)$ as follows: For any $g \in M(U, V)$ for all $u \in U$

$$\pi_\rho(g)(u) = \rho(g(u))$$

Clearly for a matching $g_1 = (S_1, T_1, f_1)$ if $\pi_\rho(g_1) = g_2 = (S_2, T_2, f_2)$ then $S_1 = S_2$. Moreover, if T_1 consists of fixpoints of ρ (i.e. $\rho(v) = v$ for all $v \in T_1$) then $g_2 = g_1$.

Two matchings (S_1, T_1, f_1) and (S_2, T_2, f_2) are said to *conflict* each other if either "$\exists k \in S_1 \cap S_2, f_1(k) \neq f_2(k)$" or "$\exists k \in T_1 \cap T_2, f_1^{-1}(k) \neq f_2^{-1}(k)$". In other words, two matchings do not conflict each other if and only if their union (as a graph) is still a matching.

For a given matching (S, T, f), we define the event $A_{S,T,f}$ as

$$A_{S,T,f} = \{\sigma \in I(U, V) |\ \sigma(i) = f(i), \forall i \in S\}.$$

An event $A \in I(U, V)$ is called to be *canonical* if $A = A_{S,T,f}$ for a matching (S, T, f). Two canonical events conflict each other if their associated matchings conflict. Note that if two events conflict each other, then they are disjoint.

Note that for any permutation ρ of V, and any matching (S, T, f), if $\pi_\rho((S, T, f)) = (S, T', f')$ then $\pi_\rho(A_{S,T,f}) = A_{S,T',f'}$.

We establish a sufficient condition for negative dependency graphs for the space of random injections by showing the following theorem.

Theorem 1 *Let $A_1, A_2, \ldots, A_n$ be canonical events in $I(U, V)$. Let G be the graph on $[n]$ defined as*

$$E(G) = \{ij \mid A_i \text{ and } A_j \text{ conflict}\}.$$

Then G is a negative dependency graph for the events $A_1, \ldots, A_m$.

Proof: We are supposed to show the inequality (1). If the condition $\wedge_{j\in S}\overline{A_j}$ has probability zero, then there is nothing to prove. So assume $\Pr(\wedge_{j\in S}\overline{A_j}) > 0$.

By (2), it suffices to show that for any index i and any set $J \subseteq \{j : A_i$ and A_j does not conflict$\}$,

$$\Pr(\wedge_{j\in J}\overline{A_j} \mid A_i) \leq \Pr(\wedge_{j\in J}\overline{A_j}). \tag{5}$$

For $1 \le k \le m$, let (S_k, T_k, f_k) be the corresponding matching of the event A_k. We first prove the following claim.

Claim: For any matching (S_i, T, f),

$$\Pr(A_{S_i,T,f}) = \Pr(A_{S_i,T_i,f_i}). \tag{6}$$

Moreover, if $J \subseteq \{j : A_i \text{ and } A_j \text{ does not conflict}\}$, we have

$$\Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T,f}) \ge \Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T_i,f_i}). \tag{7}$$

Proof of Claim: Fix a matching (S_i, T, f). Let J' be the set of indices $j \in J$ so that A_j does not conflict $A_{S_i,T,f}$. Clearly

$$(\wedge_{j\in J}\overline{A_j})A_{S_i,T,f} = (\wedge_{j\in J'}\overline{A_j})(\wedge_{j\in J\setminus J'}\overline{A_j})A_{S_i,T,f}.$$

If $j \in J \setminus J'$, then A_j conflicts to $A_{S_i,T,f}$, and so $A_{S_i,T,f} \subseteq \overline{A_j}$. Therefore

$$\overline{A_j}A_{S_i,T,f} = A_{S_i,T,f}.$$

Thus, whether $J \setminus J'$ is empty or not, we have

$$(\wedge_{j\in J\setminus J'}\overline{A_j})A_{S_i,T,f} = A_{S_i,T,f},$$

from which it follows that

$$(\wedge_{j\in J}\overline{A_j})A_{S_i,T,f} = (\wedge_{j\in J'}\overline{A_j})A_{S_i,T,f}. \tag{8}$$

Let $\rho : V \to V$ be a bijection satisfying the following: $\rho(v) = v$ for any $v \in \cup_{j\in J'}T_j$ and for $w \in T$, $\rho(w) = f_i(f^{-1}(w))$. By the definition of J' we have that for each $j \in J'$ if $u \in S_i \cap S_j$ then $f(u) = f_i(u) = f_j(u)$, therefore such a ρ clearly exists. Moreover, for each $j \in J'$, T_j consists of fixpoints of ρ, $\rho(T) = T_i$, and for $u \in S_i$, $\rho(f(u)) = f_i(u)$.

This implies that $\pi_\rho((S_i, T, f)) = (S_i, T_i, f_i)$, from which equation (6) follows. Also for each $j \in J'$ we have $\pi_\rho((S_j, T_j, f_j)) = (S_j, T_j, f_j)$. Thus, for each $j \in J'$

$$\pi_\rho(\overline{A_j}A_{S_i,T,f}) = \overline{A_j}A_{S_i,T_i,f_i}, \tag{9}$$

from which

$$\pi_\rho((\wedge_{j\in J'}\overline{A_j})A_{S_i,T,f}) = (\wedge_{j\in J'}\overline{A_j})A_{S_i,T_i,f_i} \tag{10}$$

Using equations (8) and (10) we obtain

$$\begin{aligned}
\Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T,f}) &= \Pr((\wedge_{j\in J'}\overline{A_j})A_{S_i,T,f})) \\
&= \Pr((\wedge_{j\in J'}\overline{A_j})A_{S_i,T_i,f_i}) \\
&\ge \Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T_i,f_i}).
\end{aligned}$$

The proof of the claim is finished.

For the fixed set S_i, the collection of events $\{A_{S_i,T,f} \mid (S_i,T,f) \text{ is a matching}\}$ forms a partition of the space $\Omega = I(U,V)$.

From this partition and equations (6) and (7) we get

$$\begin{aligned}
\Pr(\wedge_{j\in J}\overline{A_j}) &= \sum_{(S_i,T,f)} \Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T,f}) \\
&\geq \sum_{(S_i,T,f)} \Pr((\wedge_{j\in J}\overline{A_j})A_{S_i,T_i,f_i}) \\
&= \sum_{(S_i,T,f)} \Pr(\wedge_{j\in J}\overline{A_j} \mid A_{S_i,T_i,f_i})\Pr(A_{S_i,T_i,f_i}) \\
&= \sum_{(S_i,T,f)} \Pr(\wedge_{j\in J}\overline{A_j} \mid A_{S_i,T_i,f_i})\Pr(A_{S_i,T,f}) \\
&= \Pr(\wedge_{j\in J}\overline{A_j} \mid A_{S_i,T_i,f_i}).
\end{aligned}$$

□

3 Asymptotic Permutation Enumeration

There is a well-known asymptotic formula for the number of fixed-point-free permutations of n elements (or derangements of n elements), $n!/e$. Surprisingly, the Lovász Local Lemma gives this asymptotic formula as lower bound. Let us be given a set U of n elements.

To apply Theorem 1, set $V = U$, for $i \in U$ set $S_i = T_i = \{i\}$, define $f_i : S_i \to T_i$ by $i \mapsto i$. Set $A_i = A_{S_i,T_i,f_i}$ and observe that A_i consists of permutations that fix i. We will use empty negative dependency graph, i.e. $E(G) = \emptyset$.

For the purposes of Lemma 3 select $x_i = 1/n$. This choice is allowed, as $\Pr(A_i) = 1/n$ and the product in (3) is empty. The conclusion is that $\Pr(\wedge_i\overline{A_i}) \geq (1-\frac{1}{n})^n$, and this number converges to $1/e$.

Going further, it is known that the probability of a random permutation not having any k-cycle is asymptotically $e^{-1/k}$ [5]. With some effort, we can get this as a lower bound for the probability from Lovász Local Lemma. Let us be given a set U of n elements. To apply Theorem 1, set $V = U$, for $I \subset U$, $|I| = k$ set $S_I = T_I = I$; and consider all $f_I^\alpha : I \to I$ functions ($\alpha = 1, 2, ..., (k-1)!$) that correspond to a k-cycle on I.

Define the event $A_{I,\alpha} = A_{I,I,f_I^\alpha}$. The vertices of the negative dependency graph will be the $\binom{n}{k}(k-1)!$ events, and we join $A_{I,\alpha}$ with $A_{J,\beta}$ if $I \cap J \neq \emptyset$. Every degree in the negative dependency graph is bounded by $\left[\binom{n}{k} - \binom{n-k}{k}\right](k-1)! = \frac{(n)_k}{k} - \frac{(n-k)_k}{k}$, where $(n)_k$ is the falling factorial notation. For the purposes of Lemma 3 select $x_{I,\alpha} = x = \frac{1+\frac{c_k}{n}}{(n)_k}$, Note that $\Pr(A_{I,\alpha}) = \frac{1}{(n)_k}$, and a little calculation shows that

$$\frac{1}{(n)_k} \leq x(1-x)^{\frac{(n)_k}{k} - \frac{(n-k)_k}{k}} \tag{11}$$

by the definition of x, if c_k is sufficiently large, due to the fact that $\frac{(n)_k}{k} - \frac{(n-k)_k}{k} = O(n^{k-1})$. (Just take the logarithm of both sides of (11) that we need to prove and expand the logarithms into series.) Hence, (3), the condition of Lemma 3, holds. The conclusion of Lemma 3 is that $\Pr(\wedge_{I,\alpha}\overline{A_{I,\alpha}}) \geq \left(1 - \frac{1+\frac{c_k}{n}}{(n)_k}\right)^{\binom{n}{k}(k-1)!}$, and this number on the right hand side converges to $e^{-1/k}$.

Let us turn now to the enumeration of Latin rectangles. An $k \times n$ Latin rectangle is a sequence of k permutations of $\{1, 2, ..., n\}$ written in a matrix form, such that no column has any repeated entries. Let $L(k, n)$ denote the number of $k \times n$ Latin rectangles. The current best asymptotic formula [11] for $L(k, n)$ works for $k = o(n^{6/7})$. Without going into details of the history of the problem, the previous best range was $k = o(n^{1/2})$, with the use of the Chen-Stein method [7], [19], showing

$$L(k, n) \sim (n!)^k e^{-\binom{k}{2} - \frac{k^3}{6n}}. \tag{12}$$

Formula (12) has had an unexpected proof [17], where the inequality

$$(n!)^k \prod_{r=1}^{k-1} \left(1 - \frac{r}{n}\right)^n \leq L(k, n), \tag{13}$$

which was proved from the van der Waerden inequality for the permanent, provided the lower bound for the asymptotic formula.

Our goal now is to show (13) from Lemma 3 for $k = o(n^{1/2})$. We need the following very general lemma.

Lemma 4 *Assume that G is a negative dependency graph for the events $A_1, A_2, ..., A_n$. Assume further that $V(G)$ has a partition into classes, such that any two events in the same class have empty intersection. For any partition class J, let $B_J = \vee_{j\in J} A_j$. Now the quotient graph of G is a negative dependency graph for the events B_J.*

Proof. We have to show that if $\mathcal{K}$ is a subset of non-neighbors of J in the quotient graph, then $\Pr(B_J | \wedge_{K\in\mathcal{K}} \overline{B_K}) \leq \Pr(B_J)$. By the additivity of (conditional) probability over mutually exclusive events, it is sufficent to show that

$$\Pr(A_j | \wedge_{K\in\mathcal{K}} \overline{B_K}) \leq \Pr(A_j) \tag{14}$$

holds for every $j \in J$. However, $\wedge_{K\in\mathcal{K}}\overline{B_K} = \wedge_{i\in\cup\mathcal{K}}\overline{A_i}$, and every $i \in \cup\mathcal{K}$ is a non-neighbor of j in G, according to the definition of the quotient graph. Therefore, (14) holds as G is a negative dependency graph. □

Let us select now k permutations $\pi_1, \pi_2, ..., \pi_k$ of the elements $\{1, 2, ..., n\}$ randomly and independently, and fill in the entries $\pi_i(j)$ into a $k \times n$ matrix. We want to give a lower bound for the probability that the first $t + 1$ rows make a Latin rectangle under the condition that first t rows make a Latin rectangle. Fix an arbitrary $t \times n$ Latin rectangle now with rows $\pi_1, \pi_2, ..., \pi_t$. Define the event A_{ij} by $\pi_i(j) = \pi_{t+1}(j)$. These are canonical events. Let G be the graph whose vertices are the (i,j) entries for $j=1,2,...,n$, $i=1,2,...,t$, and every (i_1,j) is joined with every (i_2,j). The maximum degree in this graph is $t-1=o(n^{1/2})$. With the choice $x_{ij}=2/n$ these events satisfy (3) in the graph G, and therefore the graph G according to Theorem 1 is a negative dependency graph. Define the events $B_j=\vee_{1\le i\le t}A_{ij}$. Clearly $\Pr(B_j)\le t/n$. The quotient graph is empty, and is a negative dependency graph for the B_j events. Lemma 3 applies and $\Pr(\wedge_{j=1}^n \overline{B_j})\ge(1-t/n)^n$. Iterating this estimate, formula (13) follows. □

Spencer made a joke in [18], that Lovász Local Lemma 1 can prove the existence of an injection from an a-element set into a $6a$-element set, while the naive approach requires a $\Theta(a^2)$ size codomain, as it is well-known from the 'Birthday Paradox'. Now using Lemma 3 in combination with Lemma 5 below, we can show that a random function from an a-element set into an a-element set is an injection with probability at least $(\frac{1}{e}-o(1))^a$, giving a combinatorial proof to a weakened Stirling formula! (Apart from Lemma 4, this is the only result in the paper not using Theorem 1.)

We say that the events $A_1,A_2,...,A_n$ are *symmetric*, if the probability of any boolean expression of these sets do not change, if we substitute $A_{\pi(i)}$ to the place of A_i simultaneously, for any permutation π of $[n]$.

Lemma 5 *Assume that the events $A_1,A_2,...,A_n$ are symmetric, and let p_i denote $\Pr(\overline{A_1}\wedge\overline{A_2}\wedge\cdots\wedge\overline{A_i})$ for $i=1,2,...,n$ and let $p_0=1$. If the sequence is logconvex, i.e. $p_k^2\le p_{k-1}p_{k+1}$ for $k=1,2,...,n-1$, then Lemma 3 applies with an empty negative dependency graph, i.e. with $x_i=p_1$.*

Proof. Mathematical induction on the number of terms in the condition yields that $\Pr(A_1|\overline{A_2}\wedge\overline{A_3}\wedge...\wedge\overline{A_k})=1-p_k/p_{k-1}\ge 1-p_{k+1}/p_k=\Pr(A_1|\overline{A_2}\wedge\overline{A_3}\wedge...\wedge\overline{A_k}\wedge\overline{A_{k+1}})$. □

Consider a set A with $|A|=a$ and a set B with $|B|=b$, and assume $a\le b$. Consider a random function f from A to B. For $u\in A$, define the event $A_u=$ the value $f(u)$ occurs with multiplicity 2 or higher. The events A_u are symmetric. Clearly $\Pr(A_u)=1-b(b-1)^{a-1}/b^a$. Direct calculation shows that $p_i=\Pr(\overline{A_{u_1}}\wedge\overline{A_{u_2}}\cdots\overline{A_{u_i}})=i!\binom{b}{i}(b-i)^{a-i}/b^a$. The logconvexity of the p_i sequence is algebraically equivalent to $(b-k)^{2a-2k-1}\le(b-k+1)^{a-k}(b-k-1)^{a-k-1}$ for $k=1,...,a-1$. In the case $a=b$, set $n=a-k$, and the last inequality is algebraically equivalent to the well-known fact $(1+\frac{1}{n-1})^{n-1}\le(1+\frac{1}{n})^n$ for $n\ge 2$, while the case $n=1$ corresponds to $p_{a-1}^2\le p_{a-2}p_a$, which is easy to check. Hence, using Lemma 3, we obtain that the probability that a random $A\to A$ function is an injection, is at least $(1-p(A_1))^a=(1-1/a)^{a(a-1)}=(\frac{1}{e}-o(1))^a$, pretty close to the correct asymptotics $a!/a^a=\sqrt{2\pi a}e^{-a}(1+o(1))$.

4 Packing problem

A hypergraph H consists of a vertex set $V(H)$ together with a family $E(H)$ of subsets of $V(H)$, which are called edges of H. A r-uniform hypergraph, or r-graph, is a hypergraph whose edges have the same cardinality r. The complete r-graph on n vertices is denoted by $K_n^{(r)}$.

Packing problem of hypergraphs: For two r-uniform hypergraphs H_1, H_2, and an integer $n \geq \max\{|V(H_1)|, |V(H_2)|\}$, are there injections $\phi_i \colon V(H_i) \to [n]$, for $i = 1, 2$ such that $\phi_1(H_1)$ and $\phi_2(H_2)$ are edge-disjoint?

Theorem 2 *For $i = 1, 2$, assume that H_i is an r-uniform hypergraph with m_i edges, such that every edge in H_i intersects at most d_i other edges of H_i. If $(d_1+1)m_2 + (d_2+1)m_1 < \frac{1}{e}\binom{n}{r}$, then there exist injections of $V(H_1)$ and $V(H_2)$ into $K_n^{(r)}$ such that the natural images of H_1 and H_2 are edge-disjoint.*

Proof: Without loss of generality, we assume that H_2 is given as a sub-hypergraph of $K_n^{(r)}$. Consider a random injection of $V(H_1)$ into $V(K_n^{(r)})$; this injection extends to $E(H_1)$ in the natural way. Our probability space will be $I(U, V)$ with $U = V(H_1)$ and $V = [n]$. Consider two edges F_1 (of H_1) and F_2 (of H_2); and a bijection $\phi \colon F_1 \to F_2$. The events $A_{F_1,F_2,\phi}$ will be our bad events. We have

$$\Pr(A_{F_1,F_2,\phi}) = \frac{1}{r!\binom{n}{r}} = \frac{1}{(n)_r}.$$

Let G be the negative dependency graph of those $A_{F_1,F_2,\phi}$ events. An event $A_{F_1,F_2,\phi}$ conflicts another event $A_{F_1',F_2',\phi'}$ if and only if

1. Edges F_1 and F_1' have empty intersection while their images F_2 and F_2' have non-empty intersection.

2. Edges F_1 and F_1' have non-empty intersection but ϕ and ϕ' are defined differently in some intersection point.

An event $A_{F_1,F_2,\phi}$ can have at most $r!(d_2+1)m_1 - 1$ conflicts of the first type, and at most $r!(d_1+1)m_2$ conflicts of the second type, thus the maximal degree d in the negative dependency graph is at most

$$r![(d_1+1)m_2 + (d_2+1)m_1] - 1.$$

Apply Lemma 1 in the negative dependency graph setting. With positive probability, all bad events $A_{f_1,f_2,\phi}$ can be avoided simultaneously if

$$e(d+1)\Pr(A_{f_1,f_2,\phi}) < 1.$$

□

Remark: The constant coefficient $\frac{1}{e}$ in Theorem 2 can not be replaced by 2 as shown by the following example.

Let $r = 2$ and H_1 be the graph on $n = s(s-1)$ vertices consisting of $s-1$ vertex-disjoint copies of the complete graph K_s; and let H_2 be the graph on $n = s(s-1)$ vertices consisting of a single K_s and $n-s$ isolated vertices. The complement graph $\overline{H_1}$ is K_s-free by the pigeonhole principle. (In fact, $\overline{H}$ is the maximum K_s-free graph on n vertices by Turán theorem [21].) Therefore, copies of H_1 and H_2 can not be packed into K_n with disjoint edge sets. In this example, we have $d_1 = d_2 = 2(s-2)$ and $m_1 = (s-1)\binom{s}{2}$, $m_2 = \binom{s}{2}$. It is easy to see that Therefore, we have

$$\begin{aligned}(d_1+1)m_2 + (d_2+1)m_1 &= (2s-3)(s-1)\binom{s}{2} + (2s-3)\binom{s}{2} \\ &= (2s-3)\binom{s}{2} < 2\binom{s(s-1)}{2} = 2\binom{n}{2}.\end{aligned}$$

Here the last inequality holds for all $s \geq 2$ by an easy calculation.

5 Perfect Packing

For two r-uniform hypergraphs, H and G, we say that H has a perfect G-packing if there exist sub-hypergraphs $G_1, \ldots, G_k$ of H, each isomorphic to G, such that the vertex sets $V(G_1), \ldots, V(G_k)$ partition $V(H)$.

A necessary condition for the existence of perfect G-packing is that $|V(H)|$ is divisible by $|V(G)|$. We will prove the following theorem.

Theorem 3 *Suppose that two r-uniform hypergraphs G and H satisfy the following.*

1. *G has s vertices, H has n vertices, and n is divisible by s.*
2. *G has m edges, and each edge in G intersects at most d other edges of G.*
3. *For any vertex v of H, the degree of v in H is at least $(1-x)\binom{n-1}{r-1}$.*

If $x < \frac{1}{e(d+1+r^2\frac{m}{s})}$, then then H has a perfect G-packing.

A special case is that G is the r-graph with a single edge. We have $m = 1$, $d = 0$, and $s = r$.

Corollary 1 *Suppose the degree of each vertex in an r-graph H on n vertices is at least $(1 - \frac{1}{e(r+1)})\binom{n-1}{r-1}$. If n is divisible by r, then H has a perfect matching.*

Proof of Theorem 3: Let H_1 be the union of $\frac{n}{s}$ vertex-disjoint copies of G and $\overline{H}$ be the complement graph of H. Observe that H has a perfect G-packing if and only if H_1 and $H_2 = \overline{H}$ can be packed into $K_n^{(r)}$. Now we apply Theorem 2. Notice that $d_1 = d$ and $m_1 = |E(H_1)| = \frac{n}{s}|E(G)| = \frac{nm}{s}$. The degree of any vertex in $\overline{H}$ is at most $x\binom{n-1}{r-1}$ by the third condition. We have

$$d_2 \leq rx\binom{n-1}{r-1} - r + 1 \leq rx\binom{n-1}{r-1} - 1.$$

It suffices to have

$$(d_1+1)m_2 + (d_2+1)m_1 \leq (d+1)x\binom{n}{r} + \left(rx\binom{n-1}{r-1}\right)\frac{nm}{s} \leq \frac{1}{e}\binom{n}{r}.$$

The second inequality is equivalent to

$$x < \frac{1}{e(d+1+\frac{r^2 m}{s})}.$$

as desired. $\square$

6 Turán type extremal problems

For a fixed r-graph G, let $t(n,G)$ denote the smallest integer m such that every r-uniform hypergraph on n vertices with $m+1$ edges must contain a copy of G. The limit $\lim_{n\to\infty} \frac{t(n,G)}{\binom{n}{r}}$ always exists [13]. We denote it by $\pi(G)$. We have

Theorem 4 *Suppose each edge in an r-graph G intersects at most d other edges. Then we have*

$$\pi(G) \leq 1 - \frac{1}{(d+1)e}. \tag{15}$$

Proof: Consider an r-graph H on n vertices and $m = t(n,G) - 1$ edges, which do not contain a copy of G. Note that each edge of $\overline{H}$ can intersect at most $r\binom{n-1}{r-1} - r + 1$ other edges. The conclusion of Theorem 2 does not apply to G and $\overline{H}$. Therefore we must have

$$(d+1)\left(\binom{n}{r} - m\right) + \left(r\binom{n-1}{r-1} - r + 2\right)|E(G)| \geq \frac{1}{e}\binom{n}{r},$$

and by rearranging terms

$$\begin{aligned} \frac{m}{\binom{n}{r}} &\leq 1 - \frac{1}{e(d+1)} + \frac{r^2}{n}\frac{|E(G)|}{d+1} \\ &= 1 - \frac{1}{e(d+1)} + O(\frac{1}{n}) \end{aligned}$$

as desired as $n \to \infty$ while d and r are fixed. $\square$

The general upper bound for Turán density in term of the number of edges is first obtained by Sidorenko [16]

$$\pi(G) \leq 1 - \frac{1}{f-1}$$

for any r-graph G with f edges. It has slightly been improved by Keevash [12] to

$$\pi(G) < 1 - \frac{1}{f-1} - (1+O(1))(2r!^{2/r} f^{3-2/r})^{-1} \tag{16}$$

for fixed $r \geq 3$ and $f \to \infty$.

If a hypergraph G is a clique, in which any pair of edges have non-empty intersection, then we have $d = f - 1$. Inequality (16) is closer to the best known upper bound. For example, consider a special complete r-graph $K_{r+1}^{(r)}$ on $r+1$ vertices, de Caen [6], Sidorenko ([15]), Tazawa and Shirakura [20] proved

$$1 - \frac{(1+o(1))\ln r}{2r} \leq \pi(K_{r+1}^{(r)}) \leq 1 - \frac{1}{r}.$$

The upper bound was improved [8] to

$$\pi(K_{r+1}^{(r)}) \leq 1 - \frac{5r+12-\sqrt{9r^2+24r}}{2r(r+3)} = 1 - \frac{1}{r} - \frac{1}{r^2} + O(\frac{1}{r^3})$$

for odd r.

Theorem 4 only gives

$$\pi(K_{r+1}^{(r)}) \leq 1 - \frac{1}{e(r+1)}.$$

It is still quite comparable to those best known upper bounds for $\pi(K_{r+1}^{(r)})$ except for the constant coefficient e.

However, for hypergraph G with less intersection, (say, $d < \frac{f}{e}$) inequality (15) often offers a much better upper bound on $\pi(G)$ than inequality (16) does.

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