Asymptotics of the average height of 2–watermelons with a wall.

Markus Fulmek*

Fakultät für Mathematik
Universität Wien
Nordbergstraße 15, A-1090 Wien, Austria
Markus.Fulmek@Univie.Ac.At

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Abstract

We generalize the classical work of de Bruijn, Knuth and Rice (giving the asymptotics of the average height of Dyck paths of length n) to the case of p–watermelons with a wall (i.e., to a certain family of p nonintersecting Dyck paths; simple Dyck paths being the special case $p = 1$.)

An exact enumeration formula for the average height is easily obtained by standard methods and well–known results. However, straightforwardly computing the asymptotics turns out to be quite complicated. Therefore, we work out the details only for the simple case $p = 2$.

1 Introduction

The model of vicious walkers was originally introduced by Fisher [10] and received much interest, since it leads to challenging enumerative questions. Here, we consider special configurations of vicious walkers called *p–watermelons with a wall*.

Briefly stated, a p–watermelon of length n is a family $P_1, \dots P_p$ of p nonintersecting lattice paths in $\mathbb{Z}^2$, where

- P_i starts at $(0, 2i - 2)$ and ends at $(2n, 2i - 2)$, for $i = 1, \dots, p$,
- all the steps are directed north–east or south–east, i.e., lead from lattice point (i, j) to $(i + 1, j + 1)$ or to $(i + 1, j - 1)$,

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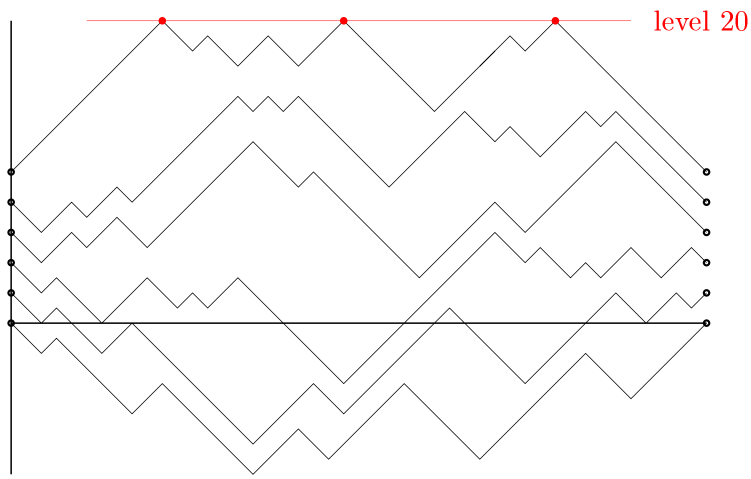
Figure 1: A 6–watermelon of length 46 and height 20



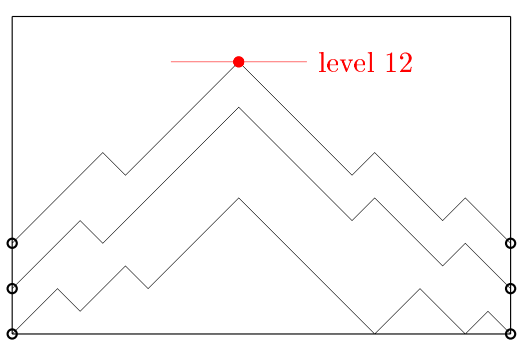
- no two paths P_i, P_j have a point in common (this is the meaning of "nonintersecting").

The *height* of a p–watermelon is the y–coordinate of the highest lattice point contained in any of its paths (since the paths are nonintersecting, it suffices to consider the lattice points contained in the highest path P_p; see Figure 1 for an illustration.)

A p–watermelon of length n *with a wall* has the additional property that none of the paths ever goes below the line $y = 0$ (since the paths are nonintersecting, it suffices to impose this condition on the lowest path P_1; see Figure 2 for an illustration.).

In [3], Bonichon and Mosbah considered (amongst other things) the *average height*

Figure 2: A 3–watermelon of length 11 and height 12 with a wall



$H(n,p)$ of p–watermelons of length n with a wall,

$$H(n,p) = \frac{1}{\#(\text{all } p\text{-watermelons of length } n \text{ with a wall})} \times \sum_h h \cdot \#(\text{all } p\text{-watermelons of length } n \text{ and height } h \text{ with a wall}),$$

and derived by computer experiments the following conjectural asymptotics [3, 4.1]:

$$H(n,p) \sim \sqrt{(1.67p - 0.06)\, 2n} + o\big(\sqrt{n}\big)\,. \tag{1}$$

The purpose of this paper is to work out the exact asymptotics for the simple special case $p = 2$ by imitating the classical reasoning of de Bruijn, Knuth and Rice [6] for the case $p = 1$ (i.e., for the average height of Dyck paths). Following this road involves rather complicated computations, even in the case $p = 2$. In particular, we shall need informations about residues and evaluations of a double Dirichlet series, which we (partly) shall obtain by imitating Riemann's representation of the zeta function ζ [7, section 1.12, (16)] in terms of Jacobi's theta function θ.

The appearance of ζ and θ in this context is not surprising from the probabilist's point of view, since in the scaling limit watermelons lead to noncolliding diffusion processes, and there are well–known connections between the zeta function and Brownian motion. For example, θ and ζ are involved in the asymptotic distribution as $n \to \infty$ of the height of 1–watermelons [18, Theorem 4, and the remark on page 181]; see Biane, Pitman and Yor [2] for a comprehensive overview.

1.1 Notational conventions

For $k, n \in \mathbb{Z}$, we shall use the notation introduced in [13] for the rising and falling factorial powers, i.e.

$$(n)^{\overline{k}} := 0 \text{ if } k < 0,\ (n)^{\overline{0}} := 1,\ (n)^{\overline{k}} := n \cdot (n+1) \cdots (n+k-1) \ \text{ if } k > 0$$

and

$$(n)^{\underline{k}} := 0 \text{ if } k < 0,\ (n)^{\underline{0}} := 1,\ (n)^{\underline{k}} := n \cdot (n-1) \cdots (n-k+1) \ \text{ if } k > 0.$$

For the binomial coefficient we adopt the convention

$$\binom{n}{k} := \begin{cases} \frac{(n)^{\underline{k}}}{k!} & \text{if } 0 \le k \le n, \\ 0 & \text{else.} \end{cases}$$

Moreover, we shall use Iverson's notation:

$$[\text{some assertion}] = \begin{cases} 1 & \text{if “some assertion” is true,} \\ 0 & \text{else.} \end{cases}$$

1.2 Organization of the material presented

This paper is organized as follows:

- In section 2, we present *exact* enumeration formulas for the average height of p–watermelons with a wall in terms of certain determinants. Moreover, we make these formulas more explicit (in terms of sums of binomial coefficients) for the simple cases $p = 1$ and $p = 2$.
- In section 3, we first review the classical reasoning for the asymptotics of the average height of 1–watermelons with a wall, which was given by de Bruijn, Knuth and Rice [6]. Then we show how this reasoning can be directly extended to the case of 2–watermelons with a wall.
- In appendix A, we summarize background information on
  - Stirling's approximation,
  - certain residues and values of the gamma and zeta function,
  - a certain double Dirichlet series and Jacobi's theta function

  which are needed in the reasoning of section 3.

1.3 Acknowledgements

I am very grateful to Professor Krätzel for pointing out to me how the poles and residues of certain Dirichlet series can be obtained in a simple way by using the reciprocity law for Jacobi's theta function, to Christian Krattenthaler for many helpful discussions, and to the anonymous referee for helpful suggestions and corrections and for drawing my attention to the paper of Biane, Pitman and Yor [2].

2 Exact enumeration

For a start, we gather some exact enumeration results.

2.1 The number of p–watermelons with a wall

The following proposition generalizes the well–known enumeration of 1–watermelons with a wall (i.e., Dyck paths) of length n, which is given by the Catalan numbers

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Proposition 1. *The number $C(n,p)$ of all p–watermelons with a wall of length n is given as*

$$C(n,p) = \prod_{j=0}^{p-1} \frac{\binom{2n+2j}{n}}{\binom{n+2j+1}{n}}. \tag{2}$$

Proof. This is a special case of Theorem 6 in [14]. $\square$

2.2 1–watermelons with a wall and height restrictions

In order to obtain the *average* height, we count p–watermelons with a wall of length n which do not *exceed* height h. To this end, we employ the following formula (see [16, p. 6, Theorem 2]):

Theorem 1. *Let u, d be nonnegative integers, and let b, t be positive integers, such that $-b < u - d < t$. The number of lattice paths from $(0,0)$ to $(u+d, u-d)$, which do not touch neither line $y = -b$ nor line $y = t$, equals*

$$\sum_{k\in\mathbb{Z}} \left(\binom{u+d}{u-k\,(b+t)} - \binom{u+d}{u-k\,(b+t)+b} \right). \tag{3}$$

Corollary 1. *Let i, j and h be integers such that $0 \le 2i, 2j \le h$. The number of all lattice paths from $(0, 2i)$ to $(2n, 2j)$ which lie* between *the lines $y = 0$ and $y = h \ge 0$ is*

$$m(n,i,j,h) := \sum_{k\in\mathbb{Z}} \left(\binom{2n}{n-i+j-k\,(h+2)} - \binom{2n}{n+i+j-k\,(h+2)+1} \right). \tag{4}$$

The special case $i = j = 0$ can be written as

$$m(n,h) := m(n,0,0,h) = \frac{1}{n+1}\binom{2n}{n} - \sum_{k\ge 1} \left(\binom{2n}{n-k\,(h+2)-1} - 2\binom{2n}{n-k\,(h+2)} + \binom{2n}{n-k\,(h+2)+1} \right). \tag{5}$$

Proof. Set $u = n - i + j$, $d = n + i - j$, $b = 2i + 1$ and $t = h + 1 - 2i$ in (3). $\square$

Corollary 2. *For $n > 0$, the number of all p–watermelons with a wall of length $2n$, which do not exceed height h, is given by the following determinant:*

$$C(n,p,h) = \left| m(n,i,j,h) \right|_{i,j=0}^{p-1}. \tag{6}$$

Proof. For $h > 2p - 2$ this follows by a straightforward application of the Lindström–Gessel–Viennot method [12].

For $0 \le h \le 2p - 2$, the determinant equals 0 (as it should: $2p - 1$ is the minimal height of a p-watermelon), since the matrix $(m(n,i,j,h))_{i,j=0}^{p-1}$ is singular in this case. More precisely, we have for all n and all j:

$$m(n,i,j,2i-1) = 0 \text{ (for } h = 2i-1 \text{ and } i \le p-1\text{)},$$
$$m(n,i,j,2i) + m(n,i+1,j,2i) = 0 \text{ (for } h = 2i \text{ and } i < p-1\text{)},$$
$$\sum_{i=0}^{p-1} (-1)^i\, m(n,i,j,2p-2) = 0 \text{ (for } h = 2p-2\text{)}.$$

$\square$

2.3 The average height of 1–watermelons with a wall

The following is a condensed version of the reasoning given in [6]. Denote by $\underline{m}(n,h)$ the number of all Dyck paths, starting at $(0,0)$ and ending at $(2n,0)$, which reach *at least* height h. By (5), we obtain

$$\begin{aligned}\underline{m}(n,h) &= m(n,n) - m(n,h-1) = C(n,1) - C(n,1,h-1)\\ &= \sum_{k\geq 1}\left(\binom{2n}{n-k\,(h+1)-1} - 2\binom{2n}{n-k\,(h+1)} + \binom{2n}{n-k\,(h+1)+1}\right).\end{aligned}$$

So, the average height of 1–watermelons with a wall (i.e.,Dyck paths) of length n is

$$\begin{aligned}H(n,1) &= \frac{1}{C(n)}\sum_{h=1}^{n}\underline{m}(n,h) = \frac{1}{C(n)}\left(-C(n) + \sum_{h=0}^{n}\underline{m}(n,h)\right)\\ &= -1 + (n+1)\left(\sum_{k\geq 1}\mathrm{d}(k)\,\frac{\left(\binom{2n}{n-k-1} - 2\binom{2n}{n-k} + \binom{2n}{n-k+1}\right)}{\binom{2n}{n}}\right),\end{aligned} \tag{7}$$

where $d(k)$ denotes the number of positive divisors of k. Introducing the notation

$$S(n,a) := \sum_{k\geq 1}\mathrm{d}(k)\,\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}}, \tag{8}$$

we arrive at

$$H(n,1) = (n+1)\Big(S(n,1) - 2\,S(n,0) + S(n,-1)\Big) - 1, \tag{9}$$

which is equivalent to equation (23) in [6].

2.4 The average height of p–watermelons with a wall

In generalization of the above notation, denote by $\underline{m}(n,p,h)$ the number of all p–watermelons of length n, which reach *at least* height h, i.e.,

$$\underline{m}(n,p,h) = C(n,p) - C(n,p,h-1)\,.$$

Clearly, we have the following exact formula for the average height of p–watermelons of length n:

$$H(n,p) = \frac{1}{C(n,p)}\sum_{h=1}^{n+2p-2}\underline{m}(n,p,h)\,. \tag{10}$$

2.4.1 The average height of 2–watermelons with a wall

Let

$$S(n,a,b) = \sum_{j\geq 1}\sum_{k\geq 1}\mathrm{d}(\gcd(j,k))\,\frac{\binom{2\,n}{n+a-j}\binom{2\,n}{n+b-k}}{\binom{2\,n}{n}^2}. \tag{11}$$

From (6), straightforward (but rather tedious) computations lead to the following formula:

$$H(n,2) = \frac{(n+1)^{\overline{2}}}{12\,(2n+1)}\Big((n+1)^{\overline{3}}\,\mathbf{S}_2(n) + \mathbf{S}_1(n)\Big) - 1, \tag{12}$$

where

$$\begin{aligned}
\mathbf{S}_1(n) = {} & \left(n^2+5n+6\right)(S(n,-3)+S(n,3)) - \\
& \left(6n^2+18n\right)(S(n,-2)+S(n,2)) + \\
& \left(15n^2+27n+6\right)(S(n,-1)+S(n,1)) - \\
& \left(20n^2+28n+24\right) S(n,0)\,, && (13)\\
\mathbf{S}_2(n) = {} & S(n,-2,-2) - S(n,-1,-3) - 2S(n,-1,-2) + S(n,-1,-1) + 2S(n,-1,0) - \\
& S(n,-1,3) + 2S(n,0,-3) - 4S(n,0,0) + 2S(n,0,3) - S(n,1,-3) - \\
& 2S(n,1,-2) + 2S(n,1,-1) + 2S(n,1,0) + S(n,1,1) - S(n,1,3) + \\
& 2S(n,2,-2) - 2S(n,2,-1) - 2S(n,2,1) + S(n,2,2). && (14)
\end{aligned}$$

3 Asymptotic enumeration

3.1 Asymptotics of the average height of 1–watermelons

In the case of 1–watermelons, the asymptotic of the average height (7) is well–known, see [11, Proposition 7.7] or [6, equation (34)]:

$$H(n,1) \simeq \sqrt{\pi n} - \frac{3}{2} + O\Big(n^{-\frac{1}{2}+\epsilon}\Big)\,. \tag{15}$$

We repeat the classical reasoning of de Bruijn, Knuth and Rice [6], in order to make clear the basic idea, which we shall also employ for the case $p=2$ later.

Proof of (15)*:* From (9) it is clear that we need to investigate the asymptotic behaviour of $S(n,a) = \sum_{k=1}^{n} \mathrm{d}(k)\,\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}}$. Note that the sums $S(n,a)$ in (9) are multiplied with a factor of order 1. So if we are interested in the asymptotics of $H(n,1)$ up to some $O(n^{-\alpha})$, we need the asymptotics for $S(n,a)$ up to $O(n^{-\alpha-1})$; for our case, $\alpha = 1-\epsilon$ is sufficient.

The basis of the following considerations is the asymptotic expansion of the quotient of binomial coefficients (30) (see appendix A.1).

3.1.1 The asymptotics of $S(n,a)$ for a fixed, $n\to\infty$

First we observe that

$$\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} = O\Big(\exp\big(-n^{2\epsilon}\big)\Big)$$

if $\left|\frac{k-a}{n}\right| \geq n^{\epsilon-\frac{1}{2}}$, i.e., if $k \geq n^{\frac{1}{2}+\epsilon} + a$ (see (30) in appendix A.1). Therefore, the sum of all terms with $k \geq n^{\frac{1}{2}+\epsilon} + a$ is negligible in (8), being $O(n^{-m})$ for all $m>0$, and we may take $\frac{k-a}{n} = O\Big(n^{\epsilon-\frac{1}{2}}\Big)$ in (30).

Next, we take (30) up to order n^{-1} and substitute $x \to \frac{k-a}{n}$: Pulling out the leading term $\mathbf{e}^{\frac{-k^2}{n}}$ and expanding the rest with respect to k gives

$$\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} = \mathbf{e}^{\frac{-k^2}{n}} \Big(1 - \frac{a^2}{n} + k \left(\frac{2\,a}{n} - \frac{a + 2\,a^3}{n^2}\right) + \frac{(1 + 4\,a^2)\;k^2}{2\,n^2} + \frac{(5\,a + 4\,a^3)\;k^3}{3\,n^3} - \frac{k^4}{6\,n^3} - \frac{a\,k^5}{3\,n^4}\Big) + O\big(n^{-2+\epsilon}\big). \tag{16}$$

Now we consider the following function

$$g(n, b) := \sum_{k\geq 1} k^b\, \mathrm{d}(k)\, \mathbf{e}^{-k^2/n}, \tag{17}$$

and observe that here the terms for $k \geq n^{1/2+\epsilon}$ are *again* negligible:

$$\sum_{k\geq n^{1/2+\epsilon}} k^b\, \mathrm{d}(k)\, \mathbf{e}^{-k^2/n} = O\big(n^{-m}\big) \text{ for all } m > 0.$$

Hence we directly obtain from (16):

$$\begin{aligned} S(n, a) = &\left(1 - \frac{a^2}{n}\right) g(n, 0) + \left(\frac{2a}{n} - \frac{2a^3 + a}{n^2}\right) g(n, 1) + \left(\frac{4a^2 + 1}{2n^2}\right) g(n, 2) \\ &+ \left(\frac{4a^3 + 5a}{3n^3}\right) g(n, 3) - \left(\frac{1}{6n^3}\right) g(n, 4) - \frac{a}{3n^4}\, g(n, 5) + O\big(n^{-2+\epsilon} g(n, 0)\big)\,. \end{aligned} \tag{18}$$

(This is equation (27) in [6].) Note that the coefficients for $g(n, b)$ are odd functions of a for odd b and obtain:

$$S(n, 1) - 2\,S(n, 0) + S(n, -1) = -\frac{2}{n}\, g(n, 0) + \frac{4}{n^2}\, g(n, 2) + O\big(n^{-2+\epsilon} g(n, 0)\big)\,. \tag{19}$$

So we reduced our problem to that of obtaining an asymptotic expansion for $g(n, b)$. Note that we need this information only for b even. It follows from the computations presented in appendix A.2.1, that $g(n, 2b) = O\big(n^{b+1/2} \log(n)\big)$, and that we have for all $m \geq 0$:

$$\begin{aligned} g(n, 0) &= \frac{1}{4}\sqrt{\pi n}\log(n) + \left(\frac{3}{4}\gamma - \frac{1}{2}\log(2)\right)\sqrt{\pi n} + \frac{1}{4} + O\big(n^{-m}\big)\,, \\ g(n, 2) &= \frac{n}{8}\sqrt{\pi n}\log(n) + \left(\frac{1}{4} + \frac{3}{8}\gamma - \frac{1}{4}\log(2)\right) n\sqrt{\pi n} + O\big(n^{-m}\big)\,. \end{aligned} \tag{20}$$

Inserting this information in (19) we immediately obtain the desired result (15). □

3.2 Asymptotics of the average height of 2–watermelons

We shall modify the reasoning from section 3.1 appropriately. In doing so, it turns out that we have to deal with the double Dirichlet series $\sum_{k,l\geq 1} \frac{k^{2a} l^{2b}}{(k^2+l^2)^s}$ for integers $a, b \geq 0$.

Proposition 3 in appendix A.3 states that this series is convergent in the half–plane $\Re(z) > a + b + 1$ and defines a meromorphic function $Z(a, b; z) : \mathbb{C} \to \mathbb{C}$ which has a simple pole at $z = a+b+1$, and an additional simple pole at $z = a+b+\frac{1}{2}$ only if $a = 0$ or $b = 0$. Hence, we can write

$$Z(a, b; z) = \frac{r_{a,b}}{z - a - b - \frac{1}{2}} + c_{a,b} + O\left(z - a - b - \frac{1}{2}\right). \tag{21}$$

Given this "implicit" definition of the numbers $c_{a,b}$ we will show:

$$\begin{aligned} H(n, 2) \simeq \sqrt{\pi n}\Big(&-2c_{0,0} + 8c_{1,0} - 9c_{1,1} - 9c_{2,0} + 15c_{2,1} \\ &+ 35c_{2,2} + 5c_{3,0} - 35c_{3,1}\Big) - \frac{3}{2} + O\left(n^{-\frac{1}{2}+\epsilon}\right). \end{aligned} \tag{22}$$

Using the representations of the constants $c_{a,b}$ by certain integrals (see (50) in appendix A.3), we obtain the following approximative asymptotics by numerical integration (carried out with Mathematica)

$$H(n, 2) \simeq 2.57758\,\sqrt{n} - \frac{3}{2} + O\left(n^{-\frac{1}{2}+\epsilon}\right). \tag{23}$$

This conforms quite well to Bonichon's and Mosbah's conjecture (1) for the case $p = 2$, which gives approximately $2.56125\,\sqrt{n}$. Straightforward computation of $H(n, 2)$ for small n (using the GNU multiple precision library in a small C++–program) indicates that the approximation is quite good. For example, considering the difference $D(n) := H(n, 2) - \left(2.57758\,\sqrt{n} - \frac{3}{2}\right)$, we obtain $D(1000) = 0.083217$, $D(2000) = 0.058859$, $D(3000) = 0.048062$ and $D(5000) = 0.037231$. Figure 3 shows a plot of $D(n)$ for $n \leq 1000$.

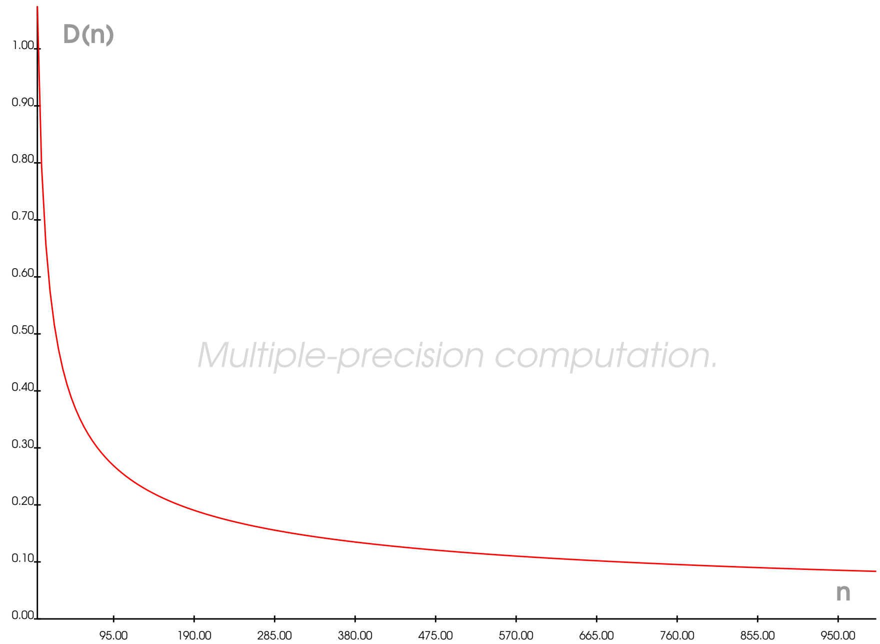
Proof of (22)*:* Note that in (12), the "single sums" $S(n, a)$ are multiplied with a rational function in n of order at most 3, while the "double sums" $S(n, a, b)$ are multiplied with a factor of order 4: So if we are interested in the asymptotics of $H(n, 2)$ up to some $O(n^{-\alpha})$, we need the asymptotics for $S(n, a)$ up to $O(n^{-\alpha-3})$ and for $S(n, a, b)$ up to $O(n^{-\alpha-4})$; for our case, $\alpha = 1 - \epsilon$ is sufficient.

3.2.1 The asymptotics of $S(n, a)$ for a fixed, $n \to \infty$

Basically, we repeat the computations from section 3.1.1. The only difference is that we need higher orders now. After some calculations, we obtain:

$$\begin{aligned} \mathbf{S}_1(n) = &-\frac{24\,(4n^2 + 20n + 89)\,g(n, 0)}{n^3} + \frac{4\,(96n^2 + 1065n + 3656)\,g(n, 2)}{n^4} \\ &- \frac{(288n^2 + 4060n + 12213)\,g(n, 4)}{n^5} + \frac{8\,(8n^2 + 107n + 335)\,g(n, 6)}{n^6} \\ &- \frac{(96n + 521)g(n, 8)}{3n^7} + \frac{10g(n, 10)}{3n^8} + O\left(n^{-4+\epsilon}g(n, 0)\right). \end{aligned} \tag{24}$$

Figure 3: The difference $D(n) := H(n,2) - \left(2.57758\sqrt{n} - \frac{3}{2}\right)$ for $n \leq 1000$. The computations were done in a C++–program using the GNU multiple precision library.



Recall that in appendix A.2.1 it is proved that $g(n,2b) = O\big(n^{b+1/2}\log(n)\big)$. Moreover, the arguments in appendix A.2.1 show that we have (in addition to (20)) for all $m \geq 0$:

$$\begin{aligned}
g(n,4) &= \frac{3n^2}{16}\sqrt{\pi n}\log(n) + \left(\frac{1}{2} + \frac{9}{16}\gamma - \frac{3}{8}\log(2)\right) n^2\sqrt{\pi n} + O\big(n^{-m}\big)\,,\\
g(n,6) &= \frac{15n^3}{32}\sqrt{\pi n}\log(n) + \left(\frac{23}{16} + \frac{45}{32}\gamma - \frac{15}{16}\log(2)\right) n^3\sqrt{\pi n} + O\big(n^{-m}\big)\,,\\
g(n,8) &= \frac{105n^4}{64}\sqrt{\pi n}\log(n) + \left(\frac{11}{2} + \frac{315}{64}\gamma - \frac{105}{32}\log(2)\right) n^4\sqrt{\pi n} + O\big(n^{-m}\big)\,,\\
g(n,10) &= \frac{945n^5}{128}\sqrt{\pi n}\log(n) + \left(\frac{1689}{64} + \frac{2835}{128}\gamma - \frac{945}{64}\log(2)\right) n^5\sqrt{\pi n} + O\big(n^{-m}\big)\,.
\end{aligned}$$

Inserting this information in (24) we immediately obtain the first part of the desired result:

$$\frac{(n+1)^{\overline{2}}}{12\,(2n+1)}\mathbf{S}_1(n) = \frac{11\sqrt{\pi n}}{6} - 1 + O\big(n^{-1/2+\epsilon}\big)\,. \tag{25}$$

3.2.2 The asymptotics of $S(n,a,b)$ for a, b fixed, $n\to\infty$

Basically, we mimic the computations from section 3.1.1. In doing so, we are led to consider the following function

$$g(n,a,b) := \sum_{k\ge 1}\sum_{l\ge 1} k^a\, l^b\, \mathrm{d}(\gcd(k,l))\, \mathbf{e}^{-\left(k^2+l^2\right)/n}.$$

Observe again that the terms for $k,l\ge n^{1/2+\epsilon}$ are negligible in this sum. For obtaining the following formula, we made use of the fact that $g(n,2a,2b)=O\big(n^{a+b+1}\big)$ (which is shown in appendix A.2.2):

$$\begin{aligned}
\mathbf{S}_2(n) &= \frac{1}{2}\left(-\frac{192}{n^4}+\frac{1632}{n^5}-\frac{8736}{n^6}\right)g(n,0,0)+\left(\frac{768}{n^5}-\frac{8928}{n^6}+\frac{61744}{n^7}\right)g(n,2,0)\\
&+\frac{1}{2}\left(-\frac{1152}{n^6}+\frac{18432}{n^7}-\frac{161488}{n^8}\right)g(n,2,2)+\left(-\frac{576}{n^6}+\frac{10336}{n^7}-\frac{99336}{n^8}\right)g(n,4,0)\\
&+\left(\frac{384}{n^7}-\frac{10784}{n^8}+\frac{138128}{n^9}\right)g(n,4,2)+\frac{1}{2}\left(\frac{512}{n^8}-\frac{7872}{n^9}+\frac{58368}{n^{10}}\right)g(n,4,4)\\
&+\left(\frac{128}{n^7}-\frac{4192}{n^8}+\frac{300624}{5n^9}\right)g(n,6,0)+\left(-\frac{256}{n^8}+\frac{6848}{n^9}-\frac{1517888}{15n^{10}}\right)g(n,6,2)\\
&+\left(\frac{2432}{3n^{10}}-\frac{62368}{5n^{11}}\right)g(n,6,4)+\frac{1}{2}\left(\frac{256}{3n^{11}}-\frac{416}{15n^{12}}\right)g(n,6,6)+\left(\frac{544}{n^9}-\frac{225488}{15n^{10}}\right)g(n,8,0)\\
&+\left(\frac{398912}{15n^{11}}-\frac{960}{n^{10}}\right)g(n,8,2)+\left(\frac{31736}{15n^{12}}-\frac{256}{3n^{11}}\right)g(n,8,4)+\frac{1328g(n,8,6)}{45n^{13}}+\frac{64g(n,8,8)}{9n^{14}}\\
&+\left(\frac{8672}{5n^{11}}-\frac{64}{3n^{10}}\right)g(n,10,0)+\left(\frac{128}{3n^{11}}-\frac{47504}{15n^{12}}\right)g(n,10,2)-\frac{7856g(n,10,4)}{45n^{13}}-\frac{32g(n,10,6)}{3n^{14}}\\
&-\frac{456g(n,12,0)}{5n^{12}}+\frac{2576g(n,12,2)}{15n^{13}}+\frac{64g(n,12,4)}{9n^{14}}+\frac{16g(n,14,0)}{9n^{13}}\\
&-\frac{32g(n,14,2)}{9n^{14}}+O\big(n^{-5+\epsilon}g(n,0,0)\big). \qquad (26)
\end{aligned}$$

From the results obtained in appendix A.2.2 and A.3.1, we easily derive the following asymptotic expansions (using the "implicit" definition of the numbers $c_{a,b}$ given in (21)):

$$\begin{aligned}
g(n,0,0) &= \frac{\pi^3 n}{24}+\frac{\sqrt{\pi n}}{4}\left(2c_{0,0}-\log(n)-\psi\left(\frac{1}{2}\right)-2\gamma\right)-\frac{1}{8}+O\big(n^{-m}\big),\\
g(n,2a,0) &= \frac{2^{-2a-3}n^a(2a)!}{a!}\left(\frac{\pi^3 n}{3}+\sqrt{\pi n}\left(4c_{a,0}-\log(n)-\psi\left(a+\frac{1}{2}\right)-2\gamma\right)\right)+O\big(n^{-m}\big),\\
g(n,2a,2b) &= \frac{2^{-2a-2b-3}n^{a+b}(2a)!(2b)!}{a!b!}\left(\frac{\pi^3 n}{3}+4\sqrt{\pi n}\,\frac{c_{a,b}\,(2a+2b)!}{(a+b)!}\right)+O\big(n^{-m}\big) \qquad (27)
\end{aligned}$$

for all $m\ge 0$. Inserting the information from (27) in (26) shows that all the $\log(n)$–terms cancel, as well as all evaluations of the digamma function ψ (see appendix A.2.1). So we obtain the second part of the desired result:

$$\frac{(n+1)^{\overline{2}}}{12\,(2n+1)}\,(n+1)^{\overline{3}}\,\mathbf{S}_2(n) = \sqrt{\pi n}\Big(-\frac{11}{6} - 2c_{0,0} + 8c_{1,0} - 9c_{1,1} - 9c_{2,0}$$
$$+ 15c_{2,1} + 35c_{2,2} + 5c_{3,0} - 35c_{3,1}\Big) + \frac{1}{2} + O\big(n^{-1/2+\epsilon}\big)\,. \quad (28)$$

Inserting the expressions (25) and (28) in (12) finally gives (22). □

A Background information and relevant results

A.1 Stirling's approximation applied to quotients of binomial coefficients

We have the following asymptotic series for $\log(\Gamma(z))$, valid for $|\arg z| < \pi - \delta$, $0 < \delta < \pi$, $|z| \to \infty$ (see [5, equation (3.10.7)]):

$$\log(\Gamma(z)) \approx \left(z - \frac{1}{2}\right)\log(z) - z + \frac{\log(2\pi)}{2} + \sum_{j=1}^{\infty} z^{1-2j}\frac{B_{2j}}{(2j)\,(2j-1)}, \quad (29)$$

where B_j denotes the j-th Bernoulli number.

Setting $x = \frac{k-a}{n}$ we thus obtain

$$\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}} = \exp\Bigg(-2n\left(\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{30} + \frac{x^8}{56} + \dots\right) + \left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \dots\right)$$
$$- \frac{1}{6n}\left(x^2 + x^4 + x^6 + x^8 + \dots\right) + \frac{1}{n^3}\left(\frac{x^2}{30} + \frac{x^4}{12} + \frac{7x^6}{45} + \frac{x^8}{4} + \dots\right)$$
$$- \frac{1}{n^5}\left(\frac{x^2}{42} + \frac{x^4}{9} + \frac{x^6}{3} + \frac{11x^8}{14} + \dots\right) + \frac{1}{n^7}\left(\frac{x^2}{30} + \frac{x^4}{4} + \frac{11x^6}{10} + \frac{143x^8}{40} + \dots\right)$$
$$- \frac{1}{n^9}\left(\frac{5x^2}{66} + \frac{5x^4}{6} + \frac{91x^6}{18} + \frac{65x^8}{3} + \dots\right) + O\big(x^2 n^{-11}\big)\Bigg). \quad (30)$$

Note that $\frac{\binom{2n}{n+a-k}}{\binom{2n}{n}}$ is zero for $|x| > 1$. The approximation given by (30) is very good if, say, $|x| \le \frac{1}{2}$.

A.2 Integral representations of the exponential function and applications

A.2.1 The asymptotics of $g(n,b)$ for b fixed, $n \to \infty$

Starting with the formula

$$\mathbf{e}^{-x} = \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} \Gamma(z)\,x^{-z}\mathrm{d}z \text{ for } c > 0, x > 1, \quad (31)$$

(see [1, (2.4.1)]) and using

$$\zeta(z)^2 = \sum_{k\ge 1} \mathrm{d}(k)\, k^{-z}, \tag{32}$$

we obtain by interchanging summation and integration

$$\begin{aligned} g(n,b) &= \sum_{k\ge 1} \frac{\mathrm{d}(k)}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\, \Gamma(z)\, k^{b-2z} \mathrm{d}z \\ &= \frac{1}{2\pi \mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\, \Gamma(z)\, \zeta(2z-b)^2\, \mathrm{d}z, \end{aligned}$$

where $c > \frac{b+1}{2}$. Denote the integrand in the above formula by $G_1(b;z)$.

For any fixed positive number q and $\Re(s) \ge -q$, we have $\zeta(s) = O\Big(|s|^{q+1/2}\Big)$ as $s \to \infty$. Since $n^z\,\Gamma(z)$ becomes small on vertical lines, we can shift the line of integration to the left as far as we want to, if we take into account the residues of our integrand $G_1(b;z)$. There is a double pole at $z = \frac{b+1}{2}$, and possibly some simple poles at $z = 0, -1, -2, \dots$.

For obtaining the residues, we use the power series expansion

$$\zeta(s) - \frac{1}{s-1} = \gamma + \sum_{n=1}^{\infty} \gamma_n\, (s-1)^n\,,$$

where γ is Euler's constant and $\gamma_n = \lim_{m\to\infty} \left(\sum_{l=1}^{m} l^{-1} (\log l)^n - (n+1)^{-1} (\log l)^{n+1}\right)$ (see [7, 1.12, (17)]), which gives the Laurent expansion at $\frac{b+1}{2}$

$$\zeta(2z-b)^2 = \frac{1}{4\left(z-\frac{b+1}{2}\right)^2} + \frac{\gamma}{\left(z-\frac{b+1}{2}\right)} + \cdots .$$

Combining this with the series expansions

$$\begin{aligned} n^z &= n^{z_0}\left(1 + \log(n)\,(z-z_0) + \dots\right), \\ \Gamma(z) &= \Gamma(z_0)\left(1 + \psi(z_0)\,(z-z_0) + \dots\right), \end{aligned}$$

for $z_0 = \frac{b+1}{2}$, where $\psi(z)$ is the digamma function (i.e., the logarithmic derivative of the gamma function $\frac{\Gamma'(z)}{\Gamma(z)}$, see [7, 1.7]), we can easily express the residue of our integrand $G_1(b;z)$ at $z = \frac{b+1}{2}$:

$$n^{\frac{b+1}{2}}\Gamma\left(\frac{b+1}{2}\right)\left(\frac{1}{4}\log(n) + \frac{1}{4}\psi\left(\frac{b+1}{2}\right) + \gamma\right). \tag{33}$$

For our purposes, we need $\psi(z)$ at positive integral or half–integral values z, which can be derived from the following information:

$$\psi(1) = -\gamma \qquad \text{(see [7, section 1.7, equation (4)])},$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log(2) \qquad \text{(see [17, p. 104])},$$

$$\psi(z+n) = \psi(z) + \sum_{j=0}^{n-1} \frac{1}{z+j} \qquad \text{(see [7, section 1.7, equation (10)])}.$$

The residue of $G_1(b;z)$ at $z=-m$ is

$$n^{-m}\frac{(-1)^m}{m!}\zeta(-2m-b)^2 = n^{-m}\frac{(-1)^m}{m!}\left(\frac{B_{2m+b+1}}{(2m+b+1)}\right)^2, \tag{34}$$

where B_k denotes the k-th Bernoulli number (see [7, 1.12, (20)]). Note that this number is non–zero only if b is odd or $m=b=0$.

The sum of (33) and (34) for all $m \geq 0$ gives an asymptotic series for $g(n,b)$.

A.2.2 The asymptotics of $g(n,a,b)$ for a, b fixed, $n \to \infty$

In the same manner as in section A.2.1, we obtain

$$\begin{aligned} g(n,a,b) &= \sum_{k,l\geq 1} \frac{\mathrm{d}(\gcd(k,l))}{2\pi\mathbf{i}} \int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\,\Gamma(z)\,k^a\,l^b\,\left(k^2+l^2\right)^{-z}\mathrm{d}z \\ &= \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\,\Gamma(z)\sum_{k,l\geq 1}\mathrm{d}(\gcd(k,l))\,k^a\,l^b\,\left(k^2+l^2\right)^{-z}\mathrm{d}z, \end{aligned}$$

where $c > \frac{a+b+1}{2}$.

For k and l fixed, set $j=\gcd(k,l)$. Then we may write $k=k_1 j$ and $l = l_1 j$ with $\gcd(k_1,l_1)=1$. This leads to

$$\begin{aligned} g(n,a,b) &= \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\,\Gamma(z) \sum_{\substack{j,k_1,l_1\geq 1\\ \gcd(k_1,l_1)=1}} \mathrm{d}(j)\; j^{a+b-2z}\,k_1^a l_1^b\left(k_1^2+l_1^2\right)^{-z}\mathrm{d}z \\ &= \frac{1}{2\pi\mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\,\Gamma(z)\,\zeta(2z-a-b)^2 \sum_{\substack{k_1,l_1\geq 1\\ \gcd(k_1,l_1)=1}} k_1^a l_1^b\left(k_1^2+l_1^2\right)^{-z}\mathrm{d}z. \end{aligned}$$

Now we get rid of the constraint $\gcd(k_1,l_1)=1$:

Proposition 2. *We have the following identity:*

$$\sum_{\substack{k,l\geq 1\\ \gcd(k,l)=1}} \frac{k^a l^b}{(k^2+l^2)^z} = \frac{1}{\zeta(2z-a-b)}\sum_{k,l\geq 1}\frac{k^a l^b}{(k^2+l^2)^z}. \tag{35}$$

Proof. By inclusion–exclusion, the left–hand side equals the sum over *all* pairs (k,l) minus the sum over all pairs (k,l), where some prime number p divides $\gcd(k,l)$, plus the sum over all pairs (k,l), where the product of two different primes p_1p_2, divides $\gcd(k,l)$, and so on:

$$\begin{aligned}\sum_{\substack{k,l\ge 1\\ \gcd(k,l)=1}} \frac{k^a l^b}{(k^2+l^2)^z} &= \sum_{k,l\ge1}\frac{k^a l^b}{(k^2+l^2)^z} - \sum_{p\text{ prime}}\sum_{k,l\ge1}\frac{(kp)^a(lp)^b}{((kp)^2+(lp)^2)^z}\\ &\qquad + \sum_{p_1\ne p_2\text{ prime}}\sum_{k,l\ge1}\frac{(kp_1p_2)^a(lp_1p_2)^b}{((kp_1p_2)^2+(lp_1p_2)^2)^z} - + \cdots\\ &= \left(1-\sum_{p\text{ prime}}\frac{1}{p^{2z-a-b}} + \sum_{p_1\ne p_2\text{ prime}}\frac{1}{(p_1p_2)^{2z-a-b}} - + \cdots\right)\sum_{k,l\ge1}\frac{k^a l^b}{(k^2+l^2)^z}\\ &= \left(\prod_{p\text{ prime}}\left(1-\frac{1}{p^{2z-a-b}}\right)\right)\sum_{k,l\ge1}\frac{k^a l^b}{(k^2+l^2)^z}.\end{aligned}$$

The product in the last line is the reciprocal of the Euler product for $\zeta(2z-a-b)$ ([9, p. 225], which proves the assertion. □

Thus, we arrive at

$$g(n,a,b) = \frac{1}{2\pi \mathbf{i}}\int_{c-\mathbf{i}\infty}^{c+\mathbf{i}\infty} n^z\,\Gamma(z)\,\zeta(2z-a-b)\sum_{k,l\ge1}\frac{k^a l^b}{(k^2+l^2)^z}\,\mathrm{d}z. \tag{36}$$

Denote the integrand in the above formula by $G_2(a,b;z)$.

Again, we may shift the line of integration to the left as far as we want to, if we take into account the residues of our integrand $G_2(b;z)$. Computing the poles and residues clearly depends on some information about the double Dirichlet series involved. This information will be provided in the next (and final) subsection.

A.3 The Dirichlet series $\sum_{k,l\ge1}\frac{k^{2a}\,l^{2b}}{(k^2+l^2)^z}$

Note that for our purposes, we only need $g(n,2a,2b)$, so the series we are interested in is

$$Z(a,b;s) := \sum_{k,l\ge1}\frac{k^{2a}l^{2b}}{(k^2+l^2)^s}. \tag{37}$$

Clearly, this is closely related to the following function:

$$\begin{aligned}Z_*(a,b;s) &:= \sum_{(k,l)\in\mathbb{Z}^2,\,(k,l)\ne(0,0)}\frac{k^{2a}l^{2b}}{(k^2+l^2)^s}\\ &= 4\cdot Z(a,b;s) + 2\cdot[b=0]\,\zeta(2s-2a) + 2\cdot[a=0]\,\zeta(2s-2b)\,. \end{aligned}\tag{38}$$

Informations on the poles and residues of $Z(a,b;s)$ could be directly derived from the work of Pierrette Cassou–Noguès [4, p. 41ff], but there is a simpler way by using the reciprocity law for Jacobi's theta function. This reasoning is a generalization of Riemann's representation (see [7, section 1.12, (16)]) of $\zeta(s)$,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s-1} + \int_1^\infty \left(t^{\frac{1-s}{2}} + t^{\frac{s}{2}}\right) t^{-1}\omega(t)\,\mathrm{d}t, \tag{39}$$

where

$$\omega(t) = \sum_{n=1}^{\infty} \mathbf{e}^{-n^2\pi t} = \frac{1}{2}\left(\theta_3(0,\mathbf{i}t) - 1\right). \tag{40}$$

Here, θ_3 denotes (one variant of) Jacobi's theta function (see [8, section 13.19, (8)])

$$\theta_3(z,t) = \sum_{n=-\infty}^{\infty} q^{n^2}\mathbf{e}^{2n\mathbf{i}z}, \tag{41}$$

where $q = \mathbf{e}^{\mathbf{i}\pi t}$.

A.3.1 Jacobi's theta function

We rewrite (41) by setting $t = y$ and $z = \pi x$ for $x, y \in \mathbb{C}$; i.e.:

$$\vartheta(x,y) = \sum_{n=-\infty}^{\infty} \mathbf{e}^{2\pi\mathbf{i}\left(xn + \frac{y}{2}n^2\right)} \tag{42}$$

This series is absolute convergent for all x and all y with $\Im(y) > 0$. Therefore, for fixed y the function $f : z \mapsto \vartheta(z,y)$ is an entire function. We have the properties

$$\begin{aligned} \vartheta(x+1,y) &= \vartheta(x,y)\,, \\ \mathbf{e}^{2\pi\mathbf{i}\left(x+\frac{y}{2}\right)}\vartheta(x+y,y) &= \vartheta(x,y)\,, \end{aligned}$$

which in fact determine the theta function up to a multiplicative factor $c(y)$ (see [15, section 2.3]). Moreover, we have the following functional equation (see [15, Theorem 2.12, equation (2.29)]):

$$\vartheta(x,y) = \vartheta\left(\frac{x}{y}, -\frac{1}{y}\right)\mathbf{e}^{-\pi\mathbf{i}\frac{x^2}{y}}\sqrt{\frac{\mathbf{i}}{y}}$$

Setting $\bar{\vartheta}(y) := \vartheta(0,\mathbf{i}y)$, we obtain as a special case the following reciprocity law, valid for all y with $\Re(y) > 0$:

$$\bar{\vartheta}(y) = \sum_{n=-\infty}^{\infty} \mathbf{e}^{-\pi n^2 y} = \sqrt{\frac{1}{y}}\cdot\bar{\vartheta}\left(\frac{1}{y}\right). \tag{43}$$

$\bar{\vartheta}(y)$ is a holomorphic function in the half plane $\Re(y) > 0$, with $\Re(y) = 0$ as essential singular line, see [15, Satz 2.13].

Interchanging summation, differentiation and integration in the appropriate places, we obtain

$$\begin{aligned}(-\pi)^{a+b}\frac{\Gamma(s)}{\pi^s}Z_*(a,b;s) &= \sum_{(k,l)\in\mathbb{Z}^2\setminus\{0\}}\Gamma(s)\,\frac{(-\pi)^{a+b}}{\pi^s}\frac{k^{2a}l^{2b}}{(k^2+l^2)^s}\\ &= \sum_{(k,l)\in\mathbb{Z}^2\setminus\{0\}}\left(\pi\left(k^2+l^2\right)\right)^{-s}\int_0^\infty t^{s-1}\mathbf{e}^{-t}(-\pi)^{a+b}k^{2a}l^{2b}\mathrm{d}t\\ &= \sum_{(k,l)\in\mathbb{Z}^2\setminus\{0\}}\int_0^\infty u^{s-1}\left(-\pi k^2\right)^a\left(-\pi l^2\right)^b\mathbf{e}^{-\pi\left(k^2+l^2\right)u}\mathrm{d}u \qquad (44)\\ &= \sum_{(k,l)\in\mathbb{Z}^2\setminus\{0\}}\int_0^\infty u^{s-1}\left(\frac{\mathrm{d}^a}{\mathrm{d}u^a}\mathbf{e}^{-\pi k^2u}\right)\left(\frac{\mathrm{d}^b}{\mathrm{d}u^b}\mathbf{e}^{-\pi l^2u}\right)\mathrm{d}u.\end{aligned}$$

For (44), we used $\Gamma(s)=\int_0^\infty t^{s-1}\mathbf{e}^{-t}\mathrm{d}t=\alpha^s\int_0^\infty u^{s-1}\mathbf{e}^{-\alpha u}\mathrm{d}u$ with $\alpha=(k^2+l^2)\,\pi$.

Clearly, $Z_*(a,b;s)=Z_*(b,a;s)$. So w.l.o.g. we may assume $a\geq b$. We have to distinguish the following two cases, where we assume $a>0$ and $b\geq 0$:

$$\frac{\Gamma(s)}{\pi^s}Z_*(0,0;s)=\int_0^\infty u^{s-1}\left(\bar\vartheta(u)^2-1\right)\mathrm{d}u, \qquad (45)$$

$$(-\pi)^{a+b}\frac{\Gamma(s)}{\pi^s}Z_*(a,b;s)=\int_0^\infty u^{s-1}\left(\left(\frac{\mathrm{d}^a}{\mathrm{d}u^a}\bar\vartheta(u)\right)\left(\frac{\mathrm{d}^b}{\mathrm{d}u^b}\bar\vartheta(u)\right)\right)\mathrm{d}u. \qquad (46)$$

A.3.1.1 Case 1 Considering (45), we basically repeat the reasoning in [15, p. 203]. From (43) we get $\frac{1}{u}\bar\vartheta\left(\frac{1}{u}\right)^2=\bar\vartheta(u)^2$. Assuming $\Re(s)>1$, we compute:

$$\begin{aligned}\int_0^\infty u^{s-1}\left(\bar\vartheta(u)^2-1\right)\mathrm{d}u &= \int_0^1 u^{s-1}\left(\frac{1}{u}\bar\vartheta\left(\frac{1}{u}\right)^2-1\right)\mathrm{d}u+\int_1^\infty u^{s-1}\left(\bar\vartheta(u)^2-1\right)\mathrm{d}u\\ &= \int_0^1 u^{s-1}\left(\frac{1}{u}\left(\bar\vartheta\left(\frac{1}{u}\right)^2-1\right)+\frac{1}{u}-1\right)\mathrm{d}u+\int_1^\infty u^{s-1}\left(\bar\vartheta(u)^2-1\right)\mathrm{d}u\\ &= -\frac{1}{s}+\frac{1}{s-1}+\int_1^\infty t^{-s}\left(\bar\vartheta(t)^2-1\right)\mathrm{d}t+\int_1^\infty u^{s-1}\left(\bar\vartheta(u)^2-1\right)\mathrm{d}u. \qquad (47)\end{aligned}$$

The integrals in the last line converge for all $s\in\mathbb{C}$ and constitute holomorphic functions, so $Z_*(0,0;s)$ only has a simple pole for $s=1$.

A.3.1.2 Case 2 Considering (46), we have to adjust the preceding method appropriately. For convenience, set $\bar\vartheta_a(u):=\frac{\mathrm{d}^a}{\mathrm{d}u^a}\bar\vartheta(u)$. Then we have:

$$\int_0^\infty u^{s-1}\left(\bar\vartheta_a(u)\,\bar\vartheta_b(u)\right)\mathrm{d}u=\int_0^1 u^{s-1}\left(\bar\vartheta_a(u)\,\bar\vartheta_b(u)\right)\mathrm{d}u+\int_1^\infty u^{s-1}\left(\bar\vartheta_a(u)\,\bar\vartheta_b(u)\right)\mathrm{d}u.$$

Now use (43) in the form

$$\bar{\vartheta}_a(u)\,\bar{\vartheta}_b(u) = \left(\frac{\mathrm{d}^a}{\mathrm{d}u^a}\left(\sqrt{\frac{1}{u}}\bar{\vartheta}\left(\frac{1}{u}\right)\right)\right)\left(\frac{\mathrm{d}^b}{\mathrm{d}u^b}\left(\sqrt{\frac{1}{u}}\bar{\vartheta}\left(\frac{1}{u}\right)\right)\right)$$

and combine this with the the formula

$$\frac{\mathrm{d}^a}{\mathrm{d}y^a}\left(\sqrt{\frac{1}{y}}\cdot f\left(\frac{1}{y}\right)\right) = (-1)^a \sum_{k=0}^{a} \frac{(2a)^{\underline{2k}}}{4^k k!}\left(\left[\frac{\mathrm{d}^{a-k}}{\mathrm{d}y^{a-k}} f\right]\left(\frac{1}{y}\right) y^{k-2a-\frac{1}{2}}\right)$$

to obtain

$$\bar{\vartheta}_a(u)\,\bar{\vartheta}_b(u) = \frac{(-1)^{a+b}\,(2a)!(2b)!}{4^{a+b}a!b!}u^{-a-b-1} +$$
$$(-1)^{a+b}\sum_{k=0}^{a}\sum_{j=0}^{b}\frac{(2a)^{\underline{2k}}\,(2b)^{\underline{2j}}}{4^{k+j}k!j!}\left(\bar{\vartheta}_{a-k}\left(\frac{1}{u}\right)\bar{\vartheta}_{b-j}\left(\frac{1}{u}\right) - [a=k\wedge b=j]\right)u^{k+j-2a-2b-1}.$$

Now in the same way as before, this gives

$$\int_0^\infty u^{s-1}\left(\bar{\vartheta}_a(u)\,\bar{\vartheta}_b(u)\right)\mathrm{d}u =$$
$$\frac{(-1)^{a+b}\,(2a)!(2b)!}{4^{a+b}a!b!\,(s-a-b-1)} + \int_1^\infty u^{s-1}\left(\bar{\vartheta}_a(u)\,\bar{\vartheta}_b(u)\right)\mathrm{d}u + (-1)^{a+b}\times$$
$$\sum_{k=0}^{a}\sum_{j=0}^{b}\frac{(2a)^{\underline{2k}}\,(2b)^{\underline{2j}}}{4^{k+j}k!j!}\int_1^\infty\left(\bar{\vartheta}_{a-k}(u)\,\bar{\vartheta}_{b-j}(u) - [a=k\wedge b=j]\right)u^{2a+2b-k-j-s}\mathrm{d}u. \quad (48)$$

Again, the integrals converge for all $z\in\mathbb{C}$ and constitute holomorphic functions, whence $Z_*(a,b;s)$ has only one simple pole at $s=a+b+1$.

We summarize all this information in the following proposition.

Proposition 3. *For arbitrary nonnegative integers a, b, the series $\sum_{k,l\geq 1}\frac{k^{2a}l^{2b}}{(k^2+l^2)^s}$ is convergent in the half–plane $\Re(z) > a+b+1$ and defines a meromorphic function $Z(a,b;z):\mathbb{C}\to\mathbb{C}$ with a simple pole at $z=a+b+1$, where the residue is $\frac{\pi(2a)!(2b)!}{4^{a+b+1}a!b!(a+b)!}$.*

If $a>0$ and $b>0$, this is the only pole.

If $a=0$ (or $b=0$), there is another simple pole at $z=b+\frac{1}{2}$ (or $z=a+\frac{1}{2}$), where the residue is $-\frac{1}{4}\left([b=0]+[a=0]\right)$.

Moreover, we have the following information on special evaluations of $Z(a,b;z)$:

$$Z(a,b;-n) = \frac{1}{4}\,[a=b=n=0]\ \text{for}\ n=0,1,2,\ldots. \quad (49)$$

For the constant term $c_{a,b}$ in the Laurent series expansion of $Z(a,b;z)$ at $z_0 = a+b+\frac{1}{2}$, we have the following formulas:

$$\begin{aligned}
c_{0,0} &= -\gamma - 1 + \frac{1}{2}\int_1^\infty t^{-\frac{1}{2}}\left(\bar\vartheta(t)^2 - 1\right)\mathrm{d}t,\\
c_{a,0} &= -\frac{\gamma}{2} - \frac{1}{2} + \frac{4^{a-1}a!\,((-1)^a+1)}{(2a)!}\int_1^\infty t^{a-\frac{1}{2}}\bar\vartheta_a(t)\,\bar\vartheta(t)\,\mathrm{d}t\\
&\quad + \sum_{k=1}^{a}\frac{4^{a-k-1}a!}{k!\,(2a-2k)!}\int_1^\infty t^{a-k-\frac{1}{2}}\left(\bar\vartheta_{a-k}(t)\,\bar\vartheta(t) - [a=k]\right)\mathrm{d}t \text{ for } a>0,\\
c_{a,b} &= \frac{4^{a+b-1}\,(a+b)!}{(2a+2b)!}\Bigg(-2\frac{(2a)!\,(2b)!}{4^{a+b}a!b!} + (-1)^{a+b}\int_1^\infty t^{a+b-\frac{1}{2}}\bar\vartheta_a(t)\,\bar\vartheta_b(t)\,\mathrm{d}t\\
&\quad + \sum_{k=0}^{a}\sum_{j=0}^{b}\frac{(2a)^{\underline{2k}}\,(2b)^{\underline{2j}}}{4^{k+j}k!j!}\\
&\quad \times \int_1^\infty\left(\bar\vartheta_{a-k}(t)\,\bar\vartheta_{b-j}(t) - [a=k\wedge b=j]\right)t^{a+b-k-j-\frac{1}{2}}\mathrm{d}t\Bigg) \text{ for } a\ge b>0. \qquad (50)
\end{aligned}$$

Proof. This information is extracted straightforwardly from (38) together with (45), (47) and (46), (48), respectively.

(The evaluation $\zeta(-2n)=0$ for nonnegative integers n, which is needed for (49), can be found in [7, 1.13, (22)].) □

References

[1] G.E. Andrews, R. Askey, and R. Roy. *Special Functions*. Cambridge University Press, Cambridge, 1999.

[2] P. Biane, J. Pitman, and M. Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bulletin of the American Math. Soc.*, 38:435–465, 2001.

[3] N. Bonichon and M. Mosbah. Watermelon uniform random generation with applications. *Theoretical Computer Science*, 307:241–256, 2003.

[4] P. Cassou-Noguès. Prolongement de certaines séries de Dirichlet. *Amer. J. Math.*, 105:13–58, 1983.

[5] N.G. de Bruijn. *Asymptotic methods in analysis.* Dover Publications, Inc., 3rd edition, 1981.

[6] N.G. de Bruijn, D.E. Knuth, and S.O. Rice. The average height of planted plane trees. In R.C. Read, editor, *Graph Theory and Computing*, pages 15–22. Academic Press, 1972.

[7] A. Erdélyi. *Higher Transcendental Functions*, volume 1. McGraw–Hill, 1953.

[8] A. Erdélyi. *Higher Transcendental Functions*, volume 2. McGraw–Hill, 1953.

[9] L. Euler. *Introductio in analysin infinitorum.* Lausanne, 1748.

[10] M.E. Fisher. Walks, walls, wetting and melting. *J. Stat. Phys.*, 34:667–729, 1984.

[11] P. Flajolet and R. Sedgewick. The average case analysis of algorithms: Mellin transform asymptotics. Technical report, Institut National de Recherche en Informatique et Automatique, 1996.

[12] I. M. Gessel and X.G. Viennot. Determinants, paths, and plane partitions. preprint, available at http://www.cs.brandeis.edu/~ira/papers/pp.pdf, 1989.

[13] R. L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison–Wesley, 1988.

[14] C. Krattenthaler, A. Guttmann, and X.G. Viennot. Vicious walkers, friendly walkers and Young tableaux II: with a wall. *J. Phys. A: Math. Gen.*, 33:8835–8866, 2000.

[15] E. Krätzel. *Analytische Funktionen in der Zahlentheorie*, volume 139 of *Teubner–Texte zur Mathematik.* B.G. Teubner, Stuttgart, 2000.

[16] Sri Gopal Mohanty. *Lattice Path Counting and Applications.* Academic Press, 1979.

[17] N. E. Nörlund. *Vorlesungen über Differenzenrechnung.* Chelsea Publishing Company, 1954.

[18] L. Takács. Remarks on random walk problems. *Publ. Math. Inst. Hung. Acad. Sci.*, 2:175–182, 1957.