Edge-bandwidth of the triangular grid

Reza Akhtar

Dept. of Mathematics and Statistics
Miami University, Oxford, OH 45056, USA

reza@calico.mth.muohio.edu

Tao Jiang

Dept. of Mathematics and Statistics
Miami University, Oxford, OH 45056, USA

jiangt@muohio.edu

Dan Pritikin

Dept. of Mathematics and Statistics
Miami University, Oxford, OH 45056, USA

pritikd@muohio.edu

Submitted: May 19, 2006; Accepted: Aug 16, 2007; Published: Oct 5, 2007
Mathematics Subject Classification: 05C78

Abstract

In 1995, Hochberg, McDiarmid, and Saks proved that the vertex-bandwidth of the triangular grid T_n is precisely $n+1$; more recently Balogh, Mubayi, and Pluhár posed the problem of determining the edge-bandwidth of T_n. We show that the edge-bandwidth of T_n is bounded above by $3n-1$ and below by $3n-o(n)$.

1 Introduction

A *labeling* of the vertices of a finite graph G is a bijective map $h : V(G) \to \{1, 2, \ldots, |V(G)|\}$. The *vertex-bandwidth* of h is defined as

$$B(G,h) = \max_{\{u,v\}\in E(G)} |h(u) - h(v)|$$

and the vertex-bandwidth (or simply bandwidth) of G is defined as

$$B(G) = \min_h B(G,h)$$

in which the minimum is taken over all labelings of $V(G)$. The *edge-bandwidth* of G is defined as

$$B'(G) = B(L(G))$$

where $L(G)$ is the line graph of G. Edge-bandwidth has been studied for several classes of graphs in various sources, among them [1], [2], [5], and [6].

In this article, we study the edge-bandwidth of the *triangular grid* T_n. For any integer $n \geq 0$, T_n is the graph whose vertices are ordered triples of nonnegative integers summing to n, with an edge connecting two triples if they agree in one coordinate and differ by 1 in the other two; see Figure 1 for an illustration of T_5, the bottom row vertices (from left to right) being $(0,5,0)$, $(1,4,0)$, $(2,3,0)$, $(3,2,0)$, $(4,1,0)$ and $(5,0,0)$. The vertex-bandwidth of T_n was studied by Hochberg, McDiarmid, and Saks; in [4], they proved that $B(T_n) = n+1$. The problem of determining $B'(T_n)$ was posed by Balogh, Mubayi, and Pluhár in [1].

Our main result is:

Theorem 1.1.

$$3n - o(n) \leq B'(T_n) \leq 3n - 1.$$

It is easy to obtain the stated upper bound on $B'(T_n)$ by considering the "top to bottom, then left to right" labeling of $E(T_n)$ as shown in Figure 1 for the case $n = 5$. This labeling may be defined by recursion, the base case T_0 being trivial as there are no edges. Now suppose $n > 0$; for each i, $0 \leq i \leq n-1$, let e_i be the edge with endpoints $(i, n-i, 0)$ and $(i, n-i-1, 1)$, f_i the edge with endpoints $(i+1, n-i-1, 0)$ and $(i, n-i-1, 1)$ and g_i the edge with endpoints $(i, n-i, 0)$ and $(i+1, n-i-1, 0)$. Observe that the subgraph of T_n induced by vertices of the form (a,b,c) with $c > 0$ is isomorphic to T_{n-1}. Label the edges of this subgraph inductively using the integers $1, 2, \ldots, \frac{3}{2}n(n-1)$. Next, use the integers $\frac{3}{2}n(n-1)+1, \ldots \frac{3}{2}n(n+1)$ to label the remaining edges in the order: $e_0, f_0, e_1, f_1, \ldots, e_{n-1}, f_{n-1}, g_0, \ldots, g_{n-1}$. The bandwidth of this edge-labeling is readily seen to be $3n-1$, so it follows that

$$B'(T_n) \leq 3n - 1.$$

The proof of the lower bound is more difficult, and constitutes the content of this article. In Section 2, we recall a general lower bound for bandwidth due to Harper and apply this in Section 4, together with several other ideas, to complete the proof of Theorem 1.1. In the proof, we also give a more precise description of the error term.

Throughout this article, we use the notation $[a,b]$ to mean $\{n \in \mathbb{Z} : a \leq n \leq b\}$ when referring to sets of indices; we define (a,b), $[a,b)$, etc. similarly. If G is a graph and

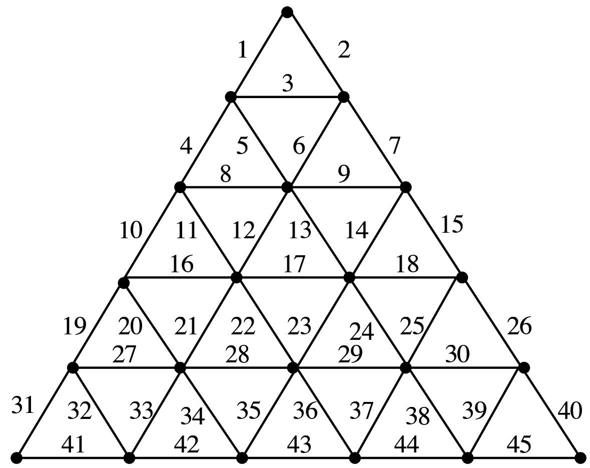

Figure 1: An edge-labeling of T_n with bandwidth $3n-1$

$S \subseteq V(G)$, the subgraph induced by S is denoted $G[S]$. If $F \subseteq E(G)$ is a set of edges, we denote by $G[F]$ the subgraph of G whose vertex set is the the set of endpoints of edges in F and whose edge set is F.

2 Lower bounds on bandwidth

Definition 2.1. *Let G be a graph and $S \subseteq V(G)$. The* boundary *of S is defined as:*

$$\partial(S) = \{v \in V(G) - S : vw \in E(G) \text{ for some } w \in S\}$$

Now suppose G is a graph and $h : V(G) \to [1, |V(G)|]$ a labeling. For $k \in [1, |V(G)|]$, let $S_k = \{v \in V(G) : h(v) \leq k\}$. The next proposition, essentially due to Harper [3], gives an elementary lower bound on bandwidth:

Proposition 2.2. *For any $k \in [1, |V(G)|]$, $B(G, h) \geq \max\{|\partial(S_k)|, |\partial(V(G) - S_k)|\}$.*

Proof.
Let $v \in \partial(S_k)$ be a vertex with maximum label; that is, $h(v) \geq h(w)$ for all $w \in \partial(S_k)$. Then $h(v) \geq k + |\partial(S_k)|$. However, v is adjacent to some vertex $u \in S_k$, so $h(u) \leq k$. Thus, $B(G, h) \geq |\partial(S_k)|$.

Likewise, let $v' \in \partial(V(G) - S_k)$ be a vertex with minimum label. Then $h(v') \leq k + 1 - |\partial(V(G) - S_k)|$. However, v' is by definition adjacent to some vertex $u' \in V(G) - S_k$, so $h(u') \geq k + 1$. Hence, $B(G, h) \geq |\partial(V(G) - S_k)|$. □

For $i \geq 2$, we define (inductively) the ith *iterated boundary* of $S \subseteq V(G)$ by

$$\partial^i(S) = \partial(\partial^{i-1}(S))$$

and the ith *shadow* by

$$\sigma^i(S) = \cup_{j=1}^i \partial^j(S).$$

A similar argument easily yields the following generalization of Proposition 2.2:

Proposition 2.3. *For any $k \in [1, |V(G)|]$ and labeling h of $V(G)$,*

$$B(G,h) \geq \max\left\{\frac{|\sigma^i(S_k)|}{i}, \frac{|\sigma^i(V(G) - S_k)|}{i}\right\}$$

Corollary 2.4. *With notation as above,*

$$B(G,h) \geq \frac{1}{2i}\big(|\sigma^i(S_k)| + |\sigma^i(V(G) - S_k)|\big).$$

Hence, a natural strategy for establishing b as a lower bound for $B(G)$ might be described as follows: given any labeling h, choose $k = k(h)$ suitably, and then apply the estimate of Corollary 2.4.

3 The Triangular Grid: Definitions

Let (i, j, k) be a vertex of the triangular grid T_n; recall that $i + j + k = n$. We typically refer to the first coordinate of such a triple as the i-coordinate, the second as the j-coordinate, and the third as the k-coordinate. We also use the notation $i(v)$ to refer to the i-coordinate of v, and so on.

We introduce some terminology to enhance the geometric intuition behind our reasoning. For each $c \in [0, n]$, let I_c (J_c, K_c) be the subgraph induced by the set of vertices whose i-coordinate (resp. j-coordinate, k-coordinate) equals c. We refer to the subgraphs I_i, J_j, and K_k as *lines*. The lines I_0, J_0, K_0 are called *sides* of T_n.

Definition 3.1. *A* connector *of T_n is a connected subgraph $S \subseteq T_n$ which contains a vertex from each side of T_n. A* tree connector *is a connector which is a tree.*

Observe that each connector of T_n has at least n vertices, and that every connector contains a tree connector.

The following principle will often be invoked without explicit mention; the proof follows immediately from the description of $E(T_n)$.

Proposition 3.2. *(Intermediate Value Principle) Let P be a v, w path in T_n. Set $m_i = \min\{i(v), i(w)\}$ and $M_i = \max\{i(v), i(w)\}$; we define m_j, M_j, m_k and M_k analogously. If $i \in [m_i, M_i]$, $j \in [m_j, M_j]$, and $k \in [m_k, M_k]$, then P contains (possibly indistinct) vertices from each of I_i, J_j, and K_k.*

4 Proof of Theorem 1.1

We now turn to the proof of the lower bound in Theorem 1.1.

Fix a choice of functions $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f(n) = o(n)$, $g(n) = o(f(n))$. For example, one might choose $f(n) = n^{\frac{2}{3}}$ and $g(n) = n^{\frac{1}{3}}$.

Suppose $n \gg 0$ and let T' be the subgraph of T_n induced by $\{(a, b, c) \in V(T_n) : a, b, c \geq g(n))\}$. Clearly $T' \cong T_{n-\lfloor 3g(n) \rfloor}$.

Let $h : E(T_n) \to [1, |E(T_n)|]$ be an edge-labeling of T_n that achieves $B'(T_n)$; that is, $B'(h) = B'(T_n)$, where $B'(h)$ denotes the maximum difference between the h-labels of two incident edges in T_n. Let $E_k = \{e \in E(T_n) : h(e) \leq k\}$, and define

$$r = \min\{k : T'[E_{k+1} \cap E(T')] \text{ is a connector of } T'\}.$$

We define a 2-coloring of $E(T_n)$ by declaring edges e with $h(e) \leq r$ to be red and the remaining edges blue. We call a vertex $v \in T_n$ *red* if all edges incident at v are red, *blue* if all edges incident at v are blue, or *mixed* otherwise. Let R (resp. B) denote the set of red (resp. blue) edges and $\mathcal{R}$ (resp., $\mathcal{B}$, $\mathcal{M}$) the set of red (resp. blue, mixed) vertices.

We recall the following Lemma from [4]:

Lemma 4.1. *([4], Lemma 6) Suppose the vertices of the triangular grid are colored with two colors. Then exactly one of the color classes contains a connector.*

Proposition 4.2. *There exists a connector S of T' such that $|V(S) - \mathcal{M}| \leq 1$.*

Proof.
Let r be as above and $C = T'[E_{r+1} \cap E(T')]$. Suppose $V(C)$ contains a blue vertex. Then v must be an endpoint of the edge labeled $r + 1$ and must lie on one of the sides of T'; in particular, there can be at most one such vertex. If such v exists, let $\mathcal{M}' = \mathcal{M} \cup \{v\}$ and $\mathcal{B}' = \mathcal{B} - \{v\}$; otherwise, let $\mathcal{M}' = \mathcal{M}$ and $\mathcal{B}' = \mathcal{B}$. In either case, $\mathcal{R} \cup \mathcal{M}'$ induces a connector of T'. On the other hand, $\mathcal{R}$ does not induce a connector of T'. Thus, by Lemma 4.1, $\mathcal{B}'$ does not induce a connector of T'. Since no vertex of $\mathcal{R}$ is adjacent to a vertex of $\mathcal{B}'$, it follows that $\mathcal{R} \cup \mathcal{B}'$ does not induce a connector of T'. Applying Lemma 4.1 again, we conclude that $\mathcal{M}'$ induces a connector of T'. □

Lemma 4.3. *Let T be any triangular grid and $V_0 \subseteq V(T)$ a subset such that $S^* = T[V_0]$ is a connector of T. If V_0 is minimal with respect to this property (that is, for every $v \in V_0$, $T[V_0 - \{v\}]$ is not a connector of T), then either S^* is a tree connector or there is some edge $e \in E(S^*)$ such that $S^* - e$ is a tree connector.*

Proof.

Let F_i, $s = 1, 2, 3$ be the three sides of T. Since S^* is a connector of T, there is a vertex $v_0 \in V(S^*)$ such that for each $i \in \{1, 2, 3\}$, there is a path in S^* from v_0 to some vertex v_i lying on F_i. The v_i are not necessarily distinct; however, $|\{v_1, v_2, v_3\}| \geq 2$. If $|\{v_1, v_2, v_3\}| = 2$, then by minimality of $|V(S^*)|$, S^* itself must be a shortest path connecting the two distinct members of $\{v_1, v_2, v_3\}$. Hence, we may assume $|\{v_1, v_2, v_3\}| = 3$.

For each $i = 1, 2, 3$, let P_i denote a shortest v_0, v_i-path in S^*. By minimality, $V(S^*) = V(P_1) \cup V(P_2) \cup V(P_3)$ and $V(P_i) \cap V(P_j) = \{v_0\}$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Let $e = \{x, y\} \in E(S^*) - (E(P_1) \cup E(P_2) \cup E(P_3))$. Since each P_i is a shortest v_0, v_i-path in S^*, it must be the case that $x \in V(P_i)$ and $y \in V(P_j)$, where $i \neq j$; clearly x and y are both distinct from v_0. If there is a vertex z on P_i that lies beween v_0 and x, then $S^* - z$ is still a connector of T, contradicting the minimality of S^*. Hence x is a neighbor of v_0. By symmetric reasoning, y is also a neighbor of v_0.

We have argued that the endpoints of an edge in $E(S^*) - (E(P_1) \cup E(P_2) \cup E(P_3))$ are neighbors of v_0, and that these two endpoints lie on distinct paths P_i and P_j. If there exist two such edges, it is easily seen $S^* - v_0$ is a still a connector of T, again contradicting minimality. Hence, S^* has at most one edge outside $E(P_1) \cup E(P_2) \cup E(P_3)$. $\square$

By Proposition 4.2 and Lemma 4.3 there exists a connector S^* of T' with $V(S^*) \subseteq \mathcal{M}'$ and $|E(S^*)| \leq |V(S^*)|$.

Proposition 4.4. *If $|V(S^*)| \geq \frac{6}{5}n + 1$, then $B'(T_n) = B'(h) \geq 3n - 1$.*

Proof.

Since $V(S^*) \subseteq \mathcal{M}'$, $|V(S^*) \cap \mathcal{M}| \geq |V(S^*)| - 1$, so every edge incident at a vertex in $V(S^*) \cap \mathcal{M}$ is in $\partial(R) \cup \partial(B)$. Since each such vertex has degree 6 in T_n, we see that

$$|\partial(R) \cup \partial(B)| \geq \sum_{v \in V(S^*) \cap \mathcal{M}} \deg v - |E(S^*)| \geq 6(|V(S^*)| - 1) - |V(S^*)| = 5|V(S^*)| - 6 \geq 6n - 1$$

Applying Corollary 2.4 (with $i = 1$) to $L(T_n)$, we obtain $B'(T_n) = B'(h) \geq 3n - 1$. $\square$

By discarding an edge if necessary, we assume henceforth that there exists a tree connector S_0 of T' such that $|V(S_0)| \leq \frac{6}{5}n$.

Definition 4.5. *Let S be a tree connector of the triangular grid T_m and $a, b \geq 0$. An (a, b)-detour in S is a 4-tuple (u, v, P, Q), where $u, v \in V(S)$, P is the unique path in S from u to v of length at least a, and Q is a path in T_m from u to v of length at most b.*

Definition 4.6. *Let S be a tree connector of T_m and (u, v, P, Q) an (a, b)-detour in S. The* shortening *of S with respect to Q, denoted $\Sigma(S, Q)$ is defined as follows:*

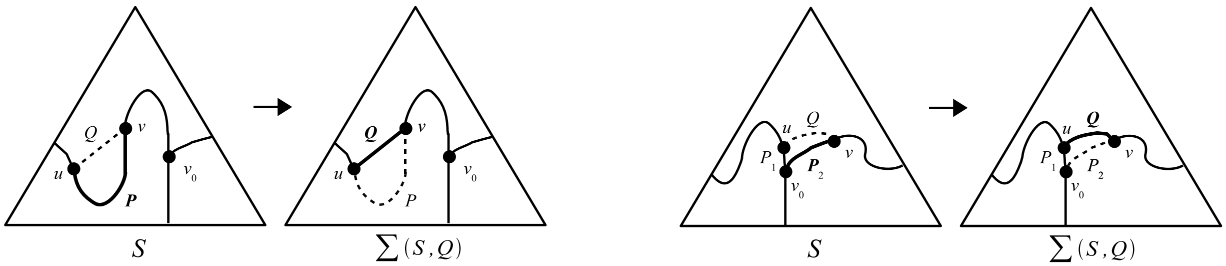

Figure 2: The shortening of S with respect to Q.

- *If v_0 does not lie along P, $\Sigma(S,Q)$ is the subgraph of T_m induced by* $V(S) \cup V(Q) - (V(P) - \{u,v\})$.

- *If v_0 lies along P, let P_1 be the portion of P between u and v_0 and P_2 the portion of P between v and v_0. If $|V(P_1)| \leq |V(P_2)|$, we define $\Sigma(S,Q) = T_m[V(S) \cup V(Q) - (V(P_2) - \{v_0\})]$; otherwise, we define* $\Sigma(S,Q) = T_m[V(S) \cup V(Q) - (V(P_1) - \{v_0\})]$.

It follows immediately from the construction that $\Sigma(S,Q)$ is a connector of T_m and that $|V(\Sigma(S,Q)) \cap V(S)| \geq m - |V(Q)|$.

Next, we show that we can use the operation of shortening to deduce the existence of a connector of T' with $n - o(n)$ vertices which does not contain a long detour.

Proposition 4.7. *Let $f(n)$, $g(n)$ be the functions chosen at the beginning of this section. Then there exists a tree connector S' of T' containing no $(2f(n), 2g(n))$-detour such that*

$$|V(S') \cap \mathcal{M}| \geq n - 3g(n) - 1 - \frac{6}{5}\frac{ng(n)}{f(n) - g(n)}.$$

Proof.

We begin by considering our tree connector S_0 and recall that $|V(S_0| \leq 6n/5$. Because S_0 is a connector of T', $|V(S_0)| \geq n - 3g(n)$.

Now consider the following inductive procedure.

- Set $i = 0$.

- If S_i contains no $(2f(n), 2g(n))$-detour, set $S' = S_i$. Otherwise, let (u, v, P, Q) be a $(2f(n), 2g(n))$-detour in S_i and define S_{i+1} to be a tree connector of T' contained in $\Sigma(S_i, Q)$.

At each iteration of this procedure, in moving from S_i to S_{i+1}, at least $f(n)$ vertices are discarded and at most $g(n)$ vertices from outside $V(S_0)$ are added. Thus, the procedure terminates after at most $\dfrac{6n/5}{f(n)-g(n)}$ iterations, and S' contains no $(2f(n), 2g(n))$-detour. The estimate on the number of mixed vertices in $V(S')$ now follows readily. $\square$

We may assume that S' consists of a vertex v_0, a path P_1 from v_0 to some vertex t_1 on the side F_1 of T', a path P_2 from v_0 to some vertex t_2 on the side F_2 of T', and a path P_3 from v_0 to some vertex t_3 on the third side F_3. We may also assume that P_1, P_2, and P_3 intersect pairwise only at v_0. Note that F_1 is a subgraph of the line $I_{g(n)}$ of T_n; similarly F_2 (F_3) is a subgraph of $J_{g(n)}$ (resp. $K_{g(n)}$).

Let w_0 be a vertex of $V(P_1) \cup V(P_2)$ with minimal k-coordinate; that is, $k(w_0) \leq k(w)$ for all $w \in V(P_1) \cup V(P_2)$. Writing $w_0 = (a, b, c)$, we have $a + b + c = n$. By the Intermediate Value Principle (Proposition 3.2), for each $i \in [g(n), a)$, P_1 contains at least one vertex with that i-coordinate; similarly for each $j \in [g(n), b)$, P_2 contains at least one vertex with that j-coordinate, and for each $k \in [g(n), c)$, P_3 contains at least one vertex with that k-coordinate.

If $x = (i, j, k) \in V(T_n)$ is any vertex and t is a positive integer, we define

$$N_I^+(x, t) = \{(i, j - s, k + s) \in V(T_n) : 0 \leq s \leq t\}.$$

Intuitively, this is the set of vertices reachable by starting at x and walking t steps along I_i in the direction away from the side K_0 of T_n. We also define:

$$N_I^-(x, t) = \{(i, j + s, k - s) \in V(T_n) : 0 \leq s \leq t\}$$

$$N_J^+(x, t) = \{(i - s, j, k + s) \in V(T_n) : 0 \leq s \leq t\}$$
$$N_J^-(x, t) = \{(i + s, j, k - s) \in V(T_n) : 0 \leq s \leq t\}$$

$$N_K^+(x, t) = \{(i + s, j - s, k) \in V(T_n) : 0 \leq s \leq t\}$$
$$N_K^-(x, t) = \{(i - s, j + s, k) \in V(T_n) : 0 \leq s \leq t\}$$

each of which has an analogous geometric interpretation.

Now define

$$\mathcal{I} = \{i \in [g(n), a) : V(S') \cap V(I_i) \cap \mathcal{M} \neq \emptyset\}.$$

This is the set of "good" indices i for which I_i contains a mixed vertex of S'. Similarly, we define

$$\mathcal{J} = \{j \in [g(n), b) : V(S') \cap V(J_j) \cap \mathcal{M} \neq \emptyset\}$$

and

$$\mathcal{K} = \{k \in [g(n), c - g(n)) : V(S') \cap V(K_k) \cap \mathcal{M} \neq \emptyset\}.$$

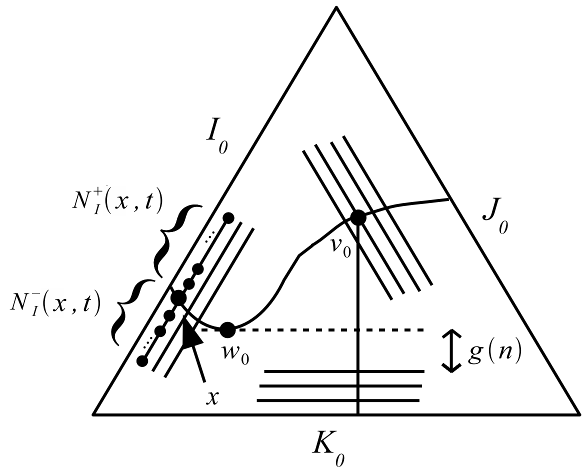

Figure 3: Illustration of $N_I^+(x,t)$ and $N_I^-(x.t)$.

Observe that by the equation $a+b+c=n$ and Proposition 4.7,

$$|\mathcal{I}|+|\mathcal{J}|+|\mathcal{K}| \geq n-4-4g(n)-\frac{6}{5}\frac{ng(n)}{f(n)-g(n)}. \tag{1}$$

For each $i \in \mathcal{I}$, let A_i^+ be the vertex of $V(I_i) \cap V(P_1) \cap \mathcal{M}$ with maximum k-coordinate and A_i^- the vertex of $V(I_i) \cap V(P_1) \cap \mathcal{M}$ with minimum k-coordinate. For each $j \in \mathcal{J}$, let B_j^+ be the vertex of $V(J_j) \cap V(P_2) \cap \mathcal{M}$ with maximum k-coordinate and B_j^- the vertex with minimum k-coordinate. Finally, for each $k \in \mathcal{K}$, let C_k^+ be the vertex of $V(K_k) \cap V(P_3) \cap \mathcal{M}$ with maximum i-coordinate and C_k^- the vertex with minimum k-coordinate.

For $i \in \mathcal{I}$, set $N_I(i) = N^+(A_i^+, g(n)) \cup N^-(A_i^-, g(n))$; for $j \in \mathcal{J}$, set $N_J(j) = N^+(B_j^+, g(n)) \cup N^-(B_j^-, g(n))$; for $k \in \mathcal{K}$, set $N_K(k) = N^+(C_k^+, g(n)) \cup N^-(C_k^-, g(n))$. Each of these newly defined sets has exactly $2g(n)+1$ members. Note also the following two facts:

- If $i_1, i_2 \in \mathcal{I}$, $i_1 \neq i_2$, then $N_I(i_1) \cap N_I(i_2) = \emptyset$, and similarly for the other two coordinates.
- For any $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $k \in \mathcal{K}$, $N_I(i) \cap N_K(k) = \emptyset = N_J(j) \cap N_K(k)$.

The first is an obvious consequence of the definitions; the second is a consequence of the choice of w_0 and the definition of the set $\mathcal{K}$.

It may be the case, however, that there is some pair $(i,j) \in \mathcal{I} \times \mathcal{J}$ such that $N_I(i) \cap N_J(j) \neq \emptyset$; this implies that there is some vertex in $V(S') \cap I_i$ which is within distance $2g(n)$ of some vertex in $V(S') \cap J_j$. Fix such a pair (i_0, j_0) and vertices $A_{i_0} \in V(I_{i_0}) \cap V(S')$, $B_{j_0} \in V(J_{j_0}) \cap V(S')$ such that $d_{T_n}(A_{i_0}, B_{j_0}) \leq 2g(n)$ and i_0 is as small as possible. Since S' is a tree, there is a unique path in S' from $w_0 = (a, b, c)$ to the vertex t_1; by Proposition 3.2, we may assume without loss of generality that A_{i_0} is on this path. Likewise, we may assume that B_{j_0} lies on the unique path in S' from w_0 to t_2. Since S' contains no $(2f(n), 2g(n))$-detour, it follows that $d_{S'}(A_{i_0}, B_{j_0}) \leq 2f(n)$. In particular, $d_{S'}(A_{i_0}, B_{j,0}) = d_{S'}(A_{i_0}, w_0) + d_{S'}(w_0, B_{j_0}) \leq (a - i_0) + (b - j_0) \leq 2f(n)$, so

$$a + b - i_0 - j_0 \leq 2f(n). \tag{2}$$

Let $\mathcal{I}' = \mathcal{I} \cap [1, i_0 - 1]$ and $\mathcal{J}' = \mathcal{J} \cap [1, j_0 - 1]$.

By construction, for any $i' \in \mathcal{I}'$ and $j' \in \mathcal{J}'$, $N_I(i') \cap N_J(j') = \emptyset$. Using the inequalities (1) and (2) and recalling that $a + b + c = n$, we have:

$$\begin{aligned}|\mathcal{I}'| + |\mathcal{J}'| + |\mathcal{K}| = |\mathcal{I}| - (a - i_0) + |\mathcal{J}| - (b - j_0) + \mathcal{K} &= |\mathcal{I}| + |\mathcal{J}| + |\mathcal{K}| - (a + b - i_0 - j_0) \\ &\geq n - 4 - 2f(n) - 4g(n) - \frac{6}{5}\frac{ng(n)}{f(n) - g(n)}.\end{aligned}$$

This immediately yields:

Proposition 4.8. *Let $V = \cup_{i \in \mathcal{I}'} N_I(i) \cup \cup_{j \in \mathcal{J}'} N_J(j) \cup \cup_{k \in \mathcal{K}} N_K(k)$. Then*

$$|V| \geq (2g(n) + 1)\left(n - 4 - 4g(n) - 2f(n) - \frac{6}{5}\frac{ng(n)}{f(n) - g(n)}\right).$$

Finally, if $v \in V$, then there is some $w \in V(S') \cap \mathcal{M}$ such that $d_{T_n}(v, w) \leq g(n)$. In particular, since w is a mixed vertex, each edge incident to a vertex in V is contained in $\sigma^{g(n)+1}(R) \cup \sigma^{g(n)+1}(B)$. Since each vertex of U has degree 6, we obtain the estimate

$$|\sigma^{g(n)+1}(R) \cup \sigma^{g(n)+1}(B)| \geq 3\,(2g(n) + 1)\left(n - 4 - 4g(n) - 2f(n) - \frac{6}{5}\frac{ng(n)}{f(n) - g(n)}\right).$$

□

Applying Corollary 2.4 to $L(T_n)$, we obtain

Corollary 4.9.

$$B'(T_n) = B'(h) \geq 3\frac{2g(n) + 1}{2(g(n) + 1)}\left(n - 4 - 4g(n) - 2f(n) - \frac{6}{5}\frac{ng(n)}{f(n) - g(n)}\right).$$

In particular, by choosing $f(n) = n^{\frac{2}{3}}$ and $g(n) = n^{\frac{1}{3}}$, we obtain, for sufficiently large n,

$$B'(T_n) \geq 3(n - 4n^{\frac{2}{3}}) = 3n - 12n^{\frac{2}{3}} = 3n - o(n).$$

Corollary 4.9 completes the proof of Theorem 1.1. □

After the manuscript was submitted, the third author noticed that in general, the upper bound of $3n - 1$ is not optimal. Indeed, when $n \geq 18$, the given labeling can be modified to produce a more complicated labeling with bandwidth $3n - 5$. However, it is not immediately clear if the method would lead to an improvement of the upper bound by more than a constant term.

5 Acknowledgements

The authors would like to thank Zevi Miller for stimulating discussions, and József Balogh, Dhruv Mubayi, and Oleg Pikhurko for helpful conversations.

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