On the number of genus one labeled circle trees

Karola Mészáros
Massachusetts Institute of Technology
karola@math.mit.edu

Submitted: Sep 25, 2005; Accepted: Sep 17, 2007; Published: Oct 5, 2007
Mathematics Subject Classification: 05C30, 05C75, 05C10, 05A99

Abstract

A genus one labeled circle tree is a tree with its vertices on a circle, such that together they can be embedded in a surface of genus one, but not of genus zero. We define an e-reduction process whereby a special type of subtree, called an e-graph, is collapsed to an edge. We show that genus is invariant under e-reduction. Our main result is a classification of genus one labeled circle trees through e-reduction. Using this we prove a modified version of a conjecture of David Hough, namely, that the number of genus one labeled circle trees on n vertices is divisible by n or $n/2$. Moreover, we explicitly characterize when each of these possibilities occur.

1 Introduction

Graphical enumeration arises in a variety of contexts in combinatorics [2], and naturally so in the realm of combinatorial objects with interesting topological properties [5]. We provide a new classification of genus one circle trees and address a question raised by Hough [3] about their number. Our study is motivated by numerous results in the study of partitions and trees of a certain genus, as well as results about the genuses of maps and hypermaps, [1], [6], [7], [8].

The following two definitions are discussed in [3] in great detail; we shall use the definition of a labeled circle tree throughout the paper, whereas we shall mostly use an alternate, less technical definition for the genus of a circle tree.

Definition 1. *A **labeled circle tree (l-c-tree)** on n points is a tree with its n vertices labeled 1 through n on a circle in a counterclockwise direction and its edges drawn as straight lines within the circle.*

Definition 2. *The* ***genus*** g ***of a l-c-tree*** T *on* n *points is defined to be* $g(\alpha) = 1 + \frac{1}{2}(n\text{-}1\text{-}z(\alpha) - z(\alpha^{-1} \cdot \sigma))$*, where* α *is the matching of the given l-c-tree* T*,* $\sigma = (1\ 2\ 3\ 4\ \ldots\ n)$*, and the function* z *gives the number of cycles of its argument;* α^{-1} *is the inverse permutation of* α*, and the multiplication of two permutations is from right to left.*

The genus of a l-c-tree can also be described as the genus of the surface with minimal genus such that the tree together with the circle it is drawn on can be drawn on the surface without crossing edges. In particular, a genus one l-c-tree is such that the tree together with the circle it is drawn on can be embedded in a surface of genus one, but not of genus zero.

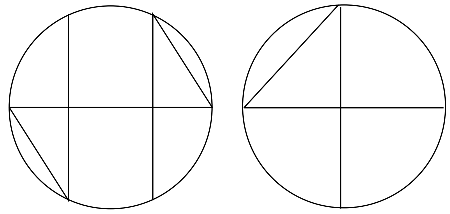
Hough [3] observed that the number of genus one labeled circle trees on n points (denoted by ***f(n)***) is divisible by n for small values of n, and hypothesized the same for all integers $n > 3$. Using our classification of all genus one labeled circle trees, we prove that either $f(n)$ is divisible by n, or it is divisible by $\frac{n}{2}$; moreover, we explicitly describe when each of these possibilities occur.

In Section 2 we discuss the necessary definitions and review a result of Marcus [4], which implies that deleting an uncrossed edge from a l-c-tree or deleting all but one of several parallel edges leads to one of two canonical reduced forms of circle trees if and only if the l-c-tree was genus one. Although Marcus' result [3] is formulated for partitions, it easily translates to l-c-trees. Since the labeling of the l-c-tree is irrelevant for the deletion of edges mentioned above, we introduce the concept of an ***unlabeled circle tree*** to which Marcus' result still applies. For understanding the interrelation between the number of genus one l-c-trees and genus one u-c-trees we explore the basic properties of u-c-trees in Section 3. We introduce a special-structured subgraph, called an ***edgelike-graph***, in Section 4, and we describe an ***e-reduction process*** in Section 5. Based on the e-reduction process we give a ***classification*** of genus one c-trees by nineteen ***reduced forms*** in Section 6. We clarify the connection between the number of l-c-trees and u-c-trees on n points in the further sections, analyzing reduced forms. Finally, using our previous results we formulate the theorem about $f(n)$'s divisibility by n or $\frac{n}{2}$.

2 Definitions and Remarks

The definition of a labeled circle tree straighforwardly extends to the definition of a ***labeled circle graph***. Indeed, replacing the word tree with graph in the definition of l-c-tree gives the desired definition of a l-c-graph. Two ***l-c-graphs*** G_1 and G_2 are said to be ***isomorphic*** if an edge e_1 with endpoints labeled i and j is in G_1 if and only if there is an edge e_2 in G_2 with endpoints i and j (we consider graphs without multiple edges). Furthermore, if a vertex labeled k is of degree zero in one of the graphs then it is of degree zero in both of them. An ***unlableled circle graph*** (***u-c-graph***) is a graph obtained by deletion of labels of a l-c-graph. Two ***u-c-graphs*** are said to be ***isomorphic*** if it is possible to label their vertices so that the obtained l-c-graphs are isomorphic.

It follows from the definition of genus that isomorphic l-c-trees have equal genuses. Thus we can define the ***genus of a u-c-tree*** T on n points to be the genus of any of the l-c-trees one obtains by labeling the vertices of T by *1* through n in a counterclockwise direction.



The left u-c-tree has exatcly three corresponding non-isomorphic l-c-trees, while the right one has four.

Figure 2.1

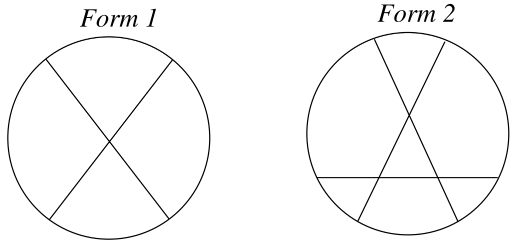


Figure 2.2: The two canonical reduced forms of genus one c-trees.

Call u-c-graphs (u-c-trees) and l-c-graphs (l-c-trees) by the common name ***c-graphs*** (***c-trees***). ***Edges*** e_1 and e_2 of a c-graph ***cross*** if they have a point in common on the drawing of the c-graph other than their endpoints. ***Edges*** e_1 and e_2 of a c-graph G are ***parallel*** if they cross the same edges of G, respectively. That the relation 'parallel' is an equivalence relation is a straightforward check of reflexivity, symmetry and transitivity.

Definition 3. *An* ***u-c-tree*** *C and a* ***l-c-tree*** *T are said to* ***correspond*** *if the u-c-tree obtained by the deletion of the labels of T is isomorphic to C.*

By the definition of genus the genuses of corresponding u-c-trees and l-c-trees are equal. Note that a u-c-tree C on n points can correspond to at most n non-isomorphic l-c-trees. In some cases the u-c-tree corresponds to exactly n non-isomorphic l-c-trees, but in some cases a u-c-tree corresponds to less then n non-isomorphic l-c-trees, Figure 2.1.

The reinterpretation of Marcus' result [3, 4]:

Proposition 1. *Performing the following two operations on a c-tree as many times as possible:*

1) deleting an edge from the c-tree, which is not crossed by any other edge

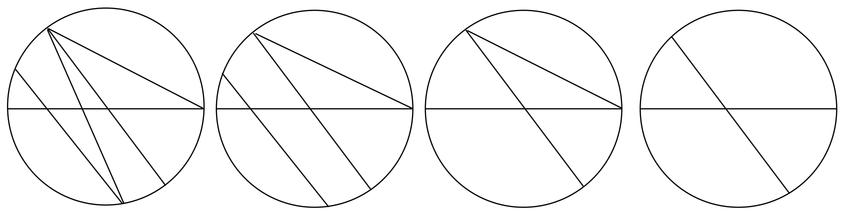
2) deleting all but one of several parallel edges

leads to Form 1 or Form 2 shown on Figure 2.2 if and only if the c-tree was genus one.

We refer to the two operations of Proposition 1 as operation 1) and operation 2).

Definition 4. *Call the u-c-graphs obtained from a u-c-tree C by executing operations 1) and 2)* ***offsprings****. A u-c-tree C* ***descends*** *to a u-c-graph, if the u-c-graph is an offspring of C. The* ***final offspring*** *of a u-c-tree C is the offspring which has no edges which could be deleted by the execution of operations 1) and 2). (Some offsprings of a u-c-tree are represented on Figure 2.3)*

A u-c-tree and its offsprings



Final offspring

Figure 2.3

In Section 3 we conclude that the final offspring of a genus one u-c-tree C is unique (up to isomorphism).

We now rephrase Proposition 1:

Proposition 1′. *The genus of a u-c-tree T is one if and only if its final offspring is Form 1 or Form 2.*

Naturally, by saying that the final offspring is Form 1 or Form 2, we mean that the final offspring is isomorphic either to Form 1 or to Form 2. We do not stress this in the future, since it is clear from the context.

We can now reformulate the question about the divisibility of $f(n)$ by n or $\frac{n}{2}$ as follows: When is the number of l-c-trees on n points, which have corresponding u-c-trees that descend to Form 1 or Form 2 (Figure 2.2) by the execution of operations 1) and 2) (these are all of the genus one l-c-trees), divisible by n and when is it divisible just by $\frac{n}{2}$ and not by n?

3 Initial Observations Concerning U-C-Trees

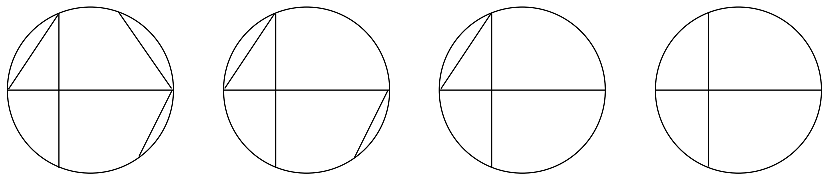
We state two simple lemmas concerning u-c-trees without proof. The proofs are based on the definitions of uncrossed and parallel edges, and are easily derived by contracition.

Lemma 1. *If an edge is uncrossed after a number of operations 1) and 2) are executed on a u-c-tree, then that edge is uncrossed in the u-c-tree, as well as in all its offsprings (if not deleted).*

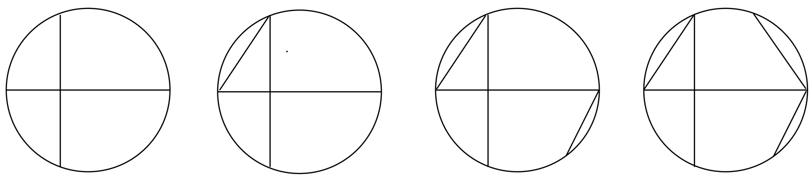
Lemma 2. *If two edges are parallel after a number of operations 1) and 2) are executed on a u-c-tree, then those two edges are parallel in the u-c-tree, as well as in all its offsprings (if not deleted).*

From Lemma 1 and Lemma 2 we conclude that the order of the execution of operations 1) and 2) and the particular choice of the order of the edges to be deleted do not affect the final offspring. By Proposition 1, after executing operations 1) and 2) on a u-c-tree until applicable, Form 1 or Form 2 are obtained if and only if the u-c-tree was genus one. Thus, it is possible to construct every genus one u-c-tree by beginning from Form 1 or Form 2 and by adding parallel edges to the ones presented in the form, and by adding uncrossed edges. Moreover, starting with these two forms and adding only parallel and uncrossed edges any u-c-tree obtained is of genus one. See Figure 3.1 for illustration. This "building" idea might serve as a basis for obtaining the exact number of genus one l-c-trees on n points.

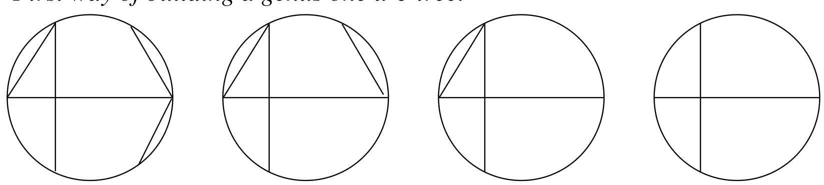
Figure 3.1



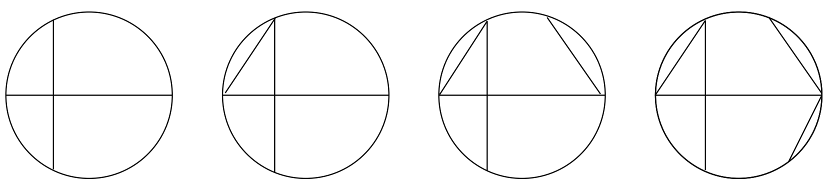
First way of executing operations 1) and 2).



First way of building a genus one u-c-tree.



Second way of executing operations 1) and 2).



Second way of building a genus one u-c-tree.

4 About Edgelike-Graphs

In this section we introduce a main concept of our work, that of an ***edgelike-graph***. As the name already suggests, these graphs behave somewhat like edges. Indeed, edgelike-graphs, or e-graphs for short, are subtrees of a given u-c-tree with the special property that collapsing an e-graph to an edge (a specified one) the obtained u-c-tree has the same genus as the one we began with.

Once the concept of e-graph is grasped, the way operations 1) and 2) act on a c-tree becomes easy to visualize and understand. A u-c-tree can be decomposed into e-graphs, in which case operations 1) and 2) act within these decomposed structures. The previous fact exhibits the correlation between the structure of a c-graph and operations 1) and 2).

Let edges $e_1, e_2, \cdots, e_k$ be parallel. If e_i and e_j are the outermost edges among $e_i, e_{i+1}, \cdots, e_j$, for all $1 \leq i \leq j \leq k$, then edges $e_1, e_2, \cdots, e_k$ are ***increasingly parallel***. Edges CD, CG, JH, FE are increasingly parallel on Figure 4.1. Edges $e_1, e_2, \ldots, e_k$ constitute a ***path*** if and only if there exist points $E_1, E_2, \ldots, E_{k+1}$ on the circle such that the endpoints of e_i

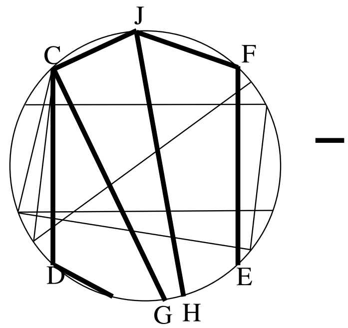


Figure 4.1

are E_i and E_{i+1} for all $1 \leq i \leq k$. We also make the convention that arc $\widehat{AB}$ is the arc between points A and B when going in a counterclockwise direction from A to B.

Definition 5. *Given a c-graph G take any crossed edge AB of it. Let increasingly parallel edges $e_1, e_2, \cdots, e_k$ be all edges of G parallel to AB (including AB itself). Let increasingly parallel edges $a_1, a_2, \cdots, a_l \in \{e_1, e_2, \cdots, e_k\}$ be all of the edges parallel to AB such that there exist a path of edges $AB = b_1$, $b_2, \cdots, b_m = a_i$ ($i \in [l]$) such that each b_j, $j \in [m]$, is either uncrossed or parallel to AB. If $a_1 = CD$ and $a_l = EF$, then the edges of G such that both of their endpoints are on arcs $\widehat{DE}$ and $\widehat{FC}$ and they are uncrossed or parallel to AB constitute the edgelike-graph, or e-graph, of G containing AB. Figure 4.1.*

Observe that there is a unique e-graph containing each crossed edge.

Definition 6. *Let the arcs $\widehat{DE}$ and $\widehat{FC}$ as in the above definition be called the arcs of an e-graph, whereas edges CD and EF the outermost edges of it. Also, call the set of crossed edges of the e-graph the set of parallel edges of the e-graph and call the set of edges of the e-graph that are uncrossed the set of uncrossed edges.*

Lemma 3. *Using the notation of Definition 5, edges $a_1, a_2, \cdots, a_l$ are all of the edges parallel to AB having both of their endpoints on arcs $\widehat{DE}$ and $\widehat{FC}$.*

Proof. Note that if $AB = e_i$, and $e_j = a_z$ for some $z \in [l]$, then any e_r, r between i and j, is equal to some a_q for some $q \in [l]$. This observation leads to the proof of the lemma. □

Lemma 4. *If $\mathcal{E}$ is an e-graph of c-graph G, then $\mathcal{E}$ consists of edges having no points in common except for their vertices.*

Proof. Suppose the opposite. Let e_1 and e_2 be two edges of $\mathcal{E}$ having a point A in common, such that A is not their endpoint. Since any edge of $\mathcal{E}$ is either uncrossed or parallel to an edge e (being uncrossed and parallel considered within G), we conclude that e_1 and e_2 are both parallel to e. However, crossing edges cannot be parallel. This contradiction proves the statement. □

Lemma 5. *If $\mathcal{E}$ is an e-graph of a u-c-tree C with arcs $\widehat{DE}$ and $\widehat{FC}$ and outermost edges CD and EF, then there is no edge of C having one of its endpoints of the open arcs $\widehat{DE}$ or $\widehat{FC}$ and the other endpoint on the open arcs $\widehat{CD}$ or $\widehat{EF}$.*

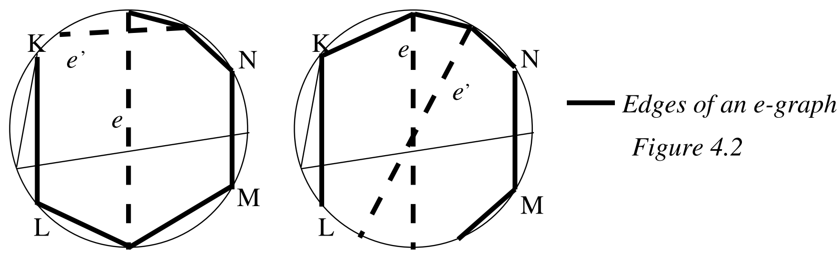


Figure 4.2

Proof. The statement of the lemma follows since CD and EF are parallel edges. □

Proposition 2. *Given an e-graph $\mathcal{E}$ of a genus one u-c-tree C, let KL and MN be its outermost edges, and the arcs $\widehat{LM}$ and $\widehat{NK}$ its arcs. Then all edges of C which have both of their endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$ are edges of $\mathcal{E}$. Conversely, only such edges are edges of an e-graph $\mathcal{E}$ of a genus one u-c-tree C.*

Proof. Let $\mathcal{U}$ and $\mathcal{P}$ be the sets of uncrossed and parallel edges of e-graph $\mathcal{E}$ described in the proposition. By the definition of e-graph $\mathcal{U}$ contains all the uncrossed edges of C with endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$ and $\mathcal{P}$ contains all edges of C parallel to KL with endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$. To prove Proposition 2 it suffices to show that there is no edge e of C with both of its endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$ such that it is not in $\mathcal{U}$ or $\mathcal{P}$. Suppose the opposite, that there was an edge e of C with both of its endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$ such that it is not in $\mathcal{U}$ or $\mathcal{P}$. If e had one of its endpoints on $\widehat{LM}$ and the other on $\widehat{NK}$, then all the edges crossing KL would cross e. Since e was not parallel to KL there would have been some edge e' which does not cross KL and MN but crosses e. Edge e' could clearly not be in $\mathcal{U}$, and it also could not be in $\mathcal{P}$, since KL and MN could not be crossed by e, given that the endpoints of e are on arcs $\widehat{LM}$ and $\widehat{NK}$. Edge e' could be an edge with both endpoints on one of the arcs $\widehat{LM}$ or $\widehat{NK}$ or with one endpoint on $\widehat{LM}$ and other endpoint on $\widehat{NK}$ (by Lemma 5 these are the only possibilities), Figure 4.2. It is clear that executing operations 1) and 2) we would not get to Form 1 or Form 2, since the cross from e and e' and from KL and some edge it crosses would remain.

On the other hand, if e had both of its endpoints on one of the arcs $\widehat{LM}$ or $\widehat{NK}$, since it was not uncrossed it would have been crossed by some edge e' and by Lemma 5 e' would have both of its endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$. All cases are depicted on Figure 4.3. The cross obtained from the crossing of e and e' and the cross from KL and some edge it crossed it would necessarily remain after executing operations 1) and 2) so we could not get to Form 1 or Form 2, thus the genus of the u-c-tree could not be one.

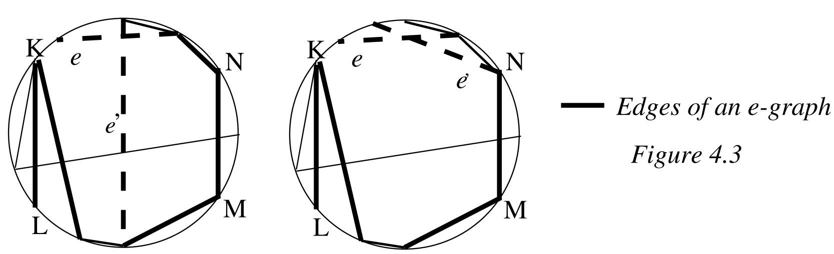
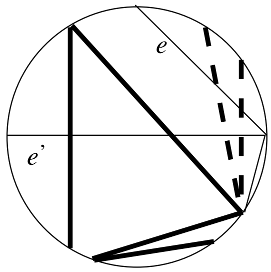


Figure 4.3



— *Edges of e-graph* $\mathcal{E}_1$

- - *Edges of e-graph* $\mathcal{E}_2$

After executing operations 1) and 2) the cross of $\mathcal{E}_1$ *and* e' *and the cross of* $\mathcal{E}_2$ *and* e *cannot be eliminated.*

Figure 4.4

Thus, all edges of C which have both of their endpoints on arcs $\widehat{LM}$ and $\widehat{NK}$ are edges of the e-graph $\mathcal{E}$. It follows by definition that only such edges are edges of the e-graph. □

Corollary. (Definition 5, Lemma 5, Proposition 2) *An e-graph of a genus one u-c-tree is a tree.* Proposition 2 also implies that given the outermost edges of an e-graph, the e-graph is uniquely determined.

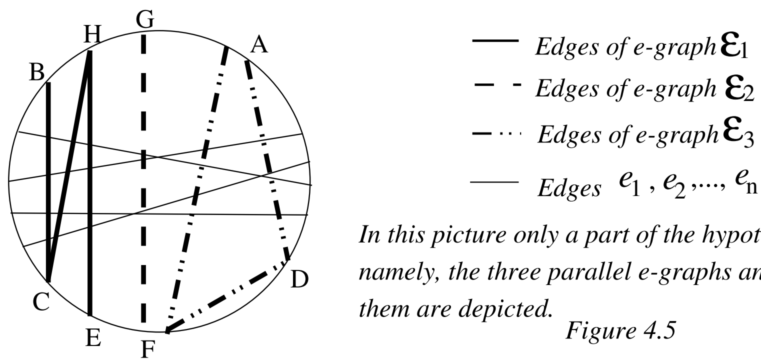
Lemma 6. *Given two e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ in a genus one u-c-tree C with sets of parallel edges $\mathcal{P}_1$ and $\mathcal{P}_2$, if $\mathcal{P}_1 = \mathcal{P}_2$, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are not different. (Two e-graphs of a c-tree are said to be different if there is an edge in one of them which does not belong to the other. When two e-graphs are not different, we also say that they are identical.)*

Proof. The set of parallel edges of an e-graph determine its outermost edges and the outermost edges determine the e-graph in a genus one u-c-tree. □

Proposition 3. *There are no two different e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ in a genus one u-c-tree C such that they have vertices in common.*

Proof. Let $\mathcal{U}_1, \mathcal{P}_1, \mathcal{U}_2, \mathcal{P}_2$ be the sets of uncrossed and parallel edges of two different e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$. From Lemma 6 $\mathcal{P}_1 \neq \mathcal{P}_2$. If the edges of $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel it is impossible that the e-graphs have common vertices by the definition of an e-graph. In case the edges of $\mathcal{P}_1$ and $\mathcal{P}_2$ are not parallel, the only vertex two e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ might have in common is an endpoint of some of their outermost edges. However, if $\mathcal{E}_1$ and $\mathcal{E}_2$ had such a point in common, there must have been some edge e which crosses, say the edges of $\mathcal{P}_2$ and does not cross the edges of $\mathcal{P}_1$. In this case the cross made by e and some edge of $\mathcal{P}_2$ as well as some cross from some edge of $\mathcal{P}_1$ and another edge, not parallel to e, must stay after executing operations 1) and 2) thus it is impossible to obtain Form 1 or Form 2 (Figure 4.4). Thus, if the u-c-tree is genus one then no two e-graphs of the u-c-tree have vertices in common. □

Given an e-graph $\mathcal{E}$ of a u-c-graph with its set of uncrossed edges $\mathcal{U}$ and set of parallel edges $\mathcal{P}$, call the elements of $\mathcal{P}$ the ***parallel edges*** and the elements of $\mathcal{U}$ the ***uncrossed edges*** of the e-graph $\mathcal{E}$. We say that an e-graph $\mathcal{E}$ is ***parallel*** to an edge e if its parallel edges are parallel to e. Similarly, e-graph $\mathcal{E}_1$ is parallel to another e-graph $\mathcal{E}_2$ if their parallel edges are parallel. We say that an edge e is ***between*** parallel e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ if both endpoint of e are on arcs $\widehat{BE}$ and $\widehat{HC}$, where edges AD, BC, EH, FG are increasingly parallel and



In this picture only a part of the hypothetical u-c-tree, namely, the three parallel e-graphs and the edges crossing them are depicted.

Figure 4.5

the arcs of $\mathcal{E}_1$ are $\widehat{AB}$, $\widehat{CD}$ and the arcs of $\mathcal{E}_2$ are $\widehat{EF}$, $\widehat{GH}$ (and if e is not an edge of $\mathcal{E}_1$ or $\mathcal{E}_2$). For example, taking e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ from Figure 4.5, an edge is between these two e-graphs if and only if both of its endpoints are on arcs $\widehat{EF}$ and $\widehat{GH}$. A ***path*** consisting of edges E_1E_2, E_2E_3, …, E_kE_{k+1}, ***connects*** two ***e-graphs*** $\mathcal{E}_1$ and $\mathcal{E}_2$ if and only if point E_1 is the intersection of $\{E_1, E_2, \ldots, E_{k+1}\}$ and the points of one of the e-graphs, and point E_{k+1} is the intersection of $\{E_1, E_2, \ldots, E_{k+1}\}$ and the points of the other of the e-graphs.

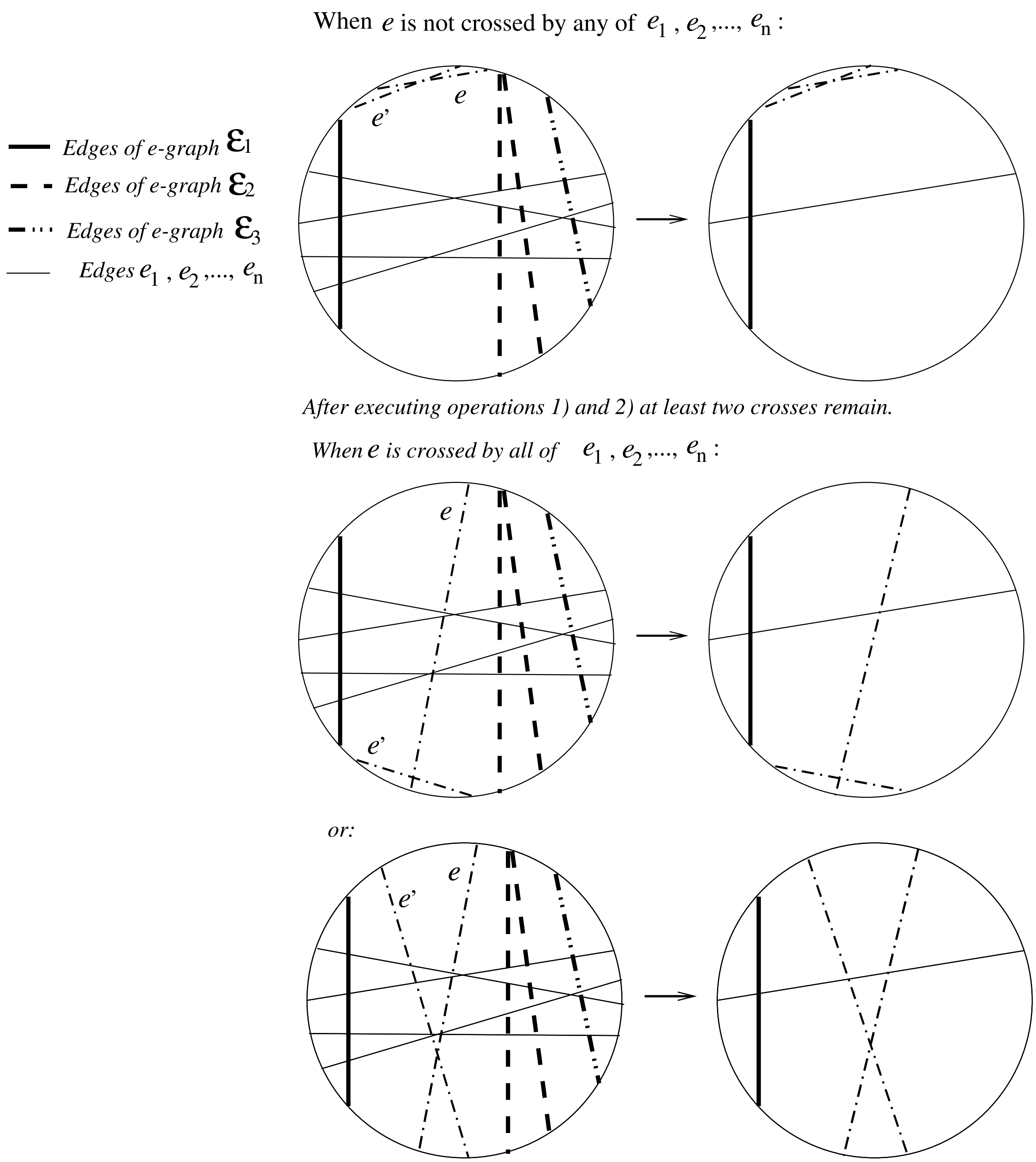
Theorem 1. *There can be at most two different e-graphs $\mathcal{E}_1$ and $\mathcal{E}_2$ of a genus one u-c-tree C such that $\mathcal{E}_1$ and $\mathcal{E}_2$ are parallel.*

Proof. The main idea of the proof is that a u-c-tree is connected, and if there were already three parallel e-graphs the u-c-tree, then it would be impossible to connect them into a connected c-graph so that the three e-graphs were really three different e-graphs, and that they were in a genus one u-c-tree. We analyze how could we possibly connect the "middle" e-graph (supposing three parallel e-graphs) to the other parts of the u-c-tree in order to obtain the desired contradiction.

Suppose the statement of Theorem 1 was false. Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be three different e-graphs of a genus one u-c-tree C parallel to each other. Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be the sets of parallel edges, and $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ be the sets uncrossed edges of $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, respectively. Let edges $e_1, e_2, \ldots, e_n$ be all of the edges crossing their parallel edges. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 = \{a_1, \ldots, a_k\}$, where $a_1, \ldots, a_k$ are increasingly parallel. If $a_1 = BC$ and $a_k = DA$, then arcs $\widehat{AB}$ and $\widehat{CD}$ are the minimal arcs such that all edges from $\mathcal{P}$ have one of their endpoints on $\widehat{AB}$ while the other on $\widehat{CD}$. Let edge BC be an edge of $\mathcal{P}_1$ and DA an edge of $\mathcal{P}_3$. Recall that different e-graphs have no common points and call $\mathcal{E}_1$ the left e-graph, $\mathcal{E}_3$ the right e-graph and $\mathcal{E}_2$ the middle e-graph. Since e-graphs $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are subtrees of u-c-tree C, they are all connected to each other within the u-c-tree. We concentrate our efforts on how could $\mathcal{E}_2$ be connected to the other parts of the u-c-tree C.

We prove that there is no path connecting $\mathcal{E}_1$ and $\mathcal{E}_2$ such that the path contains exclusively edges between $\mathcal{E}_1$ and $\mathcal{E}_2$. Analogously, there is no path connecting $\mathcal{E}_2$ and $\mathcal{E}_3$ such that the path contains exclusively edges between $\mathcal{E}_2$ and $\mathcal{E}_3$.

Suppose the opposite. Suppose that there was a path consisting of edges between $\mathcal{E}_1$ and $\mathcal{E}_2$ connecting $\mathcal{E}_2$ to $\mathcal{E}_1$. Then, either $\mathcal{E}_2$ is connected to $\mathcal{E}_1$ by only uncrossed edges and



After executing operations 1) and 2) at least two crosses remain.

Figure 4.6

edges parallel to $\mathcal{E}_1$ or $\mathcal{E}_2$ is connected to $\mathcal{E}_1$ with edges among which there are edges which are neither uncrossed nor parallel to $\mathcal{E}_1$. From the definition of an e-graph we see that only the second possibility might hold. However, if there was an edge e on the path connecting $\mathcal{E}_1$ and $\mathcal{E}_2$ which is neither uncrossed nor parallel to $\mathcal{E}_1$, then it would have been crossed by some edge which crosses none of $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ or it would have been not crossed by some which crosses these. However, if e is not crossed by some of $e_1, e_2, ..., e_n$, then it is not crossed by any of them[1], thus, since e is crossed it must be crossed by some edge which cross none of $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$. Therefore, edge e is crossed by some edge e', such that $e' \notin \{e_1, e_2, ..., e_n\}$ and e' is between $\mathcal{E}_1$ and $\mathcal{E}_2$ (so as for the three e-graphs to be parallel, and e' not to be among $e_1, e_2, ..., e_n$). All possibilities are depicted on Figure 4.6, where we did not present the whole c-tree, but only the edges that are of importance for our proof. Using operations 1) and 2) it would be impossible to obtain Form 1 or Form 2 (the cross of the three e-graphs and edges $e_1, e_2, ..., e_n$ and the other from the crossing of e and e' would remain). Therefore, $\mathcal{E}_2$ is not connected to $\mathcal{E}_1$ or $\mathcal{E}_3$ with edges between them.

Thus, if $\mathcal{E}_2$ was connected to the other parts of the u-c-tree, namely to $\mathcal{E}_1$ and $\mathcal{E}_3$, then there had to be some edge e on the path which connected $\mathcal{E}_2$ to $\mathcal{E}_1$ and $\mathcal{E}_3$ which was not between $\mathcal{E}_1$ and $\mathcal{E}_2$ or $\mathcal{E}_2$ and $\mathcal{E}_3$. However, some of such edges e would intersect some of the three e-graphs and not intersect some other, thus this would contradict that the parallel edges of $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ are parallel to each other. Therefore, there cannot be three e-graphs parallel to each other in a genus one u-c-tree. The statement of Theorem 1 is proven. $\square$

Examples of trees with two parallel e-graphs are shown in Figure 6.1, where parallel edges represent different parallel e-graphs.

5 The E-Reduction Process

In this section we describe the ***e-reduction process*** which reduces a u-c-tree C to a ***reduced form*** which carries enough information of the original u-c-tree C so that from the reduced form of a u-c-tree C we know how many e-graphs C had and how they were connected among each other.

Given a u-c-tree T perform the following ***e-reduction process***:

First Step. For all e-graphs of T do the following: given e-graph $\mathcal{E}$ in T with set $\mathcal{P}$ of parallel edges, delete all but one of the edges of $\mathcal{P}$ (operation 2)) obtaining u-c-graph T_1 from T.

Second Step. Delete all the uncrossed edges of T_1 (operation 1)) obtaining u-c-graph T_2. (Note that an edge e is uncrossed in T_1 if and only if it was uncrossed in T.)

Third Step. If in the original u-c-tree T there was a path consisting of uncrossed edges E_1E_2,..., E_kE_{k+1} connecting e-graphs $\mathcal{E}_1$ (with arcs $\mathcal{A}_1$, $\mathcal{A}_2$) and $\mathcal{E}_2$ (with arcs $\mathcal{B}_1$, $\mathcal{B}_2$) in T,

[1] Since if an edge e between $\mathcal{E}_1$ and $\mathcal{E}_2$ is crossed by some of e_i, then e's endpoints are on the two different arcs between the e-graphs, and in the case e's endpoints are on the two different arcs between the e-graphs ($\widehat{EF}$ and $\widehat{GH}$, Figure 4.5), then e is crossed by all $e_1, e_2, ..., e_n$, since the endpoints of $e_1, e_2, ..., e_n$ are left to $\mathcal{E}_1$ and right to $\mathcal{E}_3$ (left and right, referring to Figure 4.5).

then, if A_1A_2 is the edge left from the set of parallel edges of $\mathcal{E}_1$ in the *First Step* such that $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ and if B_1B_2 is the edge left from the set of parallel edges of $\mathcal{E}_2$ in the *First Step* such that $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ then add edge A_jB_i $(1 \leq j, i \leq 2)$ to u-c-graph T_2 provided E_1 belongs to arc $\mathcal{A}_j$ and E_{k+1} belongs to $\mathcal{B}_i$. Do this for all possible paths of uncrossed edges connecting two e-graphs in T. Call the u-c-graph obtained at the end T_3. Note that T_3 is a u-c-tree.

To emphasize the importance of u-c-graphs obtained after the *Second* and *Third Step* of the e-reduction process we give them special names: ***pre-reduced forms*** and ***reduced forms***, respectively, Figure 5.1. The pre-reduced form of a genus one u-c-tree T is the u-c-tree T_2 obtained by the execution of the first two steps of the e-reduction process on T. The reduced form of a genus one u-c-tree T is the u-c-tree T_3 obtained by execution of the e-reduction process on T. We say that u-c-tree T reduces to u-c-graph T_3 if the u-c-tree T_3 is the reduced form of T.

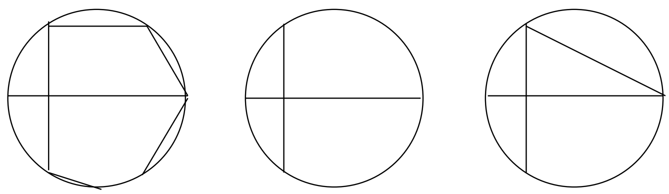


Figure 5.1

Definition 7. *Let $\mathcal{G}$ denote the set of all c-graphs. Let $\mathcal{T}_2 = \{T_2 \in \mathcal{G} \mid T_2$ is a pre-reduced form of some genus one u-c-tree $C\}$ and $\mathcal{T}_3 = \{T_3 \in \mathcal{G} \mid T_3$ is a reduced form of some genus one u-c-tree $C\}$*

Lemma 7. *Any $T_3 \in \mathcal{T}_3$ can be obtained from some $T_2 \in \mathcal{T}_2$ by addition of uncrossed edges but without addition of any vertices so that the obtained u-c-graph is a tree. Also, all possible u-c-trees T obtained by taking some u-c-graph $T_2 \in \mathcal{T}_2$ and adding exclusively uncrossed edges without adding any vertices so as to form a tree are in $\mathcal{T}_3$.*

Proof. Follows from the e-reduction process. □

Lemma 8. *Let forms T_2 and T_3 be the pre-reduced and reduced forms of u-c-tree T, respectively. Let T_2 have n_p points and n_e edges. Then T_2 is a subgraph of the tree T_3, the set of vertices of T_2 is equal to the set of vertices of T_3, and there are exactly $n_p - n_e - 1$ edges of T_3 which are not in T_2. All of these edges are uncrossed.*

Proof. The facts that T_2 is a subgraph of T_3, T_3 is a tree, the set of vertices of T_2 is equal to the set of vertices of T_3, and that the edges which belong to T_3 but do not belong to T_2 are all uncrossed follow directly from the e-reduction process. Since T_3 is a tree on n_p points it has $n_p - 1$ edges, and since T_2 has n_e edges there are exactly $n_p - n_e - 1$ in T_3 which are not in T_2. □

Lemma 9. *Let T_2 be the pre-reduced form of a genus one u-c-tree C. Then T_2 is an offspring of C.*

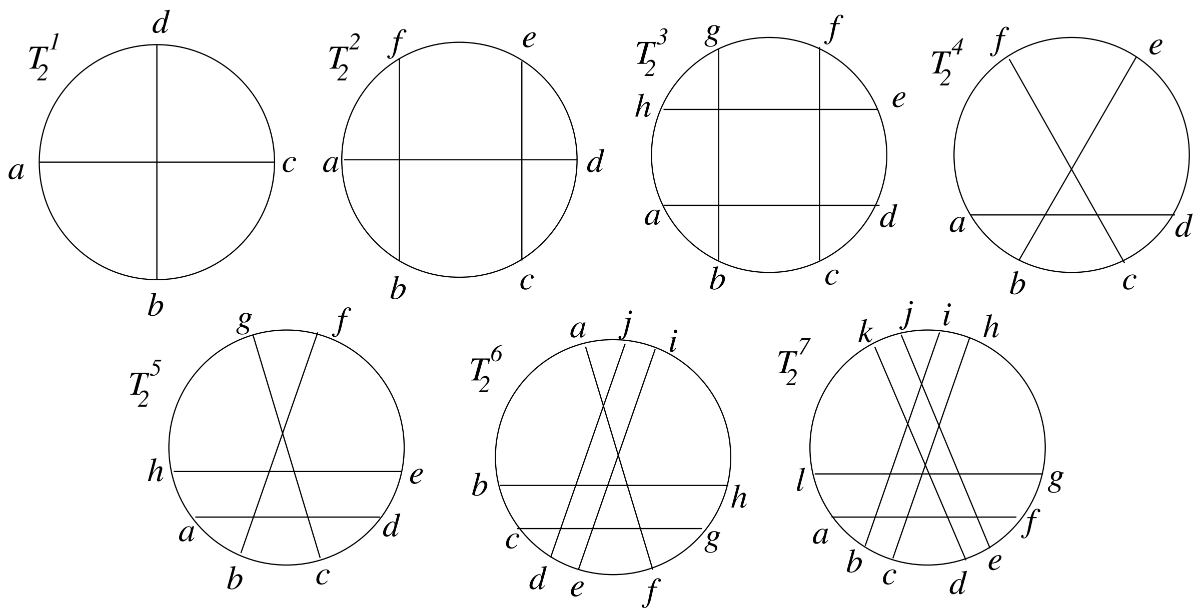


Figure 5.2

Proof. Since the *First Step* and the *Second Step* of the e-reduction process require only repeated execution of operations 1) and 2) on a u-c-tree C, the u-c-graph obtained after the *Second Step*, T_2, is an offspring of C by definition. □

Proposition 4. *The genus of the reduced form T_3 of a u-c-tree T is one if and only if the genus of T is one.*

Proof. Let T_2 be the pre-reduced form of u-c-tree T. Then T_2 is an offspring of T by Lemma 9. Note that T_2 is an offspring of T_3, since deleting the uncrossed edges of T_3 results in T_2. Thus, T and T_3 have a common offspring and so their final offspring is the same. By Proposition 1' the genus of T is one if and only if the genus of T_3 is one. □

Lemma 10. *If T_2 is a pre-reduced form of a genus one u-c-tree T, then T_2 descends to Form 1 or Form 2.*

Proof. From Lemma 9 we know that T_2 is an offspring of T. Since T is genus one if and only if its final offspring is Form 1 or Form 2, and the final offspring depends only on the starting u-c-graph, it is clear that from any offspring, so in particular from T_2, Form 1 or Form 2 can be obtained by executing operations 1) and 2) on the offspring provided T was genus one. □

Proposition 5. *If T_2 is the pre-reduced form of a genus one u-c-tree T, then T_2 is one of the 7 u-c-graphs on Figure 5.2*[2]

[2]The labels *a, b, c, d, e, f, g, h, i, j, k, l* are not part of the forms. They serve only to enable us to specify certain edges of the forms.

Proof. As a pre-reduced form T_2 contains no uncrossed edges, and in u-c-graph T_2 each e-graph of T is represented by a single edge, a representative of the set of parallel edges of the e-graph in T. Since, according to Theorem 1, there can be at most two parallel e-graphs, in T_2 there are no three edges parallel to each other. Since no different e-graphs in T had common vertices (Proposition 3), no edges of T_2 have common endpoints. By Lemma 10 T_2 descends to Form 1 or Form 2. These conditions are fulfilled exclusively in the presented 7 forms, therefore, $\mathcal{T}_2 \subseteq \{{T_2}^1, {T_2}^2, {T_2}^3, {T_2}^4, {T_2}^5, {T_2}^6, {T_2}^7\}$ as stated in the proposition. $\square$

6 Classification of All Genus One C-Trees

In this section we prove that there exist genus one u-c-trees such that their pre-reduced forms are ${T_2}^1$, ${T_2}^2$, ${T_2}^3$, ${T_2}^4$, ${T_2}^5$, ${T_2}^6$, but that there is no genus one u-c-tree having ${T_2}^7$ as its pre-reduced form, that is $\mathcal{T}_2$=$\{{T_2}^1, {T_2}^2, {T_2}^3, {T_2}^4, {T_2}^5, {T_2}^6\}$. Throughout the analysis of the seven candidates for pre-reduced forms presented in Proposition 5 we also describe all of the possible reduced forms of genus one u-c-trees. The interrelations of a u-c-tree and its reduced form enables us to get an insight into the behavior of the number of genus one l-c-trees on n points.

Theorem 2. *$\mathcal{T}_2$= $\{{T_2}^1, {T_2}^2, {T_2}^3, {T_2}^4, {T_2}^5, {T_2}^6\}$.*

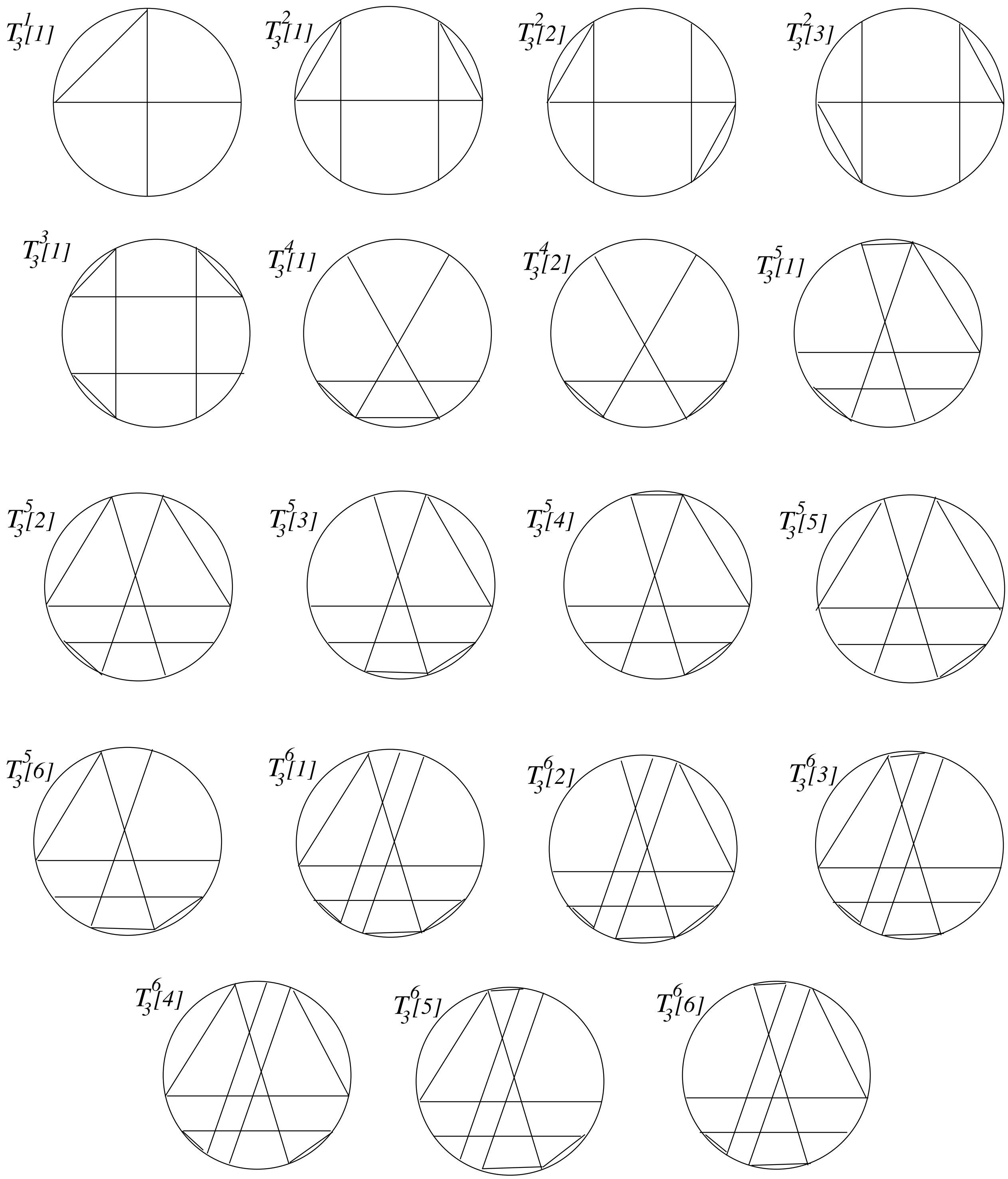
$\mathcal{T}_3$= $\{{T_3}^1[1], {T_3}^2[1], {T_3}^2[2], {T_3}^2[3], {T_3}^3[1], {T_3}^4[1], {T_3}^4[2], {T_3}^5[1], {T_3}^5[2], {T_3}^5[3], {T_3}^5[4], {T_3}^5[5], {T_3}^5[6], {T_3}^6[1], {T_3}^6[2], {T_3}^6[3], {T_3}^6[4], {T_3}^6[5], {T_3}^6[6]\}$, Figure 6.1.

Proof. We try to obtain a tree from each of the seven different candidates for pre-reduced forms from Proposition 5 by addition of uncrossed edges as described in Lemma 8, pretending the candidates were in fact pre-reduced forms. In case we are able to obtain a tree without adding vertices as Lemma 8 states, we know that the candidate was in fact a pre-reduced form, since the tree we obtain does have the candidate for its pre-reduced form and itself for its reduced form. However, in case we are unable to obtain a tree by the addition of uncrossed edges as Lemma 8 states, we know that the candidate cannot possibly be a pre-reduced form. Also, for the candidates which prove to be pre-reduced forms we give all of the reduced forms which can result from them after the execution of the *Third Step* of the e-reduction process. Since all reduced forms are obtainable from some pre-reduced form by the execution of the *Third Step* of the e-reduction process we get all of the reduced forms of genus one u-c-trees.The analysis of the seven candidates follows. We are referring to the figure of Proposition 5.

• ${T_2}^1$ has 4 points and 2 edges. In order to obtain a tree one edge needs to be added. There are 4 possibilities for uncrossed edges between the e-graphs: ab, bc, cd and da. All the 4 u-c-trees which could be obtained by adding one of these edges are isomorphic. Thus, one reduced form: ${T_3}^1[1]$ can be obtained from ${T_2}^1$.

• ${T_2}^2$ has 6 points and 3 edges, thus 2 edges need to be added to obtain a tree. There are 4 possibilities for uncrossed edges between the e-graphs: ab, cd, de and fa. (Note that bc and ef are not such edges, since bf and ce have to become different e-graphs, and if

Figure 6.1



$T_3^1[1]$
$T_3^2[1]$
$T_3^2[2]$
$T_3^2[3]$
$T_3^3[1]$
$T_3^4[1]$
$T_3^4[2]$
$T_3^5[1]$
$T_3^5[2]$
$T_3^5[3]$
$T_3^5[4]$
$T_3^5[5]$
$T_3^5[6]$
$T_3^6[1]$
$T_3^6[2]$
$T_3^6[3]$
$T_3^6[4]$
$T_3^6[5]$
$T_3^6[6]$

there would be an uncrossed edge connecting these parallel edges, they would form a single e-graph.) Edges fa and ab cannot be added at the same time to form a tree nor can cd and de together, $\{fa, ab, bf\}$ or $\{cd, de, ec\}$ would form a cycle. Thus, the 2 edges which could be added are: $\{fa, cd\}$, $\{fa, de\}$, $\{ab, cd\}$, $\{ab, de\}$. The u-c-trees obtained by adding $\{fa, de\}$ and $\{ab, cd\}$ are isomorphic. However, none of the u-c-trees obtained by adding $\{fa, de\}$, $\{fa, cd\}$, $\{ab, de\}$ are isomorphic, thus three reduced forms: ${T_3}^2[1]$, ${T_3}^2[2]$, ${T_3}^2[3]$ can be obtained from ${T_2}^2$.

• ${T_2}^3$ has 8 points and 4 edges, thus 3 edges need to be added to obtain a tree. There are 4 possibilities for uncrossed edges between the e-graphs: ab, cd, ef and gh. All the four u-c-trees which could be obtained by adding three of these edges are isomorphic. Thus, one reduced form: ${T_3}^3[1]$ can be obtained from ${T_2}^3$.

• ${T_2}^4$ has 6 points and 3 edges, thus 2 edges need to be added to obtain a tree. There are 6 possibilities for uncrossed edges between the e-graphs: ab, bc, cd, de, ef, and fa. Edges $\{ab, ed\}$, $\{bc, ef\}$, $\{cd, fa\}$ cannot added at the same time to form a tree, since $\{ab, be, ed, da\}$, $\{bc, cf, ef, eb\}$ or $\{cd, da, fa, fc\}$ would form a cycle. Thus, the 2 edges which might be added are: $\{ab, bc\}$, $\{ab, cd\}$, $\{ab, ef\}$, $\{ab, fa\}$, $\{bc, cd\}$, $\{bc, de\}$, $\{bc, fa\}$, $\{cd, de\}$, $\{cd, ef\}$, $\{de, ef\}$, $\{de, fa\}$, $\{ef, fa\}$. The u-c-trees obtained by adding $\{ab, bc\}$, $\{ab, fa\}$, $\{bc, cd\}$, $\{cd, de\}$, $\{de, ef\}$, $\{ef, fa\}$ are isomorphic; also u-c-trees obtained by adding $\{ab, cd\}$, $\{ab, ef\}$, $\{bc, de\}$, $\{bc, fa\}$, $\{cd, ef\}$, $\{de, fa\}$ are isomorphic. However, u-c-trees obtained by adding $\{ab, bc\}$, $\{ab, cd\}$ are not isomorphic, thus two reduced forms: ${T_3}^4[1]$, ${T_3}^4[2]$ can be obtained from ${T_2}^4$.

• ${T_2}^5$ has 8 points and 4 edges, thus 3 edges need to be added to obtain a tree. There are 6 possibilities for uncrossed edges between the e-graphs: ab, bc, cd, ef, fg, and gh. Since we want to obtain a tree, one of $\{ef, gh\}$ and one of $\{ab, cd\}$ must be among the added uncrossed edges in order for edges he and ad to be connected to something. Thus, two out of the three edges needed to be added must be $\{ef, ab\}$ or $\{ef, cd\}$ or $\{gh, ab\}$ or $\{gh, cd\}$. In order to obtain a tree the three added edges can be: $\{ef, ab, fg\}$, $\{ef, ab, gh\}$, $\{ef, ab, bc\}$, $\{ef, ab, cd\}$, $\{ef, cd, bc\}$, $\{ef, cd, fg\}$, $\{ef, cd, gh\}$, $\{gh, ab, bc\}$, $\{gh, ab, cd\}$, $\{gh, ab, fg\}$, $\{gh, cd, bc\}$, $\{gh, cd, fg\}$. The u-c-trees obtained by adding $\{ef, ab, fg\}$, $\{ef, ab, bc\}$ are isomorphic; also u-c-trees obtained by adding $\{ef, ab, gh\}$, $\{ef, ab, cd\}$ are isomorphic; also u-c-trees obtained by adding $\{ef, cd, bc\}$, $\{gh, ab, fg\}$ are isomorphic; also u-c-trees obtained by adding $\{ef, cd, fg\}$, $\{gh, ab, bc\}$ are isomorphic; also u-c-trees obtained by adding $\{ef, cd, gh\}$, $\{gh, ab, cd\}$ are isomorphic; also u-c-trees obtained by adding $\{gh, cd, bc\}$, $\{gh, cd, fg\}$ are isomorphic. However, u-c-trees obtained by adding $\{ef, ab, fg\}$, $\{ef, ab, gh\}$, $\{ef, cd, bc\}$, $\{ef, cd, fg\}$, $\{ef, cd, gh\}$, $\{gh, cd, bc\}$ are not isomorphic, thus six reduced forms: ${T_3}^5[1]$, ${T_3}^5[2]$,${T_3}^5[3]$, ${T_3}^5[4]$, ${T_3}^5[5]$, ${T_3}^5[6]$ can be obtained from ${T_2}^5$.

• ${T_2}^6$ has 10 points and 5 edges, thus 4 edges need to be added to obtain a tree. There are 6 possibilities for uncrossed edges between the e-graphs: ab, cd, ef, fg, hi, and ja. There are 15 possibilities to choose 4 edges out of these 6: $\{ab, cd, ef, fg\}$, $\{ab, cd, ef, hi\}$, $\{ab, cd, fg, hi\}$, $\{ab, ef, fg, hi\}$, $\{cd, ef, fg, hi\}$, $\{ab, cd, ef, ja\}$, $\{ab, cd, fg, ja\}$, $\{ab, ef, fg, ja\}$, $\{cd, ef, fg, ja\}$, $\{ab, cd, hi, ja\}$, $\{ab, ef, hi, ja\}$, $\{cd, ef, hi, ja\}$, $\{ab, fg, hi, ja\}$, $\{cd, fg, hi, ja\}$, $\{ef, fg, hi, ja\}$. Edges $\{ab, bh, hi, ie, ef, fa\}$ form a cycle thus $\{ab, hi,$

$ef\}$ cannot be in a tree. Similarly $\{cd,\ dj,\ ja,\ af,\ fg,\ gc\}$ form a cycle thus $\{cd,\ ja,\ fg\}$ cannot be in a tree. Therefore only 9 possibilities out of 15 remain: $\{ab,\ cd,\ ef,\ fg\}$, $\{ab,\ cd,\ fg,\ hi\}$, $\{cd,\ ef,\ fg,\ hi\}$, $\{ab,\ cd,\ ef,\ ja\}$, $\{ab,\ ef,\ fg,\ ja\}$, $\{ab,\ cd,\ hi,\ ja\}$, $\{cd,\ ef,\ hi,\ ja\}$, $\{ab,\ fg,\ hi,\ ja\}$, $\{ef,\ fg,\ hi,\ ja\}$. Note that the u-c-trees obtained by adding $\{ab,\ cd,\ ef,\ fg\}$, $\{ab,\ fg,\ hi,\ ja\}$ are isomorphic; also u-c-trees obtained by adding $\{cd,\ ef,\ fg,\ hi\}$, $\{ab,\ cd,\ hi,\ ja\}$ are isomorphic; also u-c-trees obtained by adding $\{ab,\ cd,\ ef,\ ja\}$, $\{ef,\ fg,\ hi,\ ja\}$ are isomorphic. However, u-c-trees obtained by adding $\{ab,\ cd,\ ef,\ fg\}$, $\{cd,\ ef,\ fg,\ hi\}$, $\{ab,\ cd,\ ef,\ ja\}$, $\{ab,\ cd,\ fg,\ hi\}$, $\{ab,\ ef,\ fg,\ ja\}$, $\{cd,\ ef,\ hi,\ ja\}$ are not isomorphic, thus six reduced forms: ${T_3}^6[1]$, ${T_3}^6[2]$,${T_3}^6[3]$, ${T_3}^6[4]$, ${T_3}^6[5]$, ${T_3}^6[6]$ can be obtained from ${T_2}^6$.

• Finally, ${T_2}^7$ has 12 points and 6 edges, thus 5 edges need to be added to obtain a tree. There are 6 possibilities for uncrossed edges between the e-graphs: ab, cd, ef, gh, ij, and kl. Note, however, that $\{ab,\ bi,\ ij,\ je,\ ef,\ fa\}$, $\{cd,\ dk,\ kl,\ lg,\ gh,\ hc\}$ are cycles, thus $\{ab,\ ij,\ ef\}$, $\{cd,\ kl,\ gh\}$ cannot be added to obatin a tree. However, there is no way to choose 5 edges out of the 6 possible so that none of the sets $\{ab,\ ij,\ ef\}$, $\{cd,\ kl,\ gh\}$ is a subset of the set of the chosen 5 edges. Therefore, no reduced form can be obtained from ${T_2}^7$.

We have analysed all possible forms from Proposition 7, and saw that some element of $\mathcal{T}_3$ can be obtained from all of ${T_2}^1$, ${T_2}^2$, ${T_2}^3$, ${T_2}^4$, ${T_2}^5$, ${T_2}^6$, but no element of $\mathcal{T}_3$ can be obtained from ${T_2}^7$. Using the result of Proposition 7 we conclude that $\mathcal{T}_2= \{{T_2}^1,\ {T_2}^2,\ {T_2}^3,\ {T_2}^4,\ {T_2}^5,\ {T_2}^6\}$, and since all elements of $\mathcal{T}_3$ are obtainable from some pre-reduced form we have that $\mathcal{T}_3= \{{T_3}^1[1],\ {T_3}^2[1],\ {T_3}^2[2],\ {T_3}^2[3],\ {T_3}^3[1],\ {T_3}^4[1],\ {T_3}^4[2],\ {T_3}^5[1],\ {T_3}^5[2],\ {T_3}^5[3],\ {T_3}^5[4],\ {T_3}^5[5],\ {T_3}^5[6],\ {T_3}^6[1],\ {T_3}^6[2],\ {T_3}^6[3],\ {T_3}^6[4],\ {T_3}^6[5],\ {T_3}^6[6]\}$. This concludes the proof of Theorem 2.

□

The nineteen reduced forms from Theorem 2 ***classify genus one l-c-trees***, namely, for all such l-c-trees the corresponding u-c-trees reduce to one of these nineteen reduced forms.

7 The Connection Between L-C-Trees and U-C-Trees

Given a genus one u-c-tree C let ***l(C)*** denote the number of non-isomorphic genus one l-c-trees corresponding to C. Alternatively, $l(C)$ is the number of non-isomorphic l-c-trees obtained from different labelings of C. If C_1, C_2, C_3, …, C_k are all of the non-isomorphic genus one u-c-trees on n points, then $f(n) = l(C_1) + l(C_2) + l(C_3) + \cdots + l(C_k)$.

Definition 8. *Given a l-c-tree T on n points let the rotation of T result in $r(T)$, the l-c-tree with edges $\{r(a)r(b) \mid ab$ is an edge of $T\}$, where $r(i) = i + 1$ for $i \in [n-1]$, and $r(n) = 1$.*

It directly follows that if T is a l-c-tree on n points, then T and $r^n(T)$ are isomorphic. When writing $T = r^m(T)$ where T is a l-c-tree it is meant that T and $r^m(T)$ are isomorphic.

Definition 9. *Given a u-c-tree C on n points with vertices evenly distributed on the circle let the rotation of C result in $R(C)$, the u-c-tree obtained by a geometrical rotation of C around the center of the circle by $\frac{2\pi}{n}$ in a counterclockwise direction, Figure 7.1.*

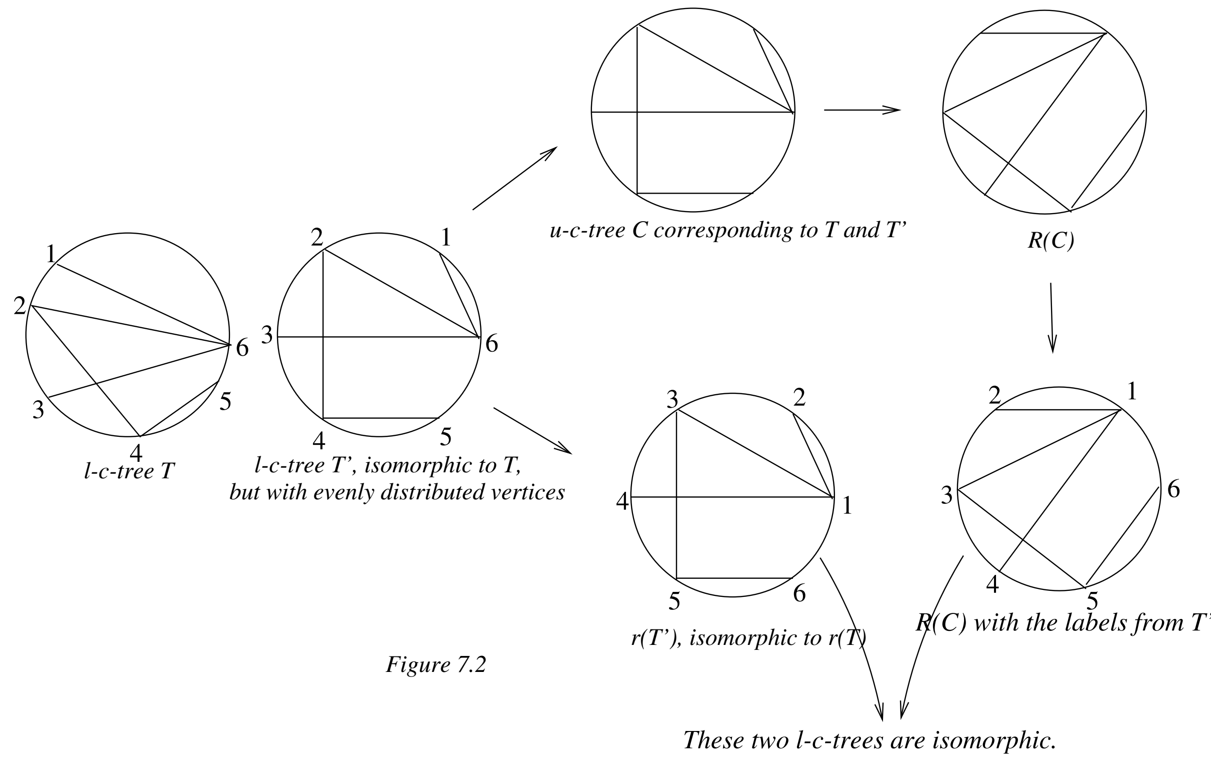


Figure 7.2

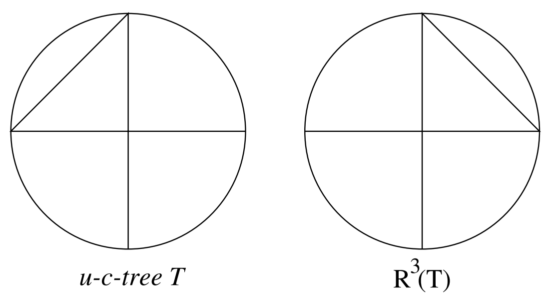


Figure 7.1

Clearly, if C is a u-c-tree on n points with evenly distributed vertices, then C coincides with $R^n(C)$ geometrically. When writing $C = R^m(C)$ where C is a u-c-tree it is meant that C and $R^m(C)$ coincide geometrically.

Proposition 6. *Given a l-c-tree T on n points, let C be a u-c-tree corresponding to T, such that C has its vertices evenly distributed on a circle. Then, $T = r^m(T)$ if and only if $C = R^m(C)$, for all $m \in \mathbb{N}$.*

Proof. Let T' be an l-c-tree isomorphic to T such that simply deleting the labels of T' results in C. Note that if we fix the labels of T', rotate C and associate the fixed labels to $R(C)$ we obtain the l-c-tree $r(T')$. (Figure 7.2.) Iterating this, it is true that associating the fixed labels of T' to $R^m(C)$ we obtain $r^m(T')$. Thus, $T' = r^m(T')$ if and only if $C = R^m(C)$.

Since T and T' are isomorphic l-c-trees, $T = r^m(T)$ if and only if $T' = r^m(T')$. Therefore, $T = r^m(T)$ if and only if $C = R^m(C)$. □

Proposition 7. *Given a l-c-tree T on n points let m be the minimal positive integer such that $T = r^m(T)$. Let C be a u-c-tree corresponding to T. Then l(C)=m and m divides n.*

Proof. Note that $T, r(T), r^2(T), \ldots, r^{m-1}(T)$ are all non-isomorphic l-c-trees, since if $r^i(T)$ was isomorphic to $r^j(T)$, $0 \leq i < j \leq m-1$ then $T = r^{j-i}(T)$, $j - i < m$, contradicting that m is the minimal positive integer such that $T = r^m(T)$. Also note that any $r^k(T)$, where $k > m-1$, is isomorphic to $r^{k'}(T)$, where k' is the smallest nonnegative remainder of k modulo m. Therefore, $T, r(T), r^2(T), \ldots, r^{m-1}(T)$ are all of the non-isomorphic l-c-trees corresponding to C, thus $l(C) = m$. Since $T = r^n(T)$, m divides n. (This follows since there exists k such that $0 \leq k \cdot m - n < m$, and since m was chosen to be the minimal positive integer with property $T = r^m(T)$, and $T = r^{k \cdot m - n}(T)$ it must be that $k \cdot m - n = 0$.)

□

Proposition 8. *f(n) is divisible by n if n is prime, n>3.*

Proof. Let $C_1, C_2, \ldots, C_k$ be all non-isomorphic u-c-trees on n points. Let T_i be one of the genus one l-c-trees corresponding to C_i for $1 \leq i \leq x$, and let m_i be the minimal positive integer such that $T_i = r^{m_i}(T_i)$. According to Proposition 7 all m_i divide n, but it is clear that m_i cannot be 1. Thus $m_i = n$ for all $1 \leq i \leq k$, and $f(n) = l(C_1)+l(C_2)+l(C_3)+\cdots+l(C_k) = k \cdot n$, is divisible by n. □

If for all genus one u-c-trees C on n points $l(C)$ was n, it would be trivial to conclude that $f(n)$ is divisible by n. However, this is not the case, as already pointed out in Section 2, Figure 2.1. We investigate the minimal number of rotations needed for a u-c-tree C to rotate into itself, which number is equal to $l(C)$ by Proposition 7.

Observation. *When a u-c-tree C is rotated into itself, then its e-graphs rotate into e-graphs, and paths of uncrossed edges connecting e-graphs rotate into paths of uncrossed edges connecting e-graphs.*

Recall that we can think of the reduced form of a u-c-tree C as of a u-c-tree where e-graphs are represented by e-graphs which have a single edge, and paths of uncrossed edges connecting e-graphs by single uncrossed edges. From the Observation we conclude that if the minimum number of rotations m for which a u-c-tree C rotates into itself, $R^m(C) = C$, is less than n, and k is the number of vertices of its reduced form, then the reduced form rotates into itself in less then k rotations (the points of the reduced form can be made to be evenly distributed on the circle, and the definition of rotation is analogous as in case of a u-c-tree).

Proposition 9. *The number of genus one l-c-trees on n points which have corresponding u-c-trees that reduce to reduced forms ${T_3}^1[1]$, ${T_3}^2[1]$, ${T_3}^3[1]$, ${T_3}^4[1]$, ${T_3}^4[2]$, ${T_3}^5[1]$, ${T_3}^5[2]$, ${T_3}^5[3]$, ${T_3}^5[4]$, ${T_3}^5[5]$, ${T_3}^5[6]$, ${T_3}^6[1]$, ${T_3}^6[2]$, ${T_3}^6[3]$ is divisible by n.*

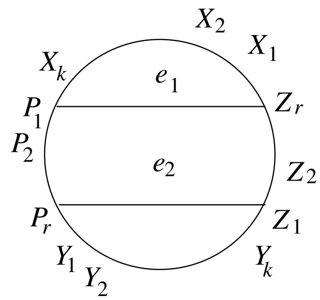


Figure 7.3

Proof. Let ***min(T)*** denote the minimum number of rotations needed for a u-c-graph T to rotate into itself. Since $min(T)$ <(the number of vertices of T) fails for all reduced forms except for ${T_3}^2[2]$, ${T_3}^2[3]$, ${T_3}^6[4]$, ${T_3}^6[5]$, and ${T_3}^6[6]$, only the genus one u-c-trees on n points having these reduced forms might rotate into themselves in less then n rotations (by Observation). Therefore, the number of genus one l-c-trees on n points which have corresponding u-c-trees that reduce to ${T_3}^1[1]$, ${T_3}^2[1]$, ${T_3}^3[1]$, ${T_3}^4[1]$, ${T_3}^4[2]$, ${T_3}^5[1]$, ${T_3}^5[2]$, ${T_3}^5[3]$, ${T_3}^5[4]$, ${T_3}^5[5]$, ${T_3}^5[6]$, ${T_3}^6[1]$, ${T_3}^6[2]$, ${T_3}^6[3]$ is divisible by n, since for each such u-c-tree C, $l(C) = n$. □

When ${T_3}^2[2]$, ${T_3}^2[3]$, ${T_3}^6[4]$, ${T_3}^6[5]$, ${T_3}^6[6]$ are rotated into themselves in less than k rotations (thinking of k as the number of vertices of a particular reduced form) in each of the reduced forms exactly one edge (which is an e-graph) rotates into itself, while all the other edges are "paired up," meaning that if edge e rotates into edge e' then edge e' rotates into edge e. Consider a u-c-tree C on n points which reduces to one of ${T_3}^2[2]$, ${T_3}^2[3]$, ${T_3}^6[4]$, ${T_3}^6[5]$, ${T_3}^6[6]$ and for which $min(C) < n$.

Based on the observation made about these reduced forms, we conclude that in C one of its e-graphs rotates into itself, while the edges not belonging to that e-graph pair up in the rotation which takes C into itself in less than n rotations.

Let the e-graph of C which rotates into itself have outermost edges e_1 and e_2. In the rotation e_1 rotates into e_2 and e_2 rotates into e_1, therefore points Z_i, P_i, X_j, Y_j rotate into points P_i, Z_i, Y_j, X_j respectively, where Z_i, P_i, X_j, Y_j, $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, k$, are as shown in Figure 7.3. Thus, $n = 2 \cdot (k + r)$ in case a u-c-tree rotates into itself in less then n rotations, and we conclude that if for a u-c-tree C on n points $min(C) < n$, then $min(C) = \frac{n}{2}$ rotations. The results we obtained in this discussion are summarized in Theorem 3:

Theorem 3. *Given a u-c-tree C on n points, where $n > 3$ is odd, C rotates into itself only after $i \cdot n$ rotations, $i \in \mathbb{N}$, thus $l(C) = n$. Given a u-c-tree C on n points, where $n > 3$ is even, if $l(C) \neq n$, then $l(C) = \frac{n}{2}$.*

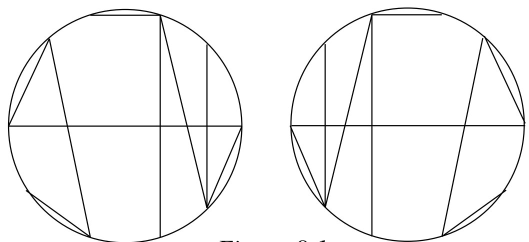
Theorem 4. *f(n) is divisible by n for n>3, n odd. For n even, n>3, f(n) is divisible by $\frac{n}{2}$.*

Proof. Theorem 4 follows from Theorem 3, since $f(n) = l(C_1) + \cdots + l(C_k)$, where $C_1, \ldots, C_k$ are all of the non-isomorphic genus one l-c-trees on n points. □

8 Reduced Forms $T_3{}^2[2]$, $T_3{}^2[3]$, $T_3{}^6[4]$, $T_3{}^6[5]$, $T_3{}^6[6]$

By Proposition 9 the number of genus one l-c-trees on n points with corresponding u-c-trees that reduce to $T_3{}^1[1]$, $T_3{}^2[1]$, $T_3{}^3[1]$, $T_3{}^4[1]$, $T_3{}^4[2]$, $T_3{}^5[1]$, $T_3{}^5[2]$, $T_3{}^5[3]$, $T_3{}^5[4]$, $T_3{}^5[5]$, $T_3{}^5[6]$, $T_3{}^6[1]$, $T_3{}^6[2]$, $T_3{}^6[3]$ is divisible by n, thus we are further interested in the number of genus one l-c-trees on n points with corresponding u-c-trees reducing to $T_3{}^2[2]$, $T_3{}^2[3]$, $T_3{}^6[4]$, $T_3{}^6[5]$, $T_3{}^6[6]$. For each u-c-tree on n points, reducing to $T_3{}^2[2]$ there is a naturally corresponding u-c-tree on n points reducing to $T_3{}^2[3]$ (due to axis symmetry, Figure 8.1) and for each u-c-tree reducing to $T_3{}^6[4]$ there is a u-c-tree reducing to $T_3{}^6[6]$ for the same reason. This statement is formalized in Proposition 10.

u-c-tree with reduced form $\boldsymbol{T_3^2}[2]$



u-c-tree with reduced form $\boldsymbol{T_3^2}[3]$

Figure 8.1

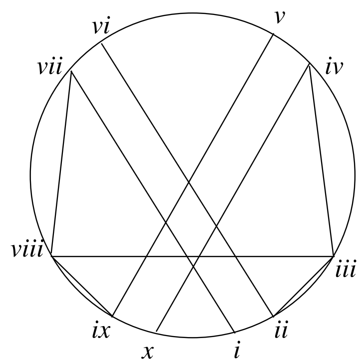
Proposition 10. *There exists a bijection b between u-c-trees on n points reducing to $T_3{}^2[2]$ ($T_3{}^6[4]$) and u-c-trees on n points reducing to $T_3{}^2[3]$ ($T_3{}^6[6]$) such that if $b(C[1]) = C[2]$, then $min(C[1]) = min(C[2])$ to rotate into itself.*

Proof. We exhibit an explicit bijection b satisfying the conditions of Proposition 10. Given a u-c-tree $C[1]$ on n points reducing to $T_3{}^2[2]$ ($T_3{}^6[4]$), label its vertices with *1* through n in a counterclockwise direction starting by labeling an arbitrary vertex with *1*. Take a circle c_2 and label n of its points in a clockwise direction with *1* through n. Draw edges ij on c_2 provided some edge of $C[1]$ was labeled with ij. Then delete the labels of the points of c_2. The obtained graph $b(C[1]) = C[2]$ is a u-c-tree reducing to $T_3{}^2[3]$ ($T_3{}^6[6]$). Bijection b satisfies the conditions specified. □

Proposition 11. *The number of genus one l-c-trees on n points with u-c-trees reducing to $T_3{}^2[2]$, $T_3{}^2[3]$ $T_3{}^6[4]$, or $T_3{}^6[6]$ is divisible by n.*

Proof. Proposition 10 proves the existence of a bijection b between u-c-trees on n points reducing to $T_3{}^2[2]$ ($T_3{}^6[4]$) and u-c-trees on n points reducing to $T_3{}^2[3]$ ($T_3{}^6[6]$) such that if $b(C[1]) = C[2]$, then $min(C[1]) = min(C[2])$. Since for any u-c-tree C, $min(C) = n$ or $min(C) = \frac{n}{2}$, it follows that $l(C[1]) + l(C[2])$ is always divisible by n (since $min(C) = l(C)$). Therefore, summing $l(C)$ over all u-c-trees C on n points reducing to $T_3{}^2[2]$, $T_3{}^2[3]$ $T_3{}^6[4]$, $T_3{}^6[6]$ we obtain a number divisible by n. The statement of the proposition follows. □

Remark. From Proposition 9 and Proposition 11 we conclude that it depends only upon the number of genus one l-c-trees T with corresponding u-c-trees C reducing to $T_3{}^6[5]$ and such that $min(C) = \frac{n}{2}$, whether $f(n)$ is divisible by n, or only by $\frac{n}{2}$. Note that this question is for n even, since for n odd we already saw that $l(C) = n$ for all genus one u-c-trees C on n points.



Reduced form $\boldsymbol{T}_3^6[5]$

Figure 9.1

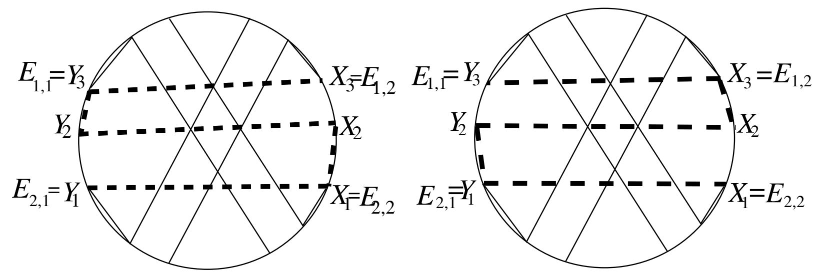
9 The Examination of Reduced Form ${T_3}^6[5]$

Let $P_n = \{C \in \mathcal{G} \mid C\ is\ a\ u\text{-}c\text{-}tree\ on\ n\ points\ reducing\ to\ {T_3}^6[5]\ such\ that\ min(C) = \frac{n}{2}\}$. From the Remark of Section 8 $f(n)$ (n even) is divisible by n if and only if $|P_n|$ is even, since $f(n) \equiv |P_n| \cdot \frac{n}{2} \pmod{n}$. If $|P_n|$ is odd, then $f(n)$ is not divisible by n but is divisible by $\frac{n}{2}$. In this section we investigate the parity of $|P_n|$.

We say that an ***e-graph reduces to an edge*** ab if and only if in the e-reduction process edge ab (or $a-b$) was the one left (not deleted) from the set of parallel edges of the e-graph. In a u-c-tree $C \in P_n$ e-graph $a-b$, $a, b \in \{i, ii, iii, \cdots, ix, x\}$, is the e-graph reducing to edge $a-b$ in the reduced form of C, Figure 9.1.

Proposition 12. *There exists a bijection h mapping the u-c-trees $C \in P_n$ with e-graph $iii - viii$ consisting of more than one edge into the u-c-trees $C \in P_n$ with e-graph $iii - viii$ consisting of more than one edge, such that if $h(C[1]) = C[2]$ then $h(C[2]) = C[1]$ and $C[1]$ and $C[2]$ are non-isomorphic.*

Proof. Given a u-c-tree $C[1] \in P_n$ we know $min(C) = \frac{n}{2}$ and thus its e-graph $iii - viii$ also rotates into itself in $\frac{n}{2}$ rotations. Let e_1 and e_2 be the e-graph's outermost edges in $C[1]$. Let points $X_1, X_2, X_3, .., X_m$ be the points on one arc of the e-graph in counterclockwise direction and let $Y_1, Y_2, Y_3, ..., Y_m$ be the points on the other arc of the e-graph in clockwise direction, so $\{e_1, e_2\} = \{X_1Y_1, X_mY_m\}$. (Note that the number of points on the two arcs of the e-graph is the same since they rotate into each other being that e_1 rotates into e_2 and vica versa. Also, $m > 1$ since we supposed the e-graph consists of more than one edge.). Leaving the edges of u-c-tree $C[1]$ the same, except changing the edges of form $(X_i,\ X_j)$ into $(Y_i,\ Y_j)$, $(Y_i,\ Y_j)$ into $(X_i,\ X_j)$ and $(X_i,\ Y_j)$ into $(Y_i,\ X_j)$ we obtain a u-c-tree $C[2]$ such that $min(C[2]) = \frac{n}{2}$, Figure 9.2. It is clear from the construction that if $h(C[1]) = C[2]$ then $h(C[2]) = C[1]$. To complete the proof, we need to show that $C[1]$ and $C[2]$ are different. Suppose the opposite. Then e-graph $iii - viii$ also coincides in the two u-c-trees. Let $e_1 = E_{1,1}E_{1,2}$ and $e_2 = E_{2,1}E_{2,2}$. Then e_1 and e_2 need to be connected by the edges of e-graph $iii - viii$ since they are the outermost edges of the e-graph. Let there be a path



Pairs of u-c-trees in case e-graph iii-viii consists of more than one edge.
edges of e-graph iii-viii
Figure 9.2

of edges of the e-graph connecting vertices E_{1,i_1} and E_{2,j_1} $(1 \leq i_1, j_1 \leq 2)$ in $C[1]$. Then in $C[2]$ vertices E_{1,i_2} and E_{2,j_2} $(1 \leq i_2, j_2 \leq 2,\ i_1 \neq i_2,\ j_1 \neq j_2)$ would be connected, and if $C[1] = C[2]$ then there would be a circle of edges containing vertices $E_{1,1}$, $E_{1,2}$, $E_{2,1}$, and $E_{2,2}$. This, however, cannot happen since in a tree there cannot be circles. $\square$

Proposition 13. *The number of u-c-trees C such that $C \in P_n$ and e-graph $iii - viii$ in C consists of more than one edge is even.*

Proof. Follows from the existence of bijection h described in Proposition 12. $\square$

Proposition 13 implies that the number of genus one l-c-trees on n points with corresponding u-c-trees that are in P_n and for which in the corresponding u-c-trees e-graph $iii - viii$ consists of more than one edge is divisible by n.

About U-C-Trees $C \in P_n$ Such That E-Graph $iii - viii$ of C Consists of a Single Edge

It remains to examine the u-c-trees $C \in P_n$ such that e-graph $iii - viii$ of C consists of a single edge. Until the end of this section it is assumed that all u-c-trees are such. The definitions assume this as well. Also, we simply denote such a u-c-tree by C. The labels $i, ii, \ldots, viii, ix, x$ refer to Figure 9.1 or to a u-c-tree C which has ${T_3}^6[5]$ as its reduced form.

When we refer to ***uncrossed edges which connect to e-graph*** $\mathcal{E}$, we mean the set of uncrossed edges E, defined recursively as follows:

• all the uncrossed edges which are not edges of $\mathcal{E}$ but have some of the endpoints of the outermost edges of $\mathcal{E}$ as their endpoints are in E

• edge e is in E if it is not in $\mathcal{E}$ and if it is uncrossed and has a common endpoint with some edge already in E

Definition 10. *Let K_1 be the subgraph of C such that it consists of e-graph $vi - ii$, and all the uncrossed edges connecting to e-graph $vi - ii$ with the restriction that they have both of their endpoints on arcs $i - iii$ and $v - vii$.*

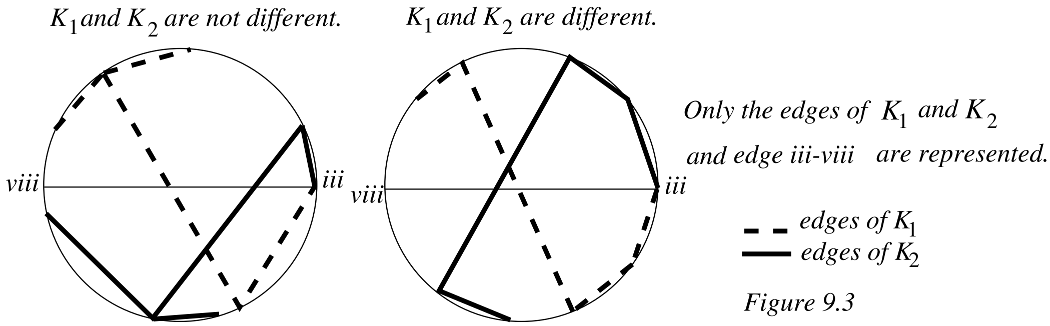


Figure 9.3

Definition 11. *Let K_2 be the subgraph of C such that it consists of e-graph $x - iv$, all of the uncrossed edges connecting to e-graph $x - iv$ with the restriction that they have both of their endpoints on arcs $ix - i$ and $iii - v$.*

Lemma 11. *E-graph $iii - viii$, K_1, and K_2 uniquely determine a u-c-tree C such that $min(C) = \frac{n}{2}$.*

Proof. When C rotates into itself in $\frac{n}{2}$ rotations, then K_1 rotates into the union of e-graph $i - vii$ and the uncrossed edges connecting to the e-graph $i - vii$ with the restriction that they have both of their endpoints on arcs $vi - viii$ and $x - ii$, and K_2 rotates into the union of e-graph $v - ix$, the uncrossed edges connecting to the e-graph $v - ix$ with the restriction that they have both of their endpoints on arcs $viii - x$ and $iv - vi$. Therefore, e-graph $iii - viii$, K_1, K_2, and the edges into which K_1 and K_2 rotate constitute all of the edges of C, thus the statement of the lemma follows. □

Definition 12. *Given C we say that K_1 and K_2 are identical if when we label the points of C with 1 through n in a clockwise direction so that iii gets labeled with 1 the edges of K_1 get the same labels as the edges of K_2 when we label the points of C with 1 through n in a counterclockwise direction so that iii gets labeled with 1. If this is not the case we say that K_1 and K_2 are different.*

Intuitively, K_1 and K_2 are different if by reflecting K_1 upon the axis of symmetry parallel to $viii - iii$ we do not get K_2, Figure 9.3.

Lemma 12. *If in a u-c-tree C K_1 is different from K_2, then there exists a bijection g mapping the set of u-c-trees C with K_1 and K_2 different into itself, so that if $g(C[1]) = C[2]$, then $g(C[2]) = C[1]$ and $C[1]$ is not isomorphic to $C[2]$.*

Proof. Intuitively, g maps u-c-tree $C[1]$ into a u-c-tree $C[2]$ such that $C[2]$ is obtained from $C[1]$ by reflecting all the edges of $C[1]$ upon edge $iii - viii$, Figure 9.4.

Formally, the following bijection g has the property described. Given $C[1]$ with K_1 and K_2 different, label its vertices with *1* through n in a counterclockwise direction starting by labeling iii with *1*. Take a circle c_2 and label n of its points in a clockwise direction with *1* through n. Draw edges ij on c_2 provided some edge of $C[1]$ was labeled with ij. Then delete the labels of the points of c_2. The obtained graph is $C[2]$. □

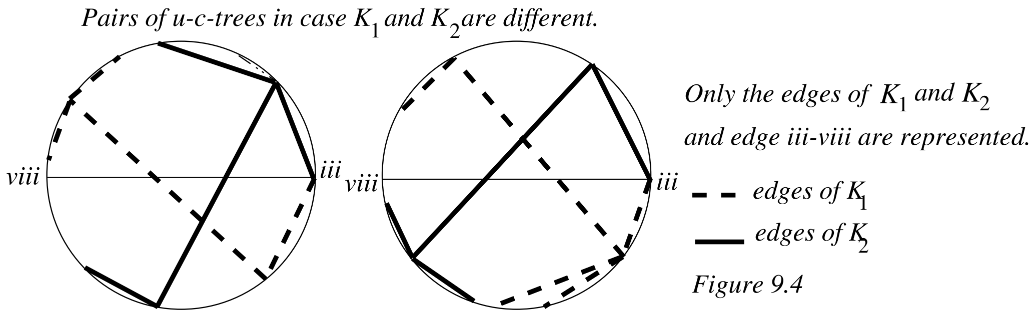


Figure 9.4

Lemma 13. *The number of u-c-trees C with K_1 and K_2 different is even.*

Proof. The statement follows from the existence of bijection g described in Lemma 12. $\square$

Therefore, it remains to determine the parity of the number p of u-c-trees in P_n, such that e-graph $iii - viii$ is a single edge, and K_1 and K_2 are identical. This can happen only if $\frac{n-2}{2}$ is even (since if K_1 and K_2 are identical, then the number of edges in K_1 and K_2 is the same, thus the sum of the number of edges in K_1 and K_2 is even, and on the other hand it is $\frac{n-2}{2}$). Therefore, in case $\frac{n-2}{2}$ is odd (which is equivalent to n divisible by 4) $p = 0$ and it follows that:

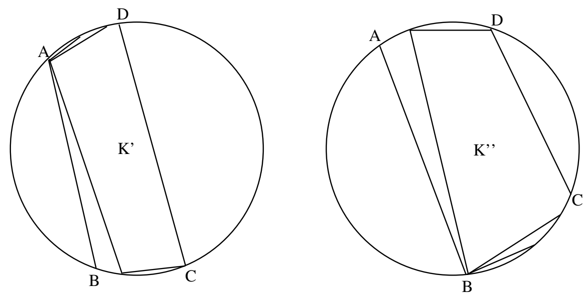
Theorem 5. *If n is divisible by 4, the parity of the number of u-c-trees on n points is even, and so $f(n)$ is divisible by n.*

For $n \equiv 2 \pmod 4$ ($\frac{n-2}{2}$ is divisible by 2) p is equal to the number of possible ways of constructing the union of e-graph $vi - ii$ and all the uncrossed edges connecting to e-graph $vi - ii$ with the restriction that they have both of their endpoints on arcs $i - iii$ and $v - vii$ using $\frac{n-2}{4}$ edges, since this uniquely determines a u-c-tree C where $iii - viii$ is a single edge and K_1 and K_2 are identical. We formalize the previous notion in the following definition.

Definition 13. *Suppose we draw e-graph $\mathcal{E}$ with parallel edges represented by vertical lines, then we call the outermost edge on the right the rightmost edges of $\mathcal{E}$. $\mathcal{K}_m = \{K \in \mathcal{G} \mid K$ is a u-c-graph with m edges consisting of an e-graph $\mathcal{E}$ and uncrossed edges connecting to the e-graph, with the special property that there is at least one uncrossed edge AX, not an edge of the e-graph $\mathcal{E}$, connecting to the e-graph whose rightmost edge is AB and X is in open arc $\widehat{AB}\}$.*

Clearly, e-graph $vi - ii$ of C is imitated by e-graph $\mathcal{E}$, that there are uncrossed edges on the arc between ii and iii is ensured by the edge AX mentioned in the definition, and the uncrossed edges connecting to e-graph $\mathcal{E}$ imitate the uncrossed edges connecting to e-graph $vi - ii$ of C with the restriction that they have both of their endpoints on arcs $i - iii$ and $v - vii$, and vice versa. From this $p = \mid \mathcal{K}_{\frac{n-2}{4}} \mid$.

Lemma 14. *There is an even number of $K \in \mathcal{K}_m$ such that the e-graph of K consists of more than one edge.*



K' and K'' when e-graph ii-vi consists of more than one edge. Only the edges of e-graph ii-vi are represented on the figure. The outermost edges of the e-graph are AB and CD.

Figure 9.5

Proof. Let $K' \in \mathcal{K}'_m$ be given, where $\mathcal{K}'_m = \{K \in \mathcal{K}_m \mid \textit{the e-graph of } K \textit{ consists of more than one edge}\}$. Define a function $k : \mathcal{K}'_m \to \mathcal{K}'_m$ as follows. Think of K' using k edges as K_1 with the property that e-graph $vi - ii$ has more than one edge. Let the points of e-graph $vi - ii$ be $X_1, X_2, X_3, \ldots, X_{m_1}$ counterclockwise on one of its arcs and $Y_1, Y_2, Y_3, \ldots, Y_{m_2}$ clockwise on the other of its arcs ($m_1 > 1$ or $m_2 > 1$). Changing edges of form (X_i, X_j) into (Y_i, Y_j), (Y_i, Y_j) into (X_i, X_j) and (X_i, Y_j) into (Y_i, X_j), and leaving the other edges of K' the same we obtain $K'' = k(K')$ different from K' (The proof that K' and K'' are different follows the lines of an analogous proof at the end of Proposition 12.). Clearly, $K' = k(K'')$. Therefore, $\mid \mathcal{K}'_m \mid$ is even, which is the statement of the lemma. See Figure 9.5. $\square$

Conclusion. Since $\mid \mathcal{K}'_{\frac{n-2}{4}} \mid$ contributes an even number to p, the only remaining case to consider is when e-graph $ii - vi$ in $K \in \mathcal{K}_{\frac{n-2}{4}}$ consists of one edge. Our goal is to determine the parity of the number of possible constructions of: union of e-graph $vi - ii$, consisting of a single edge, and all the uncrossed edges connecting to e-graph $vi - ii$ with the restriction that they have both of their endpoints on arcs $i - iii$ and $v - vii$ and that there is at least on edge on arc $ii - iii$, using $\frac{n-2}{4}$ edges.

10 L-C-Trees and Left-Right Trees

Observe that the structure described in the Conclusion of the previous section naturally decomposes into three parts, namely the edge $vi - ii$, and the uncrossed edges with endpoints on arc $i - iii$ and the uncrossed edges with endpoints on arc $v - vii$. The following two definitions capture the latter two trees, whereas the third definition is a formal way of defining the structure from the Conclusion. We note in advance that the edge AB in the definitions represents the edges $vi - ii$ from the Conclusion.

Definition 14. $\mathcal{A}$= $\{T \in \mathcal{G} \mid$ *T is a u-c-tree, such that given a specially designated edge AB, called the axis of T, the union of T and AB is a u-c-tree with all edges uncrossed, and there is no edge in T having B as its endpoint (but clearly, vertex A is a vertex of T)*$\}$.

If the axis of T is AB and A is a vertex of T, then vertex A is called the root of T. It is also said that T is rooted at A. (Note, we don't consider AB to be an edge of T.)

Definition 15. $\mathcal{B}$= $\{T \in \mathcal{G} \mid$ *T is a u-c-tree, such that given a specially designated edge AB, called the axis of T, the union of T and AB is a u-c-tree with all edges uncrossed, and there* *is no edge in T having A as endpoint (but clearly, vertex B is a vertex of T); furthermore, we require that there is at least one edge of T of form BC with C in open arc $\widehat{BA}$}. If the axis of T is AB and B is a vertex of T, then vertex B is called the root of T. It is also said that T is rooted at B. (Note, we don't consider AB to be an edge of T.)*

Definition 16. $\mathcal{L} = \{L \mid L = (AB,\, T_1,\, T_2)$, *where AB is a specially designated edge on the circle, $T_1 \in \mathcal{A}$ rooted at vertex A having axis AB, and $T_2 \in \mathcal{B}$ rooted at vertex B having axis AB; the union of AB, T_1 and T_2 is a u-c-tree with all edges uncrossed, such that T_1 and T_2 have no vertices in common*$\}$.

The number of edges of $L = (AB,\, T_1,\, T_2) \in \mathcal{L}$ is the number of edges of T_1 plus the number of edges of T_2 plus one.

As already noted at the beginning of the section, the structure defined in the conclusion of Section 9 is modeled by $L = (AB,\, T_1,\, T_2)$, indeed, AB corresponds to the edge $vi - ii$, T_1 rooted at A (vi) corresponds to the uncrossed edges with endpoints on arc $v - vii$ connecting to e-graph $vi - ii$, and T_2 rooted at B (ii) corresponds to the uncrossed edges with endpoints on arc $i - iii$ connecting to e-graph $vi - ii$. Indeed, there is a one-to-one correspondence between $L = (AB,\, T_1,\, T_2)$ with k edges and a u-c-subgraph of the u-c-tree C reducing to ${T_3}^6[5]$ consisting of $vi - ii$, and all the uncrossed edges connecting to e-graph $vi - ii$ with the restriction that they have both of their endpoints on arcs $i - iii$ and $v - vii$ and that there is at least on edge on arc $ii - iii$, using k edges. Since we are interested in the parity of the number of the latter with $k = \frac{n-2}{4}$, it suffices to investigate the parity of the number of $L = (AB,\, T_1,\, T_2)$ with $\frac{n-2}{4}$ edges in order to obtain an answer to our original question. This is the problem that we solve in the following sections.

Definition 17. *Let $\mathcal{A}_k = \{T \in \mathcal{A} \mid T$ has k edges$\}$. Let $a_k =\mid \mathcal{A}_k \mid$.*

Let $\mathcal{B}_k = \{T \in \mathcal{B} \mid T$ has k edges$\}$. Let $b_k =\mid \mathcal{B}_k \mid$.

Let $\mathcal{L}_k = \{L \in \mathcal{L} \mid L$ has $k+1$ edges$\}$. Let $l_k =\mid \mathcal{L}_k \mid$.

In terms of the just introduced symbols we are looking for the parity of l_s, where $s = \frac{n-2}{4} - 1$.

Lemma 15. $l_s = \sum_{i=0}^{s-1} a_i \cdot b_{s-i}$.

Proof. By definition, any $L \in \mathcal{L}$ is a triple $(AB,\, T_1,\, T_2)$, with a fixed edge AB and $T_1 \in \mathcal{A}$ rooted at A, $T_2 \in \mathcal{B}$ rooted at B. In order to obtain l_s we have to sum over all possible T_1 and T_2, such that the sum of the number of edges of these two forms is s. Since T_2 has at least one edge by definition, the number of edges of T_1 can vary from 0 to $s-1$. Thus, $l_s = \sum_{i=0}^{s-1} a_i \cdot b_{s-i}$. □

Definition 18. *A left-right tree T is a finite set of vertices such that:*

a. One specially designated vertex is called the root of T and a left-right delimiter $k \in \mathbb{N}\backslash\{0\}$ is specified,

b. The remaining vertices (excluding the root) are put into an ordered partition $(T_1, \ldots, T_l)$ of $l \geq 0$ disjoint non-empty sets $T_1, \ldots, T_l$, each of which is a left-right tree. The left-right *delimiter k specifies that $T_1, \dots, T_{k-1}$ are left from the root of T, and the edges connecting the root of T with the roots of $T_1, \dots, T_{k-1}$ are called left edges, while the roots of $T_k, \dots, T_l$ are said to be right from the root of T and the edges connecting the root of T and the roots of $T_k, \dots, T_l$ are called right edges. The edge connecting the root of T with the root of T_l is called the rightmost edge of the l-r-tree. The trees $T_1, \dots, T_l$ are called subtrees of the root, more precisely trees $T_1, \dots, T_{k-1}$ are left subtrees while trees $T_k, \dots, T_l$ are right subtrees of the root.*

For some simple examples of l-r-trees see Figure 11.2.

There is a straightforward bijection between the set $\mathcal{A}_k$ and the set of left-right-trees with k edges. Also, there is a straightforward bijection between the set $\mathcal{B}_k$ and the set of left-right-trees with k edges which have at least one right edge coming out of the root.

Namely, given $T \in \mathcal{A}$ ($\mathcal{B}$) with axis AB, rooted at A (B), let $C_1, \dots, C_{k-1}$ be all the vertices of T, such that AC_i (BC_i) is an edge, and C_i is in arc $\widehat{BA}$ ($\widehat{AB}$). Let $C_k, \dots, C_l$ be all the vertices of T, such that AC_i (BC_i) is an edge, and C_i is in arc $\widehat{AB}$ ($\widehat{BA}$). Arcs AC_l, $C_lC_{l-1}, \cdots, C_2C_1, C_1A$ (BC_l, $C_lC_{l-1}, \cdots, C_2C_1, C_1B$) cover the circle and are disjoint. Let $T_1, \dots, T_k$ be elements of $\mathcal{A}$, T_i rooted at vertex C_i, having axis AC_i (BC_i) and containing all the edges of T which they can possibly contain.

Then, the recursive definition of the bijection is:

a. Set vertex A' (B') to be the root of the corresponding left-right-tree T' (l-r-tree for short).

b. $(T'_1, \dots, T'_l)$ are the ordered subtrees of the root of T' with the property that $(T'_1, \dots, T'_{k-1})$ are the left subtrees, $(T'_k, \dots, T'_l)$ are the right subtrees, and T'_i is the l-r-tree corresponding to T_i. Finally, a single point corresponds to a single point.

From the bijections above we deduce that the number of l-r-trees with k edges is a_k, while the number of l-r-trees with k edges such that there is at least one right edge coming out of the root is b_k. For convenience we define the following three sets:

Definition 19. $\mathcal{A}'_k = \{T \mid T$ *is a l-r-trees with* k *edges*$\}$.

$\mathcal{B}'_k = \{T \mid T$ *is a l-r-trees with* k *edges such that there is at least one right edge coming out of the root of* $T\}$.

$\mathcal{C}'_k = \mathcal{A}'_k \backslash \mathcal{B}'_k$. *Let* $c_k = \mid \mathcal{C}'_k \mid$

Now we state the results mentioned above using these symbols. We also include the relation $c_k = a_k - b_k$ which follows directly from the definition of $\mathcal{C}'_k$.

Lemma 16. $\mid \mathcal{A}'_k \mid = a_k$, $\mid \mathcal{B}'_k \mid = b_k$, *and* $a_k = c_k + b_k$.

11 Results About a_k, b_k and c_k

As mentioned in the previous section, we are interested in the parity of the sum $l_s = \sum_{i=0}^{s-1} a_i \cdot b_{s-i}$, where $s = \frac{n-2}{4} - 1$. In order to determine this parity, we first investigate the parities of a_k and b_k.

In the following text we write $\equiv$ to mean equivalence *modulo 2*.

Observe that any $T \in \mathcal{B}'_m$ consists of the followings:

• a right edge $B'C'$ coming out of the root B' of the l-r-tree T; let $B'C'$ be the rightmost edge of T

• a $T_1 \in \mathcal{A}'_l$ rooted in C'

• a $T_2 \in \mathcal{A}'_{m-l-1}$ rooted in B'.

Thus, $b_m = \sum_{l=0}^{k-1} a_l \cdot a_{m-l-1}$. This equation immediately shows that if $m = 2k$, then $b_m \equiv 0$, and if $m = 2k+1$, then $b_m \equiv a_k^2 \equiv a_k$. That is:

Lemma 17. *$b_{2k} \equiv 0$ and $b_{2k+1} \equiv a_k$.*

Lemma 18. $a_{2k+1} \equiv 0$

Proof. We prove the statement by induction.

• Base of induction: $a_1 = 2 \equiv 0$

• Inductive hypothesis: for all $0 \leq k' < k$, $a_{2k'+1} \equiv 0$

• Inductive step:

Enumerate a_{2k+1} summing over all $a_{2k+1,m}$, $m = 1, 2, \ldots, 2k+1$, where $a_{2k+1,m}$ is the number of $T \in \mathcal{A}'_{2k+1}$ such that there are exactly m edges coming out of the root A' of T. Let $T_1, T_2, \ldots, T_m$ be the subtrees of T, where T_i is the l-r-tree rooted at the i^{th} child of A' (so that the m edges coming out of the root and $T_1, T_2, \ldots, T_m$ have no common edges, but their union is the whole l-r-tree T). Let t_i be the number of edges in T_i, for all $i \in \{1, 2, \ldots, m\}$. Then, $t_1 + t_2 + \cdots + t_m = 2k + 1 - m$. We obtain $a_{2k+1,m}$ by summing $(m+1) \cdot \prod_{i=1}^{m} a_{t_i}$ over all possible choices of $t_1, \ldots, t_m$, since $a_{2k+1,m}$ is equal to the number of ways to choose which of the m edges coming out of the root are right or left ($m+1$ ways for this) times the number of ways to construct the subtrees rooted at the children of the root ($\prod_{i=1}^{m} a_{t_i}$), and this all over the possible $t_1, \ldots, t_m$, satisfying $t_1 + t_2 + \cdots + t_m = 2k + 1 - m$.

First Claim. $a_{2k+1,2l+1} \equiv 0$

Regardless of how we fix the t_i satisfying $t_1 + t_2 + \cdots + t_{2l+1} = 2k + 1 - (2l+1)$, the sum $(2l+2) \cdot \prod_{i=1}^{2l+1} a_{t_i}$ over all possible choices of $t_1, \ldots, t_m$ will be divisible by $2l+2$, therefore, $a_{2k+1,2l+1}$ is divisible by 2.

Second Claim. $a_{2k+1,2l} \equiv 0$

There are $2l$ edges coming out of the root A', thus, $t_1 + t_2 + \cdots + t_{2l} = 2k + 1 - 2l \equiv 1$. Therefore, in every case when there are $2l$ edges coming out of the root at least one t_j must be odd, thus there exists a j such that $a_{t_j} \equiv 0$ by the inductive hypothesis (since $2l \geq 2$ it follows that $t_j < 2k+1$, so we can use the inductive hypothesis). Since $a_{2k+1,2l}$ is the sum of $(2l+1) \cdot \prod_{i=1}^{2l} a_{t_i}$ over all possible $t_1, t_2, \ldots, t_{2l}$, then this sum is $\equiv 0$, having that in each product there is some $a_{t_j} \equiv 0$.

Since $a_{2k+1} = \sum_{m=1}^{2k+1} a_{2k+1,m}$, and $a_{2k+1,m} \equiv 0$ for all m, then $a_{2k+1} \equiv 0$, and the induction is finished. ∎

Lemma 19. $a_{2k} \equiv c_k$

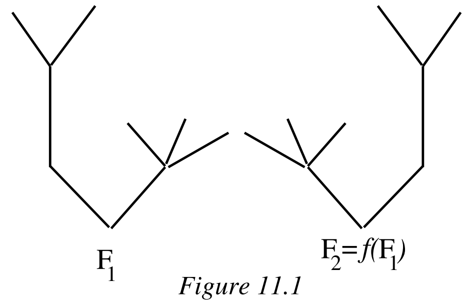


Figure 11.1

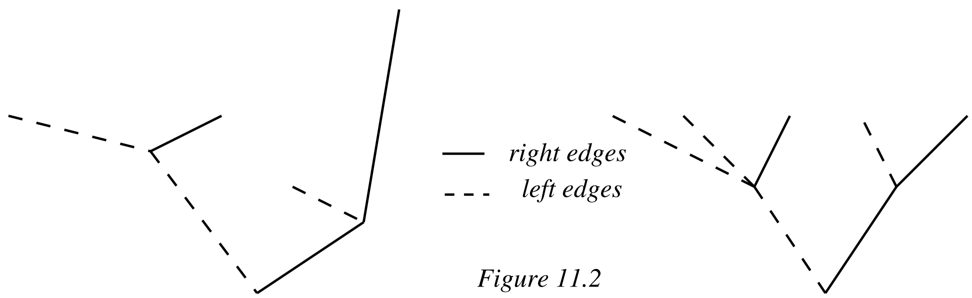


Figure 11.2

A symmetric l-r-tree. *A not symmetric l-r-tree.*

Proof. Let F_1 be a l-r-tree with $2k$ edges. Let R be the root of F_1, let d be its delimiter, and let $(T_1, \ldots, T_l)$ be the l-r-subtrees rooted at the children of the root (T_i is the l-r-tree having the i^{th} child for its root). We define operation f, "flip," as follows. Let l-r-tree $F_2 = f(F_1)$ have root R, delimiter $n - d + 2$ and the ordered partition of the remaining vertices is $(T'_l, \ldots, T'_1)$, where $T'_i = f(T_i)$. If F_1 contains just one vertex, and no edges, then $f(F_1) = F_1$. Intuitively, flipped F_1 is nothing but the l-r-tree flipped over a vertical axis going through the root, Figure 11.1. It is clear that if $f(F_1) = F_2$ then $f(F_2) = F_1$. We consider two l-r-trees identical if the edge connecting the root and the i^{th} child is left in both l-r-trees or right in both l-r-trees and if the l-r-subtrees rooted each child are identical.

Pairs of l-r-trees (F_1, F_2), where $f(F_1) = F_2$, and F_1 and F_2 are not identical contribute an even number to the number of l-r-trees with $2k$ edges, and so, the parity of a_{2k} is the parity of the number of l-r-trees F such that $f(F) = F$. In order for $f(F)$ to be identical to F, it must be that it is "symmetric," that is, the number of left edges coming out of the root is equal to the number of rigft edges coming out of the root, all subtrees rooted at children of the root are symmetric, and finally, a point is symmetric, Figure 11.2. The subgraph of a symmetric F which contains all the left edges coming out of the root and all the edges of the subtrees rooted at these left children of the root uniquely determine a symmetric F, furthermore, there is an obvious one-to-one correspondence between these subgraphs and symmetric l-r-trees. Also, there is an obvious correspondence between the subgraph described and elements of $\mathcal{C}'_k$. Thus, the parity of a_{2k} is the parity of $\mid \mathcal{C}'_k \mid$. Thus, $a_{2k} \equiv c_k$. ∎

The results obtained so far: *i)* $b_{2k} \equiv 0$; *ii)* $b_{2k+1} \equiv a_k$; *iii)* $a_k = b_k + c_k$; *iv)* $a_{2k+1} \equiv 0$;

$v)$ $a_{2k} \equiv c_k$; $vi)$ From $ii)$ $b_{4k+1} \equiv a_{2k}$ and $b_{4k+3} \equiv a_{2k+1}$; $vii)$ Using $vi)$ and $iv)$ $b_{4k+3} \equiv 0$; $viii)$ From $iii)$ and $i)$ $a_{2k} = b_{2k} + c_{2k} \equiv c_{2k}$, that is $a_{2k} \equiv c_{2k}$; $ix)$ From $iii)$ and $vi)$ $a_{4k+1} = b_{4k+1} + c_{4k+1}$ and so $a_{4k+1} \equiv b_{4k+1} + c_{4k+1} \equiv a_{2k} + c_{4k+1}$. Using $iv)$ we get $a_{2k} \equiv c_{4k+1}$; $x)$ From $iii)$ and $vii)$ $a_{4k+3} \equiv b_{4k+3} + c_{4k+3} \equiv c_{4k+3}$. Using $iv)$ $c_{4k+3} \equiv 0$. In all cases $k \geq 0$.

Summarizing this:

$a)$ $a_{2k+1} \equiv 0$; $b)$ $a_{2k} \equiv c_{4k+1} \equiv c_{2k} \equiv c_k$; $c)$ $c_{4k+3} \equiv 0$; $d)$ $b_{2k} \equiv 0$; $e)$ $b_{4k+3} \equiv 0$; $f)$ $b_{4k+1} \equiv a_{2k}$ In all cases $k \geq 0$.

We now directly attack the problem of determining the parity of $l_s = \sum_{i=0}^{s-1} a_i \cdot b_{s-i}$, where $s = \frac{n-2}{4} - 1$, $s \geq 1$.

Case 1. If $s = 2r$, for some $r \geq 1$, either i is odd, or $s-i$ is even, therefore $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv 0$ by $a)$ and $d)$. Therefore, l_s is even for $\frac{n-2}{4} - 1 = s = 2r$, where $r \geq 1$.

Result. For $n = 8r + 6$, $r \geq 1$, $f(n)$ is divisible by n.

Case 2. If $s = 2r + 1$, for some $r \geq 0$, l_s is equivalent with the sum over those i for which $i = 2k$ and $s - i = 4l + 1$, since $a_{2k+1} \equiv 0$, $b_{2k} \equiv 0$, and $b_{4k+3} \equiv 0$. We analyze this case by splitting it into more smaller cases.

Case 2.1.: $s = 4v + 1$, $2k + 4l + 1 = 4v + 1$, $v, k, l \geq 0$

Then $k = 2(v - l)$. Since l can be $0, 1, 2, \ldots, v$; k is then $2v, 2v - 2, \ldots, 0$.

Using $b)$ and $f)$ we have $a_{2k} \equiv c_k$ and $b_{4k+1} \equiv c_k$, and so $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_{2(v-i)} \cdot c_i \equiv \sum_{i=0}^{v} c_i \cdot c_{2(v-i)}$. Using $c_{2k} \equiv c_k$, $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_i \cdot c_{v-i}$. The last sum is symmetric, thus in case $v = 2v_1 + 1$, for some $v_1 \geq 0$, this sum is even, that is l_s is even for $\frac{n-2}{4} - 1 = s = 4v + 1 = 4(2v_1 + 1) + 1 = 8v_1 + 5$.

Result. For $n = 32v_1 + 26$, $v_1 \geq 0$, $f(n)$ is divisible by n.

In the case when $v = 2v_1$, $v_1 \geq 0$ we have:

Result. For $n = 32v_1 + 10$, $v_1 \geq 0 :$ $l_s \equiv c_{v_1}$, $s = \frac{n-2}{4} - 1$.

Case 2.2.: $s = 4v + 3$, $2k + 4l + 1 = 4v + 3$, $v, k, l \geq 0$

Then $k = 2(v - l) + 1$. Since l can be $0, 1, 2, \ldots, v$; k is then $2v + 1, 2v - 1, \ldots, 1$. Using $b)$ and $f)$ we have $a_{2k} \equiv c_k$ and $b_{4k+1} \equiv c_k$, and so $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1}$.

Case 2.2.1.: $v = 2v_1$, $v_1 \geq 0$.

Using $c_{4k+3} \equiv 0$ we get $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_{2i} \cdot c_{2(v-2i)+1}$. Using $c_{2k} \equiv c_k$ we get $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_i \cdot c_{2(v-2i)+1}$. Since $2(v - 2i) + 1 = 2(2v_1 - 2i) + 1 = 4(v_1 - i) + 1$, and $c_{4k+1} \equiv c_k$, we get $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_i \cdot c_{v_1-i}$. Since $\sum_{i=0}^{v_1} c_i \cdot c_{v_1-i}$ is symmetric, we have the following result: for $v_1 = 2v_2 + 1$, $v_2 \geq 0$, the sum is even, and so l_s is even for $\frac{n-2}{4} - 1 = s = 4v + 3 = 4 \cdot 2v_1 + 3 = 8(2v_2 + 1) + 3 = 16v_2 + 11$.

Result. For $n = 64v_2 + 50$, $v_2 \geq 0$, $f(n)$ is divisible by n.

In the case when $v_1 = 2v_2$ we have:

Result. For $n = 64v_2 + 18$, $v_2 \geq 0$: $l_s \equiv \sum_{i=0}^{v_1} c_i \cdot c_{v_1-i} \equiv c_{v_2}$.

Case 2.2.2.: $v = 2v_1 + 1$, $v_1 \geq 0$.

Using $c_{4k+3} \equiv 0$ from $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1}$ we get $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_{2i+1} \cdot c_{2(v-2i-1)+1} \equiv \sum_{i=0}^{v_1} c_{2i+1} \cdot c_{2(2v_1-2i)+1} \equiv \sum_{i=0}^{v_1} c_{2i+1} \cdot c_{4(v_1-i)+1}$ and using $c_{4k+1} \equiv c_k$ we have $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_{2i+1} \cdot c_{4(v_1-i)+1} \equiv \sum_{i=0}^{v_1} c_{2i+1} \cdot c_{v_1-i}$. Since, $\sum_{i=0}^{v_1} c_{2i+1} \cdot c_{v_1-i} = \sum_{i=0}^{v_1} c_{v_1-i} \cdot c_{2i+1} = \sum_{j=0}^{v_1} c_j \cdot c_{2(v_1-j)+1}$, we get that $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v_1} c_i \cdot c_{2(v_1-i)+1}$. Note that we started from $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1}$, and sums $\sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1}$ and $\sum_{i=0}^{v_1} c_i \cdot c_{2(v_1-i)+1}$

are of the same form, with the difference that the second one is of length v_1+1, while the first is of length $v+1=2v_1+2$, that is twice the second.

Let $S(v)=\sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1}$.

Observe, that we got $l_s \equiv S(v) \equiv S(v_1)$ and we still do not know what this sum is equivalent to modulo 2. Observe, that since the last two sums are of the same form, we can now begin the process of *Case 2.2.*, that is:

Case 2.2.1.: $v_1=2v_2$, if $v_2=2v_3+1$ the sum is even, but if $v_2=2v_3$ the sum is $\equiv c_{v_3}$

Case 2.2.2.: $v_1=2v_2+1$, then $\sum_{i=0}^{s-1} a_i \cdot b_{s-i} \equiv \sum_{i=0}^{v} c_i \cdot c_{2(v-i)+1} \equiv \sum_{i=0}^{v_1} c_i \cdot c_{2(v_1-i)+1} \equiv \sum_{i=0}^{v_2} c_i \cdot c_{2(v_2-i)+1}$ and so forth. Depending on v_2 we either continue the process, or get the final result in case v_2 is even.

Since in case v, v_1, v_2, etc. are odd, the sums $S(v)$, $S(v_1)$, $S(v_2)$, etc. we are considering always get twice shorter (from $v+1$ to v_1+1 to v_2+1, etc.) and so this process is finite (we cannot half an integer infinitely many times and still get an integer), i.e. at one point either the v_k we are going to consider must be even and greater than 0, and then we apply the *Case 2.2.1.* and get either that the sum is even, or that the sum $\equiv c_{v_k}$, or v_k gets equal to *1*, and in this case the sum is odd.

Result. For $n=32v_1+34$, $v_1 \geq 0$, we get a sum $S(v_1)$ for which we have to decide depending on v_1 which subcase of *Case 2.2* applies.

12 The Behavior of c_k

In order to clarify the situation in the case where $n=32v_1+10$, $v_1 \geq 0$, and $n=64v_2+18$, $v_2 \geq 0$, we have to determine how c_k behaves. We use the facts that $c_{4k+1} \equiv c_{2k} \equiv c_k$ and $c_{4k+3} \equiv 0$.

Let an integer $v>0$ of the form $4k+3$, $4k+1$ or $2k$ be given.

Let $V=v$. Therefore $c_V \equiv c_v$.

Step I:

if $v=4k+3$, we know that $c_V \equiv c_v \equiv 0$: proceed to *Step II*

if $v=4k+1$, $v>1$, we know that $c_V \equiv c_v \equiv c_{4k+1} \equiv c_k$ and $k<4k+1$: redefine $v:=k$ and proceed to *Step II*

if $v=2k$, $v>1$, we know that $c_V \equiv c_v \equiv c_{2k} \equiv c_k$ and $k<2k$: redefine $v:=k$ and proceed to *Step II*

if $v=1$, we know that $c_V \equiv c_v \equiv 1$: proceed to *Step II*

Step II:

continue *Step I* until $v=4k+3$ for some k, that is $c_V \equiv 0$, or $v=1$, that is $c_V \equiv 1$.

It is clear that by executing the process described above, v must become either 1, or of form $4k+3$ in a finite number of steps.

Since only those c_v, for which v gets to 1 are odd, we aim for determining the form of these v.

Define:

- unary operation m on an integer k: $m(k)=2 \cdot k$
- unary operation M on an integer k: $M(k)=4 \cdot k+1$

It follows form the process described above that all integers v for which $c_v \equiv 1$ can be created by any number and any order of operations m and M on 1, e.g.

$v = M \circ M \circ m \circ m \circ m \circ M \circ m \circ m(1) =$
$M \circ M \circ m \circ m \circ m \circ M \circ m(2) =$
$M \circ M \circ m \circ m \circ m \circ M(4) =$
$M \circ M \circ m \circ m \circ m(17) =$
$M \circ M \circ m \circ m(34) =$
$M \circ M \circ m(68) =$
$M \circ M(136) =$
$M(545) = 2181$

$\Rightarrow c_{2181} \equiv 1$, thus, for example, for $n = 32 \cdot 2181 + 10 = 69802$ the number of genus one c-trees on 69802 points is not divisible by 69802, only by $69802 \div 2 = 34901$.

It is easy to see, that all v obtained by finitely many operations M and m on 1 are of the following form:

$1 \cdot 2^{l_0} + 4 \cdot 2^{l_1} + 4^2 \cdot 2^{l_2} + 4^3 \cdot 2^{l_3} + \cdots + 4^i \cdot 2^{l_i} + \cdots + 4^f \cdot 2^{l_f}$, where
$0 \leq l_0 \leq l_1 \leq l_2 \leq l_3 \leq \cdots \leq l_i \leq \cdots \leq l_f$, $l_i \in \mathbb{N}$, $i \in \{0, 1, 2, \ldots, f\}$.

If v satisfies the previous or if $v = 0$ ($c_0 \equiv 1$), we will say v is a ***negligent number*** (or nn for short).

Therefore, we have proven that $(c_v \equiv 1) \Leftrightarrow (v \text{ is } nn)$. For all other v, $c_v \equiv 0$. From this we have that:

Results. For $n = 32v_1 + 10$, $v_1 \geq 0$, where v_1 is a nn l_s is odd, that is $f(n)$ is not divisible by n but is divisible by $\frac{n}{2}$. For $n = 32v_1 + 10$, $v_1 \geq 0$, where v_1 is not a nn l_s is even, that is $f(n)$ is divisible by n.

For $n = 64v_2 + 18$, $v_2 \geq 0$, where v_2 is a nn l_s is odd, that is $f(n)$ is not divisible by n but is divisible by $\frac{n}{2}$. For $n = 64v_2 + 18$, $v_2 \geq 0$, where v_2 is not a nn l_s is even, that is $f(n)$ is divisible by n.

13 What If $n = 32k + 34$?

In Section 11 we have seen that for $n = 32v_1 + 34$, $v_1 \geq 0$, we get a sum $S(v_1)$ for which we have to decide depending on v_1 which subcase of *Case 2.2* applies in order to determine the parity of l_s. We decide this now.

We have that $\frac{n-2}{4} - 1 = s = 4v + 3$ and $v = 2v_1 + 1$, $v_1 \geq 0$. There are two possibilities:

First Possibility:

$v = 2v_1 + 1$, $v_1 = 2v_2 + 1$, $v_2 = 2v_3 + 1, \ldots$, $v_{k-1} = 2v_k + 1$, $v_k = 2v_{k+1}$, $v_{k+1} = 2v_{k+2} + 1$, $k \geq 1$. It is easy to see, that $v = 1 + 2 + 2^2 + \cdots + 2^{k-2} + 2^{k-1} + 2^k \cdot v_k = 2^k - 1 + 2^k \cdot v_k$, thus $s = 16 \cdot 2^k \cdot v_{k+2} + 3 \cdot 4 \cdot 2^k - 1$, that is $n = 64 \cdot 2^k \cdot v_{k+2} + 3 \cdot 16 \cdot 2^k + 2$, $k \geq 1$. In this case l_s is even.

Second Possibility:

$v = 2v_1 + 1$, $v_1 = 2v_2 + 1$, $v_2 = 2v_3 + 1, \ldots$, $v_{k-1} = 2v_k + 1$, $v_k = 2v_{k+1}$, $v_{k+1} = 2v_{k+2}$, $k \geq 1$, $v_{k+2} > 0$. It is easy to see, that $v = 1 + 2 + 2^2 + \cdots + 2^{k-2} + 2^{k-1} + 2^k \cdot v_k = 2^k - 1 + 2^k \cdot v_k$, thus $s = 16 \cdot 2^k \cdot v_{k+2} + 4 \cdot 2^k - 1$, that is $n = 64 \cdot 2^k \cdot v_{k+2} + 16 \cdot 2^k + 2$, $v_{k+2} > 0$. In this case

$l_s \equiv c_{v_{k+2}}$, and including the case when $v_{k+2} = 0$ we get that if v_{k+2} is a nn then l_s is odd, otherwise l_s is even.

Now that we have obtained the parity of l_s in every case, we can state the main theorem of our work, namely when $f(n)$ is divisible by n and when it is divisible only by $\frac{n}{2}$.

14 The Theorem

The number of genus one l-c-trees on n points is divisible by n or $\frac{n}{2}$ for all integers $n > 3$.

The number of genus one l-c-trees on n points ($n > 3$) is not divisible by n, but is divisible by $\frac{n}{2}$ if and only if n is of form:

$32v_1 + 10$, $v_1 \geq 0$, and v_1 is a negligent number; $64v_2 + 18$, $v_2 \geq 0$, and v_2 is a negligent number; $64 \cdot 2^k \cdot v_{k+2} + 16 \cdot 2^k + 2$,$k \geq 1$, and v_{k+2} is a negligent number.

For all other n the number of genus one l-c-trees on n points ($n > 3$) is divisible by n.

Proof. The proof is given in sections 1 to 13. □

Acknowledgments

I am grateful to Professor Richard P. Stanley for drawing my attention to this problem and for his advice and encouragement throughout this research. I would also like to thank Thomas Lam who read and commented on drafts of this paper. This research was supported by the UROP Office and the Lord Foundation.

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