Compact hyperbolic Coxeter n-polytopes with $n+3$ facets

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Abstract

We use methods of combinatorics of polytopes together with geometrical and computational ones to obtain the complete list of compact hyperbolic Coxeter n-polytopes with $n+3$ facets, $4 \leq n \leq 7$. Combined with results of Esselmann this gives the classification of all compact hyperbolic Coxeter n-polytopes with $n+3$ facets, $n \geq 4$. Polytopes in dimensions 2 and 3 were classified by Poincaré and Andreev.

1 Introduction

A polytope in the hyperbolic space $\mathbb{H}^n$ is called a *Coxeter polytope* if its dihedral angles are all integer submultiples of π. Any Coxeter polytope P is a fundamental domain of the discrete group generated by reflections in the facets of P.

There is no complete classification of compact hyperbolic Coxeter polytopes. Vinberg [V1] proved there are no such polytopes in $\mathbb{H}^n, n \geq 30$. Examples are known only for $n \leq 8$ (see [B1], [B2]).

In dimensions 2 and 3 compact Coxeter polytopes were completely classified by Poincaré [P] and Andreev [A]. Compact polytopes of the simplest combinatorial type, the simplices, were classified by Lannér [L]. Kaplinskaja [K] (see also [V2]) listed simplicial prisms, Esselmann [E2] classified the remaining compact n-polytopes with $n+2$ facets.

In the paper [ImH] Im Hof classified polytopes that can be described by Napier cycles. These polytopes have at most $n+3$ facets. Concerning polytopes with $n+3$ facets, Esselmann proved the following theorem ([E1, Th. 5.1]):

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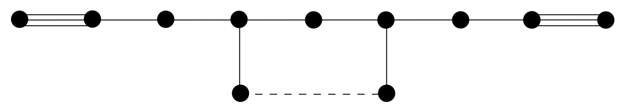
Let P be a compact hyperbolic Coxeter n-polytope bounded by $n+3$ facets. Then $n \leq 8$; if $n = 8$, then P is the polytope found by Bugaenko in [B2]. This polytope has the following Coxeter diagram:



In this paper, we expand the technique derived by Esselmann in [E1] and [E2] to complete the classification of compact hyperbolic Coxeter n-polytopes with $n+3$ facets. The aim is to prove the following theorem:

Main Theorem. *Tables 4.8–4.11 contain all Coxeter diagrams of compact hyperbolic Coxeter n-polytopes with $n+3$ facets for $n \geq 4$.*

The paper is organized as follows. In Section 2 we recall basic definitions and list some well-known properties of hyperbolic Coxeter polytopes. We also emphasize the connection between combinatorics (Gale diagram) and metric properties (Coxeter diagram) of hyperbolic Coxeter polytope. In Section 3 we recall some technical tools from [V1] and [E1] concerning Coxeter diagrams and Gale diagrams, and introduce notation suitable for investigating of large number of diagrams. Section 4 is devoted to the proof of the main theorem. The most part of the proof is computational: we restrict the number of Coxeter diagrams in consideration, and use a computer check after that. The bulk is to find an upper bound for the number of diagrams, and then to reduce the number to make the computation short enough.

This paper is a completely rewritten part of my Ph.D. thesis (2004) with several errors corrected. I am grateful to my advisor Prof. E. B. Vinberg for his help. I am also grateful to Prof. R. Kellerhals who brought the papers of F. Esselmann and L. Schlettwein to my attention, and to the referee for useful suggestions.

2 Hyperbolic Coxeter polytopes and Gale diagrams

In this section we list essential facts concerning hyperbolic Coxeter polytopes, Gale diagrams of simple polytopes, and Coxeter diagrams we use in this paper. Proofs, details and definitions in general case may be found in [G] and [V2]. In the last part of this section we present the main tools used for the proof of the main theorem.

We write *n-polytope* instead of "n-dimensional polytope" for short. By *facet* we mean a face of codimension one.

2.1 Gale diagrams

An n-polytope is called *simple* if any its k-face belongs to exactly $n-k$ facets. Proposition 2.2 implies that any compact hyperbolic Coxeter polytope is simple. From now on we consider simple polytopes only.

Every combinatorial type of simple n-polytope with d facets can be represented by its *Gale diagram* G. This consists of d points $a_1, \dots, a_d$ on the $(d-n-2)$-dimensional unit sphere in $\mathbb{R}^{d-n-1}$ centered at the origin.

The combinatorial type of a simple convex polytope can be read off from the Gale diagram in the following way. Each point a_i corresponds to the facet f_i of P. For any subset J of the set of facets of P the intersection of facets $\{f_j \,|\, j \in J\}$ is a face of P if and only if the origin is contained in the interior of $\mathrm{conv}\{a_j \,|\, j \notin J\}$.

The points $a_1, \dots, a_d \in \mathbb{S}^{d-n-2}$ compose a Gale diagram of some n-dimensional polytope P with d facets if and only if every open half-space H^+ in $\mathbb{R}^{d-n-1}$ bounded by a hyperplane H through the origin contains at least two of the points $a_1, \dots, a_d$.

We should notice that the definition of Gale diagram introduced above is "dual" to the standard one (see, for example, [G]): usually Gale diagram is defined in terms of vertices of polytope instead of facets. Notice also that the definition above concerns simple polytopes only, and it takes simplices out of consideration: usually one means the origin of $\mathbb{R}^1$ with multiplicity $n+1$ by the Gale diagram of an n-simplex, however we exclude the origin since we consider simple polytopes only, and the origin is not contained in G for any simple polytope except simplex.

We say that two Gale diagrams G and G' are *isomorphic* if the corresponding polytopes are combinatorially equivalent.

If $d = n+3$ then the Gale diagram of P is two-dimensional, i.e. nodes a_i of the diagram lie on the unit circle.

A *standard Gale diagram* of simple n-polytope with $n+3$ facets consists of vertices $v_1, \dots, v_k$ of regular k-gon (k is odd) in $\mathbb{R}^2$ centered at the origin which are labeled according to the following rules:

1) Each label is a positive integer, the sum of labels equals $n+3$.

2) The vertices that lie in any open half-space bounded by a line through the origin have labels whose sum is at least two.

Each point v_i with label μ_i corresponds to μ_i facets $f_{i,1}, \dots, f_{i,\mu_i}$ of P. For any subset J of the set of facets of P the intersection of facets $\{f_{j,\gamma} \,|\, (j,\gamma) \in J\}$ is a face of P if and only if the origin is contained in the interior of $\mathrm{conv}\{v_j \,|\, (j,\gamma) \notin J\}$.

It is easy to check (see, for example, [G, Sec. 6.3]) that any two-dimensional Gale diagram is isomorphic to some standard diagram. Two simple n-polytopes with $n+3$ facets are combinatorially equivalent if and only if their standard Gale diagrams are congruent.

2.2 Coxeter diagrams

Any Coxeter polytope P can be represented by its Coxeter diagram.

An *abstract Coxeter diagram* is a one-dimensional simplicial complex with weighted edges, where weights are either of the type $\cos\frac{\pi}{m}$ for some integer $m \geq 3$ or positive real numbers no less than one. We can suppress the weights but indicate the same information by labeling the edges of a Coxeter diagram in the following way:

• if the weight equals $\cos\frac{\pi}{m}$ then the nodes are joined by either an $(m-2)$-fold edge or a simple edge labeled by m;
• if the weight equals one then the nodes are joined by a bold edge;
• if the weight is greater than one then the nodes are joined by a dotted edge labeled by its weight.

A *subdiagram* of Coxeter diagram is a subcomplex with the same as in Σ. The *order* $|\Sigma|$ is the number of vertices of the diagram Σ.

If Σ_1 and Σ_2 are subdiagrams of a Coxeter diagram Σ, we denote by $\langle\Sigma_1,\Sigma_2\rangle$ a subdiagram of Σ spanned by all nodes of Σ_1 and Σ_2. We say that a node of Σ *attaches* to a subdiagram $\Sigma_1\subset\Sigma$ if it is joined with some nodes of Σ_1 by edges of any type.

Let Σ be a diagram with d nodes u_1,...,u_d. Define a symmetric $d\times d$ matrix $\mathrm{Gr}(\Sigma)$ in the following way: $g_{ii}=1$; if two nodes u_i and u_j are adjacent then g_{ij} equals negative weight of the edge u_iu_j; if two nodes u_i and u_j are not adjacent then g_{ij} equals zero.

By signature and determinant of diagram Σ we mean the signature and the determinant of the matrix $\mathrm{Gr}(\Sigma)$.

An abstract Coxeter diagram Σ is called *elliptic* if the matrix $\mathrm{Gr}(\Sigma)$ is positive definite. A Coxeter diagram Σ is called *parabolic* if the matrix $\mathrm{Gr}(\Sigma)$ is degenerate, and any subdiagram of Σ is elliptic. Connected elliptic and parabolic diagrams were classified by Coxeter [C]. We represent the list in Table 2.1.

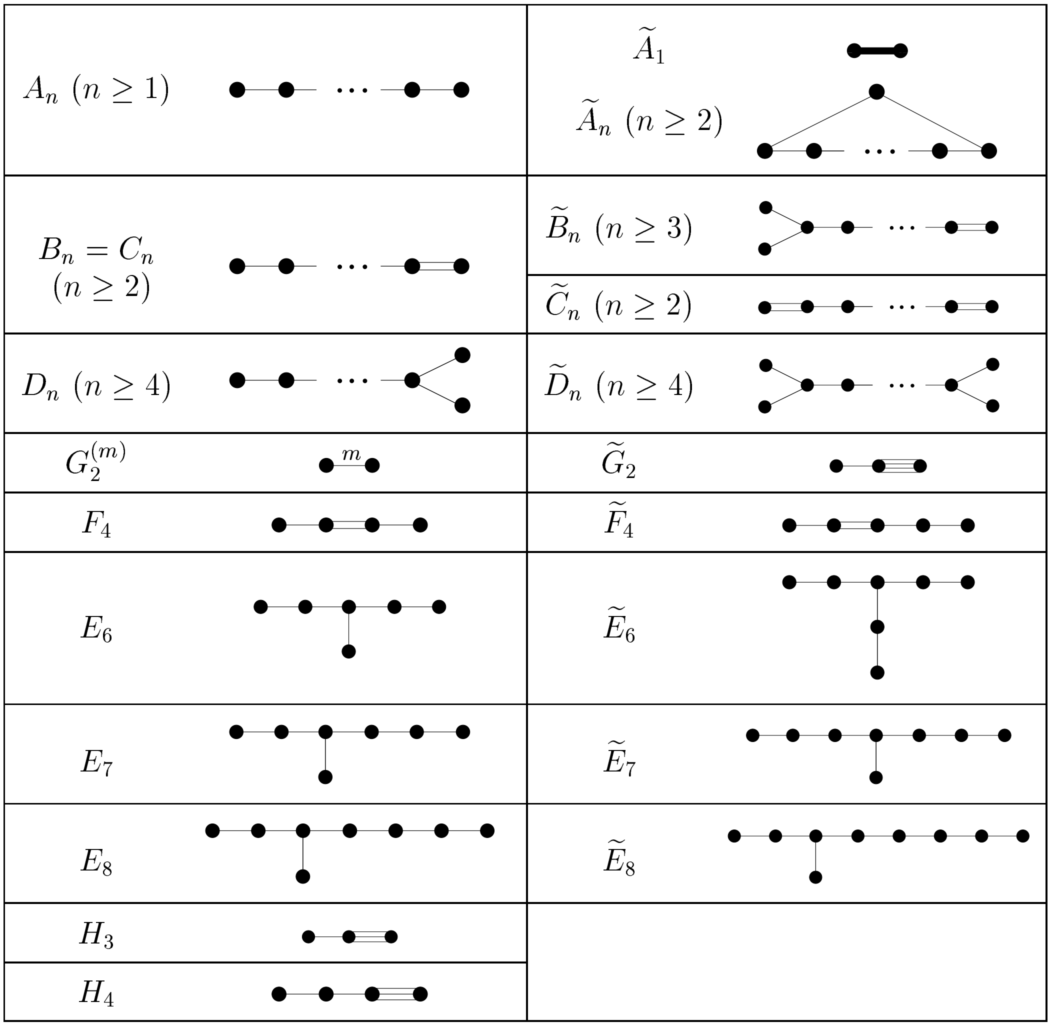
A Coxeter diagram Σ is called a *Lannér diagram* if any subdiagram of Σ is elliptic, and the diagram Σ is neither elliptic nor parabolic. Lannér diagrams were classified by Lannér [L]. We represent the list in Table 2.2. A diagram Σ is *superhyperbolic* if its negative inertia index is greater than 1.

By a *simple* (resp., *multiple*) edge of Coxeter diagram we mean an $(m-2)$-fold edge where m is equal to (resp., greater than) 3. The number $m-2$ is called the *multiplicity* of a multiple edge. Edges of multiplicity greater than 3 we call *multi-multiple* edges. If an edge u_iu_j has multiplicity $m-2$ (i.e. the corresponding facets form an angle $\frac{\pi}{m}$), we write $[u_i,u_j]=m$.

A *Coxeter diagram* $\Sigma(P)$ *of Coxeter polytope* P is a Coxeter diagram whose matrix $\mathrm{Gr}(\Sigma)$ coincides with Gram matrix of outer unit normals to the facets of P (referring to the standard model of hyperbolic n-space in $\mathbb{R}^{n,1}$). In other words, nodes of Coxeter diagram correspond to facets of P. Two nodes are joined by either an $(m-2)$-fold edge or an m-labeled edge if the corresponding dihedral angle equals $\frac{\pi}{m}$. If the corresponding facets are parallel the nodes are joined by a bold edge, and if they diverge then the nodes are joined by a dotted edge (which may be labeled by hyperbolic cosine of distance between the hyperplanes containing these facets).

If $\Sigma(P)$ is the Coxeter diagram of P then nodes of $\Sigma(P)$ are in one-to-one correspondence with elements of the set $I=\{1,\dots,d\}$. For any subset $J\subset I$ denote by $\Sigma(P)_J$ the subdiagram of $\Sigma(P)$ that consists of nodes corresponding to elements of J.

Table 2.1: Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively.



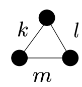
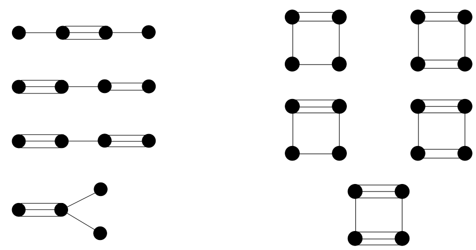
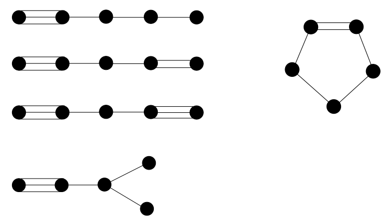
2.3 Hyperbolic Coxeter polytopes

In this section by polytope we mean a (probably non-compact) intersection of closed half-spaces.

Proposition 2.1 ([V2], Th. 2.1). *Let* $\mathrm{Gr} = (g_{ij})$ *be indecomposable symmetric matrix of signature* $(n, 1)$*, where* $g_{ii} = 1$ *and* $g_{ij} \leq 0$ *if* $i \neq j$*. Then there exists a unique (up to isometry of* $\mathbb{H}^n$*) convex polytope* $P \subset \mathbb{H}^n$ *whose Gram matrix coincides with* Gr*.*

Let Gr be the Gram matrix of the polytope P, and let $J \subset I$ be a subset of the set of facets of P. Denote by Gr_J the Gram matrix of vectors $\{e_i \mid i \in J\}$, where e_i is outward unit normal to the facet f_i of P (i.e. $\mathrm{Gr}_J = \mathrm{Gr}(\Sigma(P)_J)$). Denote by $|J|$ the number of elements of J.

Table 2.2: Lannér diagrams.

order	diagrams
2	
3	 $(2 \leq k, l, m < \infty, \frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1)$
4	
5	

Proposition 2.2 ([V2], Th. 3.1). *Let $P \subset \mathbb{H}^n$ be an acute-angled polytope with Gram matrix* Gr*, and let J be a subset of the set of facets of P. The set*

$$q = P \cap \bigcap_{i \in J} f_i$$

is a face of P if and only if the matrix Gr_J *is positive definite. Dimension of q is equal to $n - |J|$.*

Notice that Prop. 2.2 implies that the combinatorics of P is completely determined by the Coxeter diagram $\Sigma(P)$.

Let A be a symmetric matrix whose non-diagonal elements are non-positive. A is called *indecomposable* if it cannot be transformed to a block-diagonal matrix via simultaneous permutations of columns and rows. We say A to be *parabolic* if any indecomposable component of A is positive semidefinite and degenerate. For example, a matrix $\mathrm{Gr}(\Sigma)$ for any parabolic diagram Σ is parabolic.

Proposition 2.3 ([V2], cor. of Th. 4.1, Prop. 3.2 and Th. 3.2). *Let $P \subset \mathbb{H}^n$ be a compact Coxeter polytope, and let* Gr *be its Gram matrix. Then for any $J \subset I$ the matrix* Gr_J *is not parabolic.*

Corollary 2.1 reformulates Prop. 2.3 in terms of Coxeter diagrams.

Corollary 2.1. *Let $P \subset \mathbb{H}^n$ be a compact Coxeter polytope, and let Σ be its Coxeter matrix. Then any non-elliptic subdiagram of Σ contains a Lannér subdiagram.*

Proposition 2.4 ([V2], Prop. 4.2). *A polytope P in $\mathbb{H}^n$ is compact if and only if it is combinatorially equivalent to some compact convex n-polytope.*

The main result of paper [FT] claims that if P is a compact hyperbolic Coxeter n-polytope having no pair of disjoint facets, then P is either a simplex or one of the seven polytopes with $n+2$ facets described in [E1]. As a corollary, we obtain the following proposition.

Proposition 2.5. *Let $P \subset \mathbb{H}^n$ be a compact Coxeter polytope with at least $n+3$ facets. Then P has a pair of disjoint facets.*

2.4 Coxeter diagrams, Gale diagrams, and missing faces

Now, for any compact hyperbolic Coxeter polytope we have two diagrams which carry the complete information about its combinatorics, namely Gale diagram and Coxeter diagram. The interplay between them is described by the following lemma, which is a reformulation of results listed in Section 2.3 in terms of Coxeter diagrams and Gale diagrams.

Lemma 2.1. *A Coxeter diagram Σ with nodes $\{u_i \,|\, i=1,\dots,d\}$ is a Coxeter diagram of some compact hyperbolic Coxeter n-polytope with d facets if and only if the following two conditions hold:*

1) Σ is of signature $(n,1,d-n-1)$;

2) there exists a $(d-n-1)$-dimensional Gale diagram with nodes $\{v_i \,|\, i=1,\dots,d\}$ and one-to-one map $\psi : \{u_i \,|\, i=1,\dots,d\} \to \{v_i \,|\, i=1,\dots,d\}$ such that for any $J \subset \{1,\dots,d\}$ the subdiagram Σ_J of Σ is elliptic if and only if the origin is contained in the interior of $\operatorname{conv}\{\psi(v_i) \,|\, i \notin J\}$.

Let P be a simple polytope. The facets $f_1,\dots,f_m$ of P compose a *missing face* of P if $\bigcap_{i=1}^{m} f_i = \emptyset$ but any proper subset of $\{f_1,\dots,f_m\}$ has a non-empty intersection.

Proposition 2.6 ([FT], Lemma 2). *Let P be a simple d-polytope with $d+k$ facets $\{f_i\}$, let $G=\{a_i\} \subset \mathbb{S}^{k-2}$ be a Gale diagram of P, and let $I \subset \{1,\dots,d+k\}$. Then the set $M_I=\{f_i \,|\, i \in I\}$ is a missing face of P if and only if the following two conditions hold:*

(1) there exists a hyperplane H through the origin separating the set $\widehat{M_I}=\{a_i \,|\, i \in I\}$ from the remaining points of G;

(2) for any proper subset $J \subset I$ no hyperplane through the origin separates the set $\widehat{M_J}=\{a_i \,|\, i \in J\}$ from the remaining points of G.

Remark. Suppose that P is a compact hyperbolic Coxeter polytope. The definition of missing face (together with Cor. 2.1) implies that for any Lannér subdiagram $L \subset \Sigma(P)$ the facets corresponding to L compose a missing face of P, and any missing face of P corresponds to some Lannér diagram in $\Sigma(P)$.

Now consider a compact hyperbolic Coxeter n-polytope P with $n+3$ facets with standard Gale diagram G (which is a k-gon, k is odd) and Coxeter diagram Σ. Denote by $\Sigma_{i,j}$ a subdiagram of Σ corresponding to $j-i+1 \pmod k$ consecutive nodes $a_i, \dots, a_j$ of G (in the sense of Lemma 2.1). If $i=j$, denote $\Sigma_{i,i}$ by Σ_i.

The following lemma is an immediate corollary of Prop. 2.6.

Lemma 2.2. *For any $i \in \{0, \dots, k-1\}$ a diagram $\Sigma_{i+1, i+\frac{k-1}{2}}$ is a Lannér diagram. All Lannér diagrams contained in Σ are of this type.*

It is easy to see that the collection of missing faces completely determines the combinatorics of P. In view of Lemma 2.2 and the remark above, this means that in Lemma 2.1 for given Coxeter diagram we need to check the signature and correspondence of Lannér diagrams to missing faces of some Gale diagram.

Example. Suppose that there exists a compact hyperbolic Coxeter polytope P with standard Gale diagram G shown in Fig. 2.1(a). What can we say about Coxeter diagram $\Sigma = \Sigma(P)$?

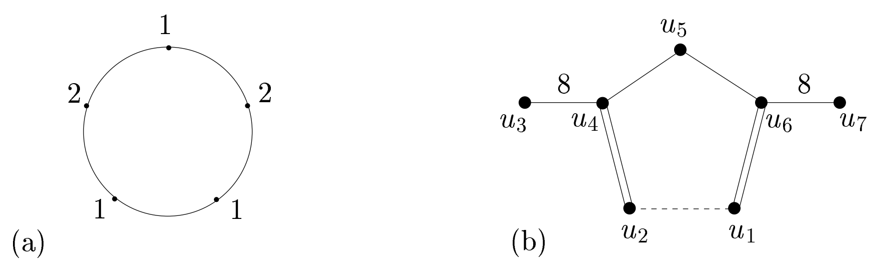


Figure 2.1: (a) A standard Gale diagram G and (b) a Coxeter diagram of one of polytopes with Gale diagram G

The sum of labels of nodes of Gale diagram G is equal to 7, so P is a 4-polytope with 7 facets. Thus, Σ is spanned by nodes $u_1, \dots, u_7$, and its signature equals $(4,1,2)$. Further, G is a pentagon. By Lemma 2.2, Σ contains exactly 5 Lannér diagrams, namely $\langle u_1, u_2 \rangle$, $\langle u_2, u_3, u_4 \rangle$, $\langle u_3, u_4, u_5 \rangle$, $\langle u_5, u_6, u_7 \rangle$, and $\langle u_6, u_7, u_1 \rangle$.

Now consider the Coxeter diagram Σ shown in Fig. 2.1(b). Assigning label $1+\sqrt{2}$ to the dotted edge of Σ, we obtain a diagram of signature $(4,1,2)$ (this may be shown by direct calculation). Therefore, there exist 7 vectors in $\mathbb{H}^4$ with Gram matrix $\mathrm{Gr}(\Sigma)$. It is easy to see that Σ contains exactly 5 Lannér diagrams described above. Thus, Σ is a Coxeter diagram of some compact 4-polytope with Gale diagram G.

Of course, Σ is just an *example* of a Coxeter diagram satisfying both conditions of Lemma 2.1 with respect to given Gale diagram G. In the next two sections we will show how to list *all* compact hyperbolic Coxeter polytopes of given combinatorial type.

3 Technical tools

From now on by polytope we mean a compact hyperbolic Coxeter n-polytope with $n+3$ facets, and we deal with standard Gale diagrams only.

3.1 Admissible Gale diagrams

Suppose that there exists a compact hyperbolic Coxeter polytope P with k-angled Gale diagram G. Since the maximal order of Lannér diagram equals five, Lemma 2.2 implies that the sum of labels of $\frac{k-1}{2}$ consecutive nodes of Gale diagram does not exceed five. On the other hand, by Lemma 2.5, P has a missing face of order two. This is possible in two cases only: either G is a pentagon with two neighboring vertices labeled by 1, or G is a triangle one of whose vertices is labeled by 2 (see Prop. 2.6). Table 3.1 contains all Gale diagrams satisfying one of two conditions above with at least 7 and at most 10 vertices, i.e. Gale diagrams that may correspond to compact hyperbolic Coxeter n-polytopes with $n+3$ facets for $4 \le n \le 7$.

3.2 Admissible arcs

Let P be an n-polytope with $n+3$ facets and let G be its k-angled Gale diagram. By Lemma 2.2, for any $i \in \{0,\dots,k-1\}$ the diagram $\Sigma_{i+1,i+\frac{k-1}{2}}$ is a Lannér diagram. Denote by

$$\lfloor x_1,\dots,x_l \rfloor_{\frac{k-1}{2}}, \quad l \le k$$

an arc of length l of G that consists of l consecutive nodes with labels $x_1,\dots,x_l$. By writing $J = \lfloor x_1,\dots,x_l \rfloor_{\frac{k-1}{2}}$ we mean that J is the set of facets of P corresponding to these nodes of G. The index $\frac{k-1}{2}$ means that for any $\frac{k-1}{2}$ consecutive nodes of the arc (i.e. for any arc $I = \left\lfloor x_{i+1},\dots,x_{i+\frac{k-1}{2}} \right\rfloor_{\frac{k-1}{2}}$) the subdiagram Σ_I of $\Sigma(P)$ corresponding to these nodes is a Lannér diagram (i.e. I is a missing face of P).

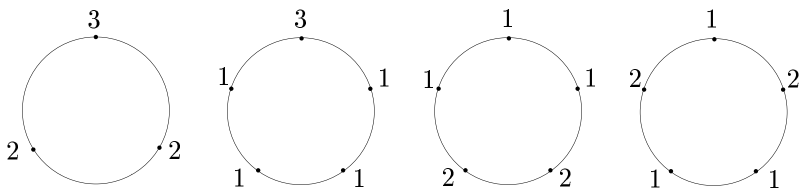
By Cor. 2.1, any diagram $\Sigma_J \subset \Sigma(P)$ corresponding to an arc $J = \lfloor x_1,\dots,x_l \rfloor_{\frac{k-1}{2}}$ satisfies the following property: any subdiagram of Σ_J containing no Lannér diagram is elliptic. Clearly, any subdiagram of $\Sigma(P)$ containing at least one Lannér diagram is of signature $(k,1)$ for some $k \le n$. As it is shown in [E1], for some arcs J there exist a few corresponding diagrams Σ_J only. In the following lemma, we recall some results of Esselmann [E1] and prove similar facts concerning some arcs of Gale diagrams listed in Table 3.1. This will help us to restrict the number of Coxeter diagrams that may correspond to some of Gale diagrams listed in Table 3.1.

Lemma 3.1. *The diagrams presented in the middle column of Table 3.2 are the only diagrams that may correspond to arcs listed in the left column.*

Proof. At first, notice that for any J as above (i.e. J consists of several consecutive nodes of Gale diagram) the diagram Σ_J must be connected. This follows from the fact that any Lannér diagram is connected, and that Σ_J is not superhyperbolic.

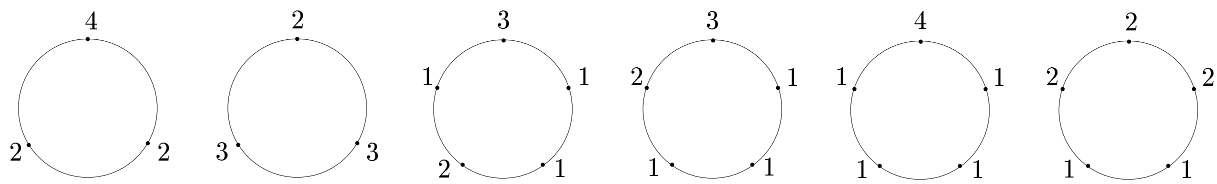
Table 3.1: Gale diagrams that may correspond to compact Coxeter polytopes (see Section 3.1)

<u>$n = 4$</u>



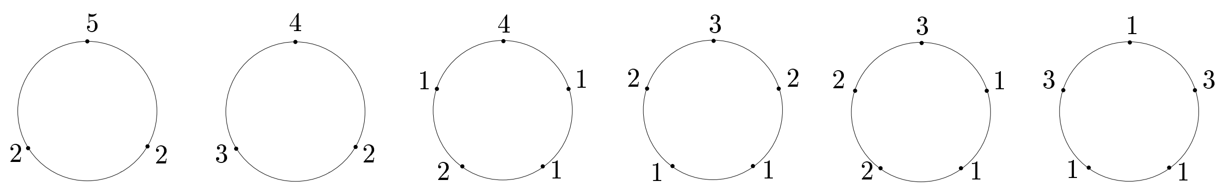
G_{232} G_{11311} G_{21112} G_{12121}

<u>$n = 5$</u>



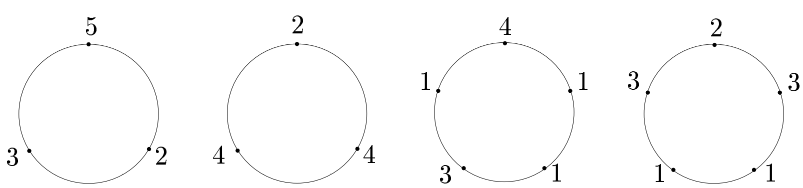
G_{242} G_{323} G_{21311} G_{12311} G_{11411} G_{12221}

<u>$n = 6$</u>



G_{252} G_{342} G_{21411} G_{12321} G_{22311} G_{13131}

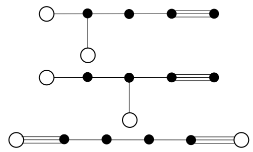
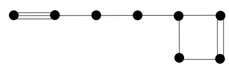
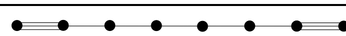
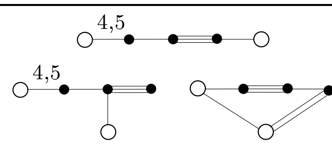
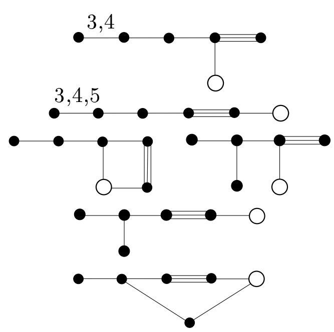
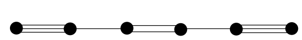
<u>$n = 7$</u>



G_{352} G_{424} G_{31411} G_{13231}

Now we restrict our considerations to items 8–11 only. For none of these J the diagram Σ_J contains a Lannér diagram of order 2 or 3. Since Σ_J is connected and does not contain parabolic subdiagrams, this implies that Σ_J does not contain neither dotted nor multi-multiple edges. Thus, we are left with finitely many possibilities only, that allows us to use a computer check: there are several (from 5 to 7) nodes, some of them joined by edges of multiplicity at most 3. We only need to check all possible diagrams for the number of

Table 3.2: Possible diagrams Σ_J for some arcs J. White nodes correspond to endpoints of arcs having multiplicity one

	J	all possibilities for Σ_J	reference (if any)
1	$\lfloor x,y \rfloor_1$, $x \geq 4, y \geq 3$	$\varnothing$	[E1], Lemma 4.7
2	$\lfloor 1,4,1 \rfloor_2$		[E1], Lemma 5.3
3	$\lfloor 3,2,2 \rfloor_2$	$\varnothing$	[E1], Lemma 5.7
4	$\lfloor 4,1,3 \rfloor_2$		[E1], Lemma 5.9
5	$\lfloor 3,1,4,1 \rfloor_2$	$\varnothing$	[E1], Folgerung 5.10
6	$\lfloor 2,3,2 \rfloor_2$		[E1], Lemma 5.12
7	$\lfloor 3,2,3 \rfloor_2$		[E1], Lemma 5.12
8	$\lfloor 1,3,1 \rfloor_2$		
9	$\lfloor 1,3,2 \rfloor_2$		
10	$\lfloor 2,2,2 \rfloor_2$		
11	$\lfloor 3,1,3 \rfloor_2$	$\varnothing$	

Lannér diagrams of all orders and for parabolic subdiagrams. Namely, in items 8, 10 and 11 we look for diagrams of order 5, 6 and 7 containing exactly 2 Lannér subdiagrams of order 4 (and containing neither other Lannér diagrams nor parabolic subdiagrams), and in item 9 we look for diagrams of order 6 containing exactly one Lannér subdiagram of order 4 and exactly one Lannér diagram of order 5. Notice also that we do not need to check the signature of obtained diagrams: all them are certainly non-elliptic, and since any of them contains exactly two Lannér diagrams which have at least one node in common, by excluding this node we obtain an elliptic diagram.

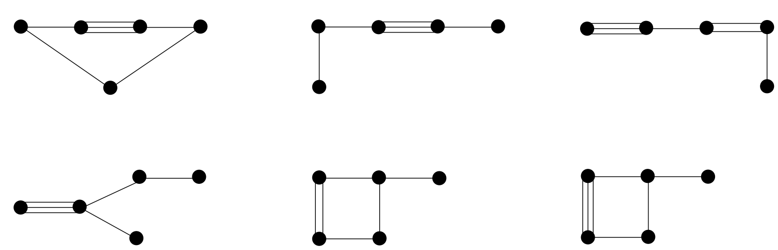
However, the computation described above is really huge. In what follows we describe case-by-case how to reduce these computations to a few minutes of hand-calculations.

• **Item 8** ($J = \lfloor 1,3,1 \rfloor_2$). We may consider Σ_J as a Lannér diagram L of order 4 together with one vertex attached to L to compose a unique additional Lannér diagram which should be of order 4, too. There are 9 possibilities for L only (Table 2.2).

• **Item 9** ($J = \lfloor 1,3,2 \rfloor_2$). The considerations follow the preceding ones, but we take as L a Lannér diagram of order 5. Again, there are few possibilities for L only (namely five: see Table 2.2).

• **Item 10** ($J = \lfloor 2,2,2 \rfloor_2$). Again, Σ_J contains a Lannér diagram L of order 4. One of the two remaining nodes of Σ_J must be attached to L. Denote this node by v. The diagram $\langle L, v \rangle \subset \Sigma_J$ consists of five nodes and contains a unique Lannér diagram which is of order 4. All such diagrams are listed in [E1, Lemma 3.8] (see the first two rows of Tabelle 3, the case $|\mathcal{N}_F| = 1$, $|\mathcal{L}_F| = 4$). We reproduce this list in Table 3.3.

Table 3.3: One of these diagrams should be contained in Σ_J for $J = \lfloor 2,2,2 \rfloor_2$



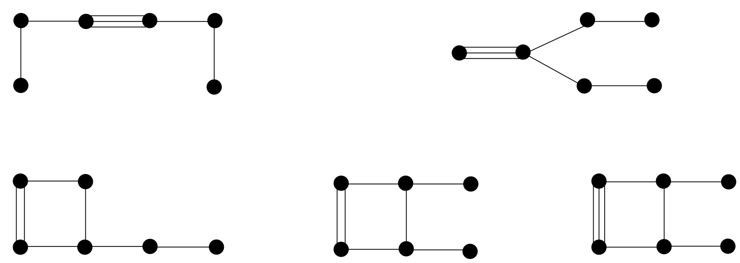
One can see that there are six possibilities only. Now to each of them we attach the remaining node to compose a unique new Lannér diagram which should be of order 4.

• **Item 11** ($J = \lfloor 3,1,3 \rfloor_2$). The considerations are very similar to the preceding case. Σ_J contains a Lannér diagram L of order 4. One of the three remaining nodes of Σ_J must be attached to L. Denote this node by v. Now, one of the two remaining nodes attaches to $\langle L, v \rangle \subset \Sigma_J$. Denote it by u. The diagram $\langle L, v, u \rangle \subset \Sigma_J$ consists of six nodes and contains a unique Lannér diagram which is of order 4. All such diagrams are listed in [E1, Lemma 3.8] (see Tabelle 3, the first two rows of page 27, the case $|\mathcal{N}_F| = 2$, $|\mathcal{L}_F| = 4$). We reproduce this list in Table 3.4.

There are five possibilities only. As above, we attach to each of them the remaining node to compose a unique new Lannér diagram which should be of order 4.

$\square$

Table 3.4: One of these diagrams should be contained in Σ_J for $J = \lfloor 3, 1, 3 \rfloor_2$



3.3 Local determinants

In this section we list some tools derived in [V1] to compute determinants of Coxeter diagrams. We will use them to show that some (infinite) series of Coxeter diagrams are superhyperbolic.

Let Σ be a Coxeter diagram, and let T be a subdiagram of Σ such that $\det(\Sigma \setminus T) \neq 0$. A *local determinant* of Σ on a subdiagram T is

$$\det(\Sigma, T) = \frac{\det \Sigma}{\det(\Sigma \setminus T)}.$$

Proposition 3.1 ([V1], Prop. 12). *If a Coxeter diagram Σ consists of two subdiagrams Σ_1 and Σ_2 having a unique vertex v in common, and no vertex of $\Sigma_1 \setminus v$ attaches to $\Sigma_2 \setminus v$, then*

$$\det(\Sigma, v) = \det(\Sigma_1, v) + \det(\Sigma_2, v) - 1.$$

Proposition 3.2 ([V1], Prop. 13). *If a Coxeter diagram Σ is spanned by two disjoint subdiagrams Σ_1 and Σ_2 joined by a unique edge $v_1 v_2$ of weight a, then*

$$\det(\Sigma, \langle v_1, v_2 \rangle) = \det(\Sigma_1, v_1) \det(\Sigma_2, v_2) - a^2.$$

Denote by $\mathcal{L}_{p,q,r}$ a Lannér diagram of order 3 containing subdiagrams of the dihedral groups $G_2^{(p)}$, $G_2^{(q)}$ and $G_2^{(r)}$. Let v be the vertex of $\mathcal{L}_{p,q,r}$ that does not belong to $G_2^{(r)}$, see Fig. 3.1. Denote by $D\,(p, q, r)$ the local determinant $\det(\mathcal{L}_{p,q,r}, v)$.

It is easy to check (see e.g. [V1]) that

$$D\,(p, q, r) = 1 - \frac{\cos^2(\pi/p) + \cos^2(\pi/q) + 2\cos(\pi/p)\cos(\pi/q)\cos(\pi/r)}{\sin^2(\pi/r)}.$$

Notice that $|D\,(p, q, r)|$ is an increasing function on each of p, q, r tending to infinity while r tends to infinity.

4 Proof of the Main Theorem

The plan of the proof is the following. First, we show that there is only a finite number of combinatorial types (or Gale diagrams) of polytopes we are interested in, and we list

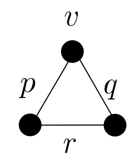


Figure 3.1: Diagram $\mathcal{L}_{p,q,r}$

these Gale diagrams. This was done in Table 3.1. For any Gale diagram from the list we should find all Coxeter polytopes of given combinatorial type. For that, we try to find all Coxeter diagrams with the same structure of Lannér diagrams as the structure of missing faces of the Gale diagram is, and then check the signature. Our task is to be left with finite number of possibilities for each of Gale diagrams, and use a computer after that. Some computations involve a large number of cases, but usually it takes a few minutes of computer's thought. In cases when it is possible to hugely reduce the computations by better estimates we do that, but we follow that by long computations to avoid mistakes.

Lemma 4.1. *The following Gale diagrams do not correspond to any hyperbolic Coxeter polytope:* G_{342}*,* G_{22311}*,* G_{13131}*,* G_{352}*,* G_{424}*,* G_{31411}*.*

Proof. The statement follows from Lemma 3.1. Indeed, the diagram G_{342} contains an arc $J = \lfloor 3, 4 \rfloor_1$. The corresponding Coxeter diagram Σ_J should be of order 7, should contain exactly two Lannér diagrams of order 3 and 4 which do not intersect, and should have negative inertia index at most one. Item 1 of Table 3.2 implies that there is no such Coxeter diagram Σ_J. Thus, G_{342} is not a Gale diagram of any hyperbolic Coxeter polytope.

Similarly, Item 1 of Table 3.2 also implies the statement of the lemma for diagrams G_{352} and G_{424}. Item 3 implies the statement for G_{22311}, Item 11 implies the statement for G_{13131}, and Item 5 implies the statement for the diagram G_{31411}.

□

In what follows we check the 14 remaining Gale diagrams case-by-case. We start from larger dimensions.

4.1 Dimension 7

In dimension 7 we have only one diagram to consider, namely G_{13231}.

Lemma 4.2. *There are no compact hyperbolic Coxeter* 7*-polytopes with* 10 *facets.*

Proof. Suppose that there exists a compact hyperbolic Coxeter polytope P with Gale diagram G_{13231}. This Gale diagram contains an arc $J = \lfloor 3, 2, 3 \rfloor_2$. According to Lemma 3.1 (Item 7 of Table 3.2) and Lemma 2.2, the Coxeter diagram Σ of P consists of a subdiagram Σ_J shown in Fig. 4.1, and two nodes u_9, u_{10} joined by a dotted edge. By Lemma 2.1, the subdiagrams $\langle u_{10}, u_1, u_2, u_3 \rangle$ and $\langle u_6, u_7, u_8, u_9 \rangle$ are Lannér diagrams, and no other

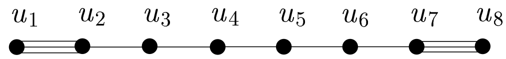
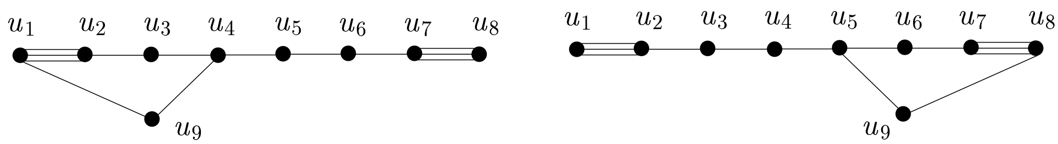


Figure 4.1: A unique diagram Σ_J for $J = \lfloor 3, 2, 3 \rfloor_2$

Lannér subdiagram of Σ contains u_9 or u_{10}. In particular, Σ does not contain Lannér subdiagrams of order 3.

Consider the diagram $\Sigma' = \langle \Sigma_J, u_9 \rangle$. It is connected and contains neither Lannér diagrams of order 2 or 3, nor parabolic diagrams. Therefore, Σ' does not contain neither dotted nor multi-multiple edges. Moreover, by the same reason the node u_9 may attach to nodes u_1, u_2, u_7 and u_8 by simple edges only. It follows that there are finitely many possibilities for the diagram Σ'. Further, since the diagram Σ' defines a collection of 9 vectors in 8-dimensional space $\mathbb{R}^{7,1}$, the determinant of Σ' is equal to zero. A few seconds computer check shows that the only diagrams satisfying conditions listed in this paragraph are the following ones:



However, the left one contains a Lannér diagram $\langle u_2, u_1, u_9, u_4, u_5 \rangle$, and the right one contains a Lannér diagram $\langle u_7, u_8, u_9, u_5, u_4 \rangle$, which is impossible since u_9 does not belong to any Lannér diagram of order 5.

□

4.2 Dimension 6

In dimension 6 we are left with three diagrams, namely G_{252}, G_{21411}, and G_{12321}.

Lemma 4.3. *There is only one compact hyperbolic Coxeter polytope with Gale diagram* G_{12321}. *Its Coxeter diagram is the lowest one shown in Table 4.9.*

Proof. Let P be a compact hyperbolic Coxeter polytope with Gale diagram G_{12321}. This Gale diagram contains an arc $J = \lfloor 2, 3, 2 \rfloor_2$. According to Lemma 3.1 (Item 6 of Table 3.2) and Lemma 2.2, the Coxeter diagram Σ of P consists of a subdiagram Σ_J shown in Fig. 4.2, and two nodes u_8, u_9 joined by a dotted edge. By Lemma 2.1, the subdiagrams $\langle u_8, u_1, u_2 \rangle$

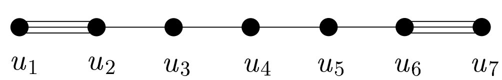


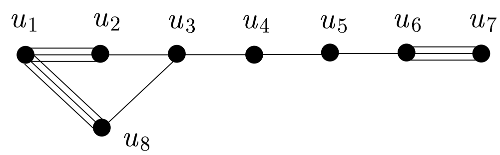
Figure 4.2: A unique diagram Σ_J for $J = \lfloor 2, 3, 2 \rfloor_2$

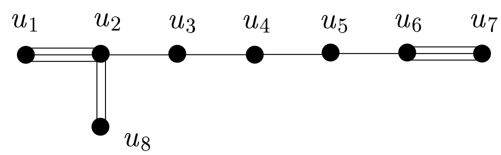
and $\langle u_6, u_7, u_9 \rangle$ are Lannér diagrams, and no other Lannér subdiagram of Σ contains u_8 or u_9. So, we need to check possible multiplicities of edges incident to u_8 and u_9.

Consider the diagram $\Sigma' = \langle \Sigma_J, u_8 \rangle$. It is connected, contains neither Lannér diagrams of order 2 nor parabolic diagrams, and contains a unique Lannér diagram of order 3, namely $\langle u_8, u_1, u_2 \rangle$. Therefore, Σ' does not contain dotted edges, and the only multi-multiple edge that may appear should join u_8 and u_1.

On the other hand, the signature of Σ_J is $(6, 1)$. This implies that the corresponding vectors in $\mathbb{R}^{6,1}$ form a basis, so the multiplicity of the edge $u_1 u_8$ is completely determined by multiplicities of edges joining u_8 with the remaining nodes of Σ_J. Since these edges are neither dotted nor multi-multiple, we are left with a finite number of possibilities only. We may reduce further computations observing that u_8 does not attach to $\langle u_4, u_5, u_6, u_7 \rangle$ (since the diagram $\langle u_8, u_4, u_5, u_6, u_7 \rangle$ should be elliptic), and that multiplicities of edges $u_8 u_2$ and $u_8 u_3$ are at most two and one respectively.

Therefore, we have the following possibilities: $[u_8, u_2] = 2, 3, 4$, and, independently, $[u_8, u_3] = 2, 3$. For each of these six cases we should attach the node u_8 to u_1 satisfying the condition $\det \Sigma' = 0$. An explicit calculation shows that there are two diagrams listed below.





The left one contains a Lannér diagram $\langle u_1, u_8, u_3, u_4, u_5 \rangle$, which is impossible. At the same time, the right one contains exactly Lannér diagrams prescribed by Gale diagram.

Similarly, the node u_9 may be attached to Σ_J in a unique way, i.e. by a unique edge $u_9 u_6$ of multiplicity two. Thus, Σ must look like the diagram shown in Fig. 4.3.

Now we write down the determinant of Σ as a quadratic polynomial of the weight d of the dotted edge. An easy computation shows that

$$\det \Sigma = \frac{\sqrt{5} - 2}{32} \Big(d - (\sqrt{5} + 2) \Big)^2 .$$

The signature of Σ for $d = \sqrt{5} + 2$ is equal to $(6, 1, 2)$, so we obtain that this diagram corresponds to a Coxeter polytope.

□

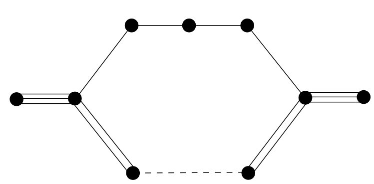
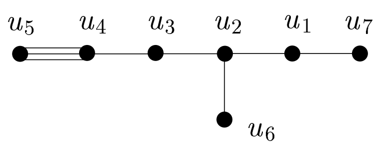


Figure 4.3: Coxeter diagram of a unique Coxeter polytope with Gale diagram G_{12321}

Lemma 4.4. *There are two compact hyperbolic Coxeter polytopes with Gale diagram G_{21411}. Their Coxeter diagrams are shown in the upper row of Table 4.9.*

Proof. Let P be a compact hyperbolic Coxeter polytope with Gale diagram G_{21411}. This Gale diagram contains an arc $J = \lfloor 1, 4, 1 \rfloor_2$. Hence, the Coxeter diagram Σ of P contains a diagram Σ_J which coincides with one of the three diagrams shown in Item 2 of Table 3.2. Further, Σ contains two Lannér diagrams of order 3, one of which (say, L) intersects Σ_J. Denote the common node of that Lannér diagram L and Σ_J by u_1, the 5 remaining nodes of Σ_J by $u_2, \dots, u_6$ (in a way that u_6 is marked white in Table 3.2, i.e. it belongs to only one Lannér diagram of order 5), and denote the two remaining nodes of L by u_7 and u_8. Since L is connected, we may assume that u_7 is joined with u_1. Notice that u_1 is also a node marked white in Table 3.2, elsewhere it belongs to at least three Lannér diagrams in Σ.

Consider the diagram $\Sigma' = \langle \Sigma_J, u_7 \rangle$. It is connected, and all Lannér diagrams contained in Σ' are contained in Σ_J. In particular, Σ' does not contain neither dotted nor multi-multiple edges. Hence, we have only finite number of possibilities for Σ'. More precisely, to each of the three diagrams Σ_J shown in Item 2 of Table 3.2 we must attach a node u_7 without making new Lannér (or parabolic) diagrams, and all edges must have multiplicities at most 3. In addition, u_7 is joined with u_1. The last condition is restrictive, since we know that u_1 and u_6 are the nodes of Σ_J marked white in Table 3.2. A direct computation (using the technique described in Section 3.2) leads us to the two diagrams Σ'_1 and Σ'_2 (up to permutation of indices 2, 3, 4 and 5 which does not play any role) shown in Fig. 4.4.

$\Sigma'_1 =$ 

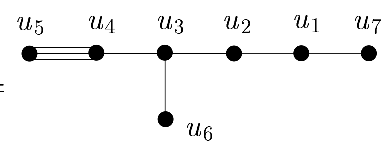
$\Sigma'_2 =$ 

Figure 4.4: Two possibilities for diagram Σ', see Lemma 4.4

Now consider the diagram $\Sigma'' = \langle \Sigma', u_8 \rangle = \langle \Sigma_J, u_7, u_8 \rangle = \langle \Sigma_J, L \rangle$. As above, u_8 may attach to Σ_J by edges of multiplicity at most 3, so the only multi-multiple edge that may appear in Σ'' is u_8u_7. Since both diagrams Σ'_1 and Σ'_2 have signature $(6, 1)$, the corresponding vectors in $\mathbb{R}^{6,1}$ form a basis, so the multiplicity of the edge u_8u_7 is completely determined by multiplicities of edges joining u_8 with the remaining nodes of Σ'. Thus, there is a finite number of possibilities for Σ''. To reduce the computations note that u_8 is not joined with $\langle u_2, u_3, u_4, u_5 \rangle$ (since the diagram $\langle u_2, u_3, u_4, u_5, u_8 \rangle$ must be elliptic). Attaching u_8 to Σ'_2, we do not obtain any diagram with zero determinant and prescribed Lannér diagrams. Attaching u_8 to Σ'_1, we obtain the two diagrams Σ''_1 and Σ''_2 shown in Fig. 4.5.

The remaining node of Σ, namely u_9, is joined with u_6 by a dotted edge. It is also contained in a Lannér diagram $\langle u_7, u_8, u_9 \rangle$ of order 3, but no other Lannér diagram contains u_9. Since u_7 attaches to u_1, we see that all edges joining u_9 with $\Sigma' \setminus u_6$ are neither dotted nor multi-multiple. On the other hand, for both diagrams Σ''_1 and Σ''_2, the diagram $\Sigma'' \setminus u_6$ has signature $(6, 1)$. Hence, the weight of edge u_9u_8 is completely determined by multiplicities of edges joining u_9 with the remaining nodes of $\Sigma'' \setminus u_6$, so we are left

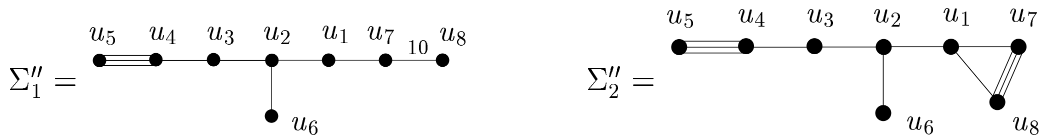


Figure 4.5: Two possibilities for diagram Σ'', see Lemma 4.4

with finitely many possibilities for $\Sigma'' \setminus u_6$. Again, we note that u_9 is not joined with $\langle u_2, u_3, u_4, u_5 \rangle$. Now we attach u_9 to u_1 and to u_7 by edges of multiplicities from 0 (i.e. no edge) to 3, and then compute the weight of the edge u_9u_8 to obtain $\det(\Sigma \setminus u_6) = 0$. This weight is equal to $\cos\frac{\pi}{m}$ for integer m only in case of the diagrams shown in Fig. 4.6.

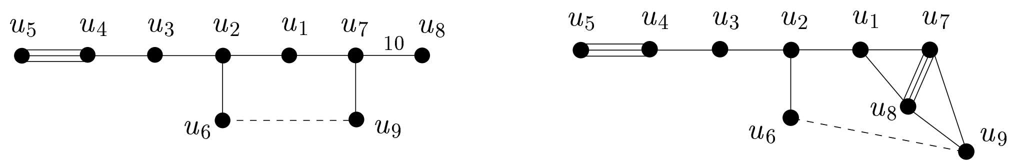


Figure 4.6: Coxeter diagrams of Coxeter polytopes with Gale diagram G_{21411}

The last step is to find the weight of the dotted edge u_9u_6 to satisfy the signature condition, i.e. the signature should equal $(6,1,2)$. We write the determinant of Σ as a quadratic polynomial of the weight d of the dotted edge, and compute the root. An easy computation shows that for both diagrams the signature of Σ for $d = \frac{1+\sqrt{5}}{2}$ is equal to $(6,1,2)$, so we obtain that these two diagrams correspond to Coxeter polytopes. One can note that the right polytope can be obtained by gluing two copies of the left one along the facet corresponding to the node u_8.

$\square$

Lemma 4.5. *There are no compact hyperbolic Coxeter polytopes with Gale diagram G_{252}.*

Proof. Suppose that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{252}. The Coxeter diagram Σ of P contains a Lannér diagram $L_1 = \langle u_1, \dots u_5 \rangle$ of order 5, and two diagrams of order 2, denote them $L_2 = \langle u_6, u_8 \rangle$ and $L_3 = \langle u_7, u_9 \rangle$. The diagram $\langle L_1, L_2 \rangle$ is connected, otherwise it is superhyperbolic. Thus, we may assume that u_6 attaches to L_1. Similarly, we may assume that u_7 attaches to L_1.

Therefore, the diagram $\Sigma' = \langle L_1, u_6, u_7 \rangle$ consists of a Lannér diagram L_1 of order 5 and two additional nodes which attach to L_1, and these nodes are not contained in any Lannér diagram. According to [E1, Lemma 3.8] (see Tabelle 3, page 27, the case $|\mathcal{N}_F| = 2$, $|\mathcal{L}_F| = 5$), Σ' must coincide with the diagram (up to permutation of indices of nodes of L_1) shown in Fig. 4.7.

Consider the diagram $\Sigma''_1 = \langle \Sigma', u_8 \rangle = \Sigma \setminus u_9$. The node u_8 is joined with u_6 by a dotted edge. The diagram $\Sigma''_1 \setminus u_6$ contains a unique Lannér diagram, L_1. If u_8 attaches to L_1, $\Sigma''_1 \setminus u_6$ should coincide with Σ'. Thus, u_8 does not attach to $\langle u_1, \dots, u_4 \rangle$, and $[u_8, u_5] = 2$ or 3. It is also easy to see that $[u_8, u_7] \leq 4$. Since the signature of Σ' is $(6,1)$, the weight

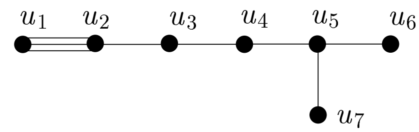


Figure 4.7: The diagram Σ', see Lemma 4.5

of the edge u_8u_6 is completely determined by multiplicities of edges joining u_8 with the remaining nodes of Σ'. Hence, we have a finite number of possibilities for Σ''_1. To reduce the computations observe that either $[u_5, u_8]$ or $[u_7, u_8]$ must equal 2. We are left with only 4 cases: the pair $([u_5, u_8], [u_7, u_8])$ coincides with one of $(2, 2), (2, 3), (2, 4)$ or $(3, 2)$. For each of them we compute the weight of u_8u_6 by solving the equation $\det \Sigma''_1 = 0$. Each of these equations has one positive and one negative solution, but the positive solution in case of $([u_5, u_8], [u_7, u_8]) = (2, 4)$ is less than one, so it cannot be a weight of a dotted edge. Therefore, we have three cases $([u_5, u_8], [u_7, u_8]) = (2, 2), (2, 3)$ or $(3, 2)$, for which the weight of u_8u_6 is equal to $\frac{\sqrt{2}\sqrt{4+\sqrt{5}}}{\sqrt{11}}$, $\frac{-3\sqrt{5}+7+4\sqrt{10-4\sqrt{5}}}{\sqrt{-9+5\sqrt{5}}}$, and $\frac{5+4\sqrt{5}}{11}$ respectively.

By symmetry, we obtain the same cases for the diagram $\Sigma''_2 = \langle \Sigma', u_9 \rangle = \Sigma \setminus u_8$, and the same values of the weight of the edge u_9u_7 when $([u_5, u_9], [u_6, u_9]) = (2, 2), (2, 3)$ and $(3, 2)$ respectively. Now, we have only 9 cases to attach nodes u_8 and u_9 to Σ' (in fact, there are only six up to symmetry). For each of these cases we compute the weight of the edge u_8u_9 by solving the equation $\det \Sigma = 0$. None of these solutions is equal to $\cos \frac{\pi}{m}$ for integer m, which contradicts the fact that the diagram $\langle u_8, u_9 \rangle$ is elliptic. This contradiction proves the lemma.

$\square$

4.3 Dimension 5

In dimension 5 we must consider six Gale diagrams, namely G_{242}, G_{323}, G_{21311}, G_{12311}, G_{11411}, and G_{12221}.

Lemma 4.6. *There is only one compact hyperbolic Coxeter polytope with Gale diagram G_{12221}. Its Coxeter diagram is the left one shown in the first row of Table 4.10.*

Proof. The proof is similar to the proof of Lemma 4.3. We assume that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{12221}. This Gale diagram contains an arc $J = \lfloor 2, 2, 2 \rfloor_2$. According to Lemma 3.1 (Item 10 of Table 3.2) and Lemma 2.2, the Coxeter diagram Σ of P consists of the subdiagram Σ_J shown in Fig. 4.8, and two

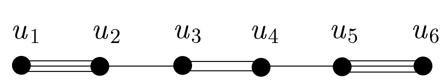


Figure 4.8: A unique diagram Σ_J for $J = \lfloor 2, 2, 2 \rfloor_2$

nodes u_7, u_8 joined by a dotted edge. By Lemma 2.1, the subdiagrams $\langle u_7, u_1, u_2 \rangle$ and

$\langle u_5, u_6, u_8 \rangle$ are Lannér diagrams, and no other Lannér subdiagram of Σ contains u_7 or u_8. So, we need to check possible multiplicities of edges incident to u_7 and u_8.

Again, we consider the diagram $\Sigma' = \langle \Sigma_J, u_7 \rangle$. It is connected, does not contain dotted edges, and its determinant is equal to zero. Furthermore, observe that u_7 does not attach to $\langle u_2, u_3, u_4, u_5 \rangle$ (since the diagram $\langle u_7, u_2, u_3, u_4, u_5 \rangle$ should be elliptic), and u_7 does not attach to u_6 (since the diagram $\langle u_7, u_4, u_5, u_6 \rangle$ should be elliptic). Therefore, u_7 is joined with u_1 only. Solving the equation $\det \Sigma' = 0$, we find that $[u_7, u_1] = 4$.

By symmetry, we obtain that u_8 is not joined with $\langle u_1, u_2, u_3, u_4, u_5 \rangle$, and $[u_8, u_6] = 4$. Thus, we have the Coxeter diagram Σ shown in Fig. 4.9. Assigning the weight $d =$

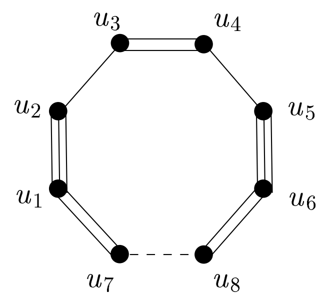


Figure 4.9: Coxeter diagram of a unique Coxeter polytope with Gale diagram G_{12221}

$\sqrt{2}(\sqrt{5}+1)/4$ to the dotted edge, we see that the signature of Σ is equal to $(5, 1, 2)$, so we obtain that this diagram corresponds to a Coxeter polytope.

□

Before considering the diagram G_{11411}, we make a small geometric excursus, the first one in this purely geometric paper.

The combinatorial type of polytope defined by Gale diagram G_{11411} is twice truncated 5-simplex, i.e. a 5-simplex in which two vertices are truncated by hyperplanes very close to the vertices. If we have such a polytope P with acute angles, it is easy to see that we are always able to truncate the polytope again by two hyperplanes in the following way: we obtain a combinatorially equivalent polytope P'; the two truncating hyperplanes do not intersect initial truncating hyperplanes and intersect exactly the same facets of P the initial ones do; the two truncating hyperplanes are orthogonal to all facets of P they do intersect.

The difference between polytopes P and P' consists of two small polytopes, each of them is combinatorially equivalent to a product of 4-simplex and segment, i.e. each of these polytopes is a simplicial prism. Of course, it is a Coxeter prism, and one of the bases is orthogonal to all facets of the prism it does intersect. All such prisms were classified by Kaplinskaja in [K]. Simplices truncated several times with orthogonality condition described above were classified by Schlettwein in [S]. Twice truncated simplices from the second list are the right ones in rows 1, 3, and 5 of Table 4.10.

Therefore, to classify all Coxeter polytopes with Gale diagram G_{11411} we only need to do the following. We take a twice truncated simplex from the second list, it has two "right" facets, i.e. facets which make only right angles with other facets. Then we find all the prisms that have "right" base congruent to one of "right" facets of the truncated simplex, and glue these prisms to the truncated simplex by "right" facets in all possible ways.

The result is presented in Table 4.10. All polytopes except the left one from the first row have Gale diagram G_{11411}. The polytopes from the fifth row are obtained by gluing one prism to the right polytope from this row, the polytopes from the third and fourth rows are obtained by gluing prisms to the right polytope from the third row, and the polytopes from the first and second rows are obtained by gluing prisms to the right polytope from the first row. The number of glued prisms is equal to the number of edges inside the maximal cycle of Coxeter diagram. Hence, we come to the following lemma:

Lemma 4.7. *There are* 15 *compact hyperbolic Coxeter* 5*-polytopes with* 8 *facets with Gale diagram* G_{11411}*. Their Coxeter diagrams are shown in Table 4.10.*

Proof. In fact, the lemma has been proved above. Here we show how to verify the previous considerations without any geometry and without referring to classifications from [K] and [S]. Since the procedure is very similar to the proof of Lemma 4.6, we provide only a plan of necessary computations without details.

Let P be a compact hyperbolic Coxeter polytope P with Gale diagram G_{11411}. This Gale diagram contains an arc $J = \lfloor 1, 4, 1 \rfloor_2$, so the Coxeter diagram Σ of P consists of one of the diagrams Σ_J presented in Item 2 of Table 3.2 and two nodes u_7 and u_8 joined by a dotted edge.

Choose one of three diagrams Σ_J. Consider the diagram $\Sigma' = \langle \Sigma_J, u_7 \rangle$. It is connected, contains a unique dotted edge, no multi-multiple edges, and its determinant is equal to zero. So, we are able to find the weight of the dotted edge joining u_7 with Σ_J depending on multiplicities of the remaining edges incident to u_7. The weight of this edge should be greater than one. Of course, we must restrict ourselves to the cases when non-dotted edges incident to u_7 do not make any new Lannér diagram together with Σ_J. The number of such cases is really small.

Further, we do the same for the diagram $\Sigma'' = \langle \Sigma_J, u_8 \rangle$, and we find all possible such diagrams together with the weight of the dotted edge joining u_8 with Σ_J. Then we are left to determine the weight of the dotted edge u_7u_8 for any pair of diagrams Σ' and Σ''. It occurs that this weight is always greater than one.

Doing the procedure described above for all the three possible diagrams Σ_J, we obtain the complete list of compact hyperbolic Coxeter 5-polytopes with 8 facets with Gale diagram G_{11411}. The computations completely confirm the result of considerations previous to the lemma.

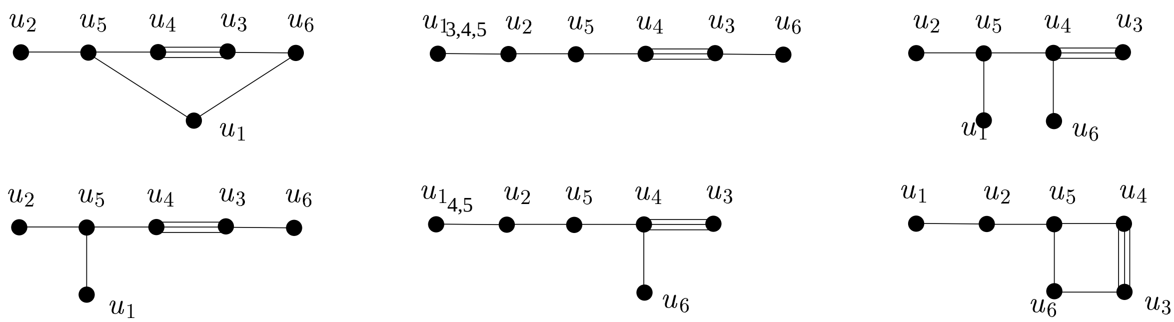
□

In the remaining part of this section we show that Gale diagrams G_{242}, G_{323}, G_{21311}, and G_{12311} do not give rise to any Coxeter polytope.

Lemma 4.8. *There are no compact hyperbolic Coxeter polytopes with Gale diagram* G_{12311}*.*

Proof. Suppose that there exists a compact hyperbolic Coxeter polytope P with Gale diagram G_{12311}. This Gale diagram contains an arc $J = \lfloor 2,3,1 \rfloor_2$. According to Lemma 3.1 (Item 9 of Table 3.2) and Lemma 2.2, the Coxeter diagram Σ of P consists of one of the nine subdiagrams Σ_J shown in Table 4.1, and two nodes u_7, u_8 joined by a dotted edge.

Table 4.1: All possible diagrams Σ_J for $J = \lfloor 2,3,1 \rfloor_2$



By Lemma 2.1, the subdiagrams $\langle u_7, u_1, u_2 \rangle$ and $\langle u_6, u_8 \rangle$ are Lannér diagrams, and no other Lannér subdiagram of Σ contains u_7 or u_8.

Consider the diagram $\Sigma' = \langle \Sigma_J, u_7 \rangle$. It is connected, does not contain dotted edges, and its determinant is equal to zero. Observe that the diagram $\langle u_2, u_3, u_4, u_5 \rangle$ is of the type H_4. Since the diagram $\langle u_7, u_2, u_3, u_4, u_5 \rangle$ is elliptic, this implies that u_7 is not joined with $\langle u_2, u_3, u_4, u_5 \rangle$. Furthermore, notice that the diagram $\langle u_3, u_4, u_6 \rangle$ is of the type H_3. Since the diagram $\langle u_7, u_3, u_4, u_6 \rangle$ is elliptic, we obtain that $[u_7, u_6] = 2$ or 3. Thus, for each of 9 diagrams Σ_J we have 2 possibilities of attaching u_7 to $\Sigma_J \setminus u_1$. Solving the equation $\det \Sigma' = 0$, we compute the weight of the edge $u_7 u_1$. In all 18 cases the result is not of the form $\cos \frac{\pi}{m}$ for positive integer m, which proves the lemma.

$\square$

Lemma 4.9. *There are no compact hyperbolic Coxeter polytopes with Gale diagram G_{21311}.*

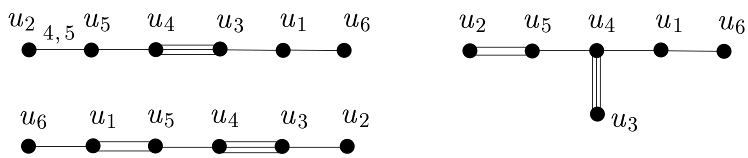
Proof. Suppose that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{21311}. This Gale diagram contains an arc $J = \lfloor 1,3,1 \rfloor_2$. Therefore, the Coxeter diagram Σ of P contains one of the five subdiagrams Σ_J, shown in Item 8 of Table 3.2.

On the other hand, Σ contains a Lannér diagram L of order 3 intersecting Σ_J. Denote by u_1 the intersection node of L and Σ_J, and denote by u_6 and u_7 the remaining nodes of L. Since L is connected, we may assume that u_6 attaches to u_1. Denote by u_2 the node of Σ_J different from u_1 and contained in only one Lannér diagram of order 4, and denote by u_3, u_4, u_5 the nodes of Σ_J contained in two Lannér diagrams of order 4.

Consider the diagram $\Sigma_0 = \langle \Sigma_J, u_6 \rangle \setminus u_2$. It is connected, has order 5, and contains a unique Lannér diagram which is of order 4. All such diagrams are listed in [E1, Lemma 3.8] (see the first two rows of Tabelle 3, the case $|\mathcal{N}_F| = 1$, $|\mathcal{L}_F| = 4$). We have reproduced this list in Table 3.3.

Consider the diagram $\Sigma_1 = \langle \Sigma_J, u_6 \rangle = \langle \Sigma_J, \Sigma_0 \rangle$. Comparing the lists of possibilities for Σ_J and Σ_0, it is easy to see that Σ_1 coincides with one of the four diagrams listed in Table 4.2 (up to permutation of indices 3, 4 and 5). Now consider the diagram $\Sigma' =$

Table 4.2: All possibilities for diagram Σ_1, see Lemma 4.9



$\langle \Sigma_J, L \rangle = \langle \Sigma_1, u_7 \rangle$. It is connected, does not contain dotted edges, its determinant is equal to zero, and the only multi-multiple edge may join u_7 and u_6. To reduce further computations notice, that the diagram $\langle u_7, u_3, u_4, u_5 \rangle$ is elliptic, so u_7 does not attach to $\langle u_3, u_4 \rangle$, and may attach to u_5 by simple edge only. Moreover, since the diagrams $\langle u_7, u_2, u_4, u_5 \rangle$ and $\langle u_7, u_1, u_4, u_5 \rangle$ are elliptic, u_7 is not joined with u_5. Furthermore, since the diagrams $\langle u_7, u_1, u_4, u_5 \rangle$ and $\langle u_7, u_1, u_3, u_4 \rangle$ are elliptic, $[u_7, u_1] = 2$ or 3. Considering elliptic diagrams $\langle u_7, u_2, u_4, u_5 \rangle$ and $\langle u_7, u_2, u_3, u_4 \rangle$, we obtain that $[u_7, u_2]$ is also at most 3. Then for all 4 diagrams Σ_1 and all admissible multiplicities of edges $u_7 u_1$ and $u_7 u_2$ we compute the weight of the edge $u_7 u_6$. We obtain exactly two diagrams Σ' where this weight is equal to $\cos \frac{\pi}{m}$ for some positive integer m, these diagrams are shown in Fig. 4.10. We are left to attach the node u_8 to Σ'. Consider the diagram $\Sigma'' = \Sigma \setminus u_2$.

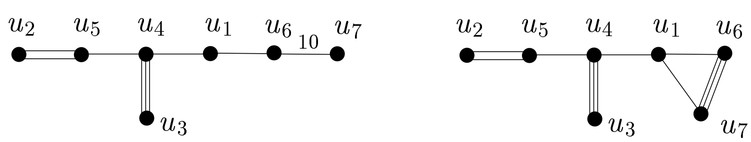


Figure 4.10: All possibilities for diagram Σ', see Lemma 4.9

As usual, it is connected, does not contain dotted edges, its determinant is equal to zero, and the only multi-multiple edge that may appear is $u_8 u_7$. Furthermore, the diagram $\langle u_3, u_4, u_1, u_6 \rangle$ is of the type H_4, and the diagram $\langle u_8, u_3, u_4, u_1, u_6 \rangle$ is elliptic. Thus, u_8 does not attach to $\langle u_3, u_4, u_1, u_6 \rangle$. The diagram $\langle u_3, u_4, u_5 \rangle$ is of the type H_3, and since the diagram $\langle u_8, u_3, u_4, u_5 \rangle$ should be elliptic, this implies that $[u_8, u_5] = 2$ or 3. Now for both diagrams $\Sigma' \setminus u_2 \subset \Sigma''$ we compute the weight of the edge $u_8 u_5$. In all four cases this weight is not equal to $\cos \frac{\pi}{m}$ for any positive integer m, that finishes the proof.

□

Lemma 4.10. *There are no compact hyperbolic Coxeter polytope with Gale diagram G_{323}.*

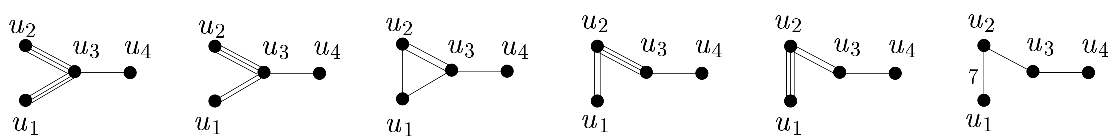
Proof. Suppose that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{323}. The Coxeter diagram Σ of P consists of two Lannér diagrams L_1 and L_2 of order 3, and one Lannér diagram L_3 of order 2. Any two of these Lannér diagrams are joined in Σ, and any subdiagram of Σ not containing one of these three diagrams is elliptic.

Consider the diagram $\Sigma_{12}=\langle L_1,L_2\rangle$. Due to [E2, p. 239, Step 4], we have three cases:

(1) L_1 and L_2 are joined by two simple edges having a common vertex, say in L_2;
(2) L_1 and L_2 are joined by a unique double edge;
(3) L_1 and L_2 are joined by a unique simple edge.

We fix the following notation: $L_1=\langle u_1,u_2,u_3\rangle$, $L_2=\langle u_4,u_5,u_6\rangle$, $L_3=\langle u_7,u_8\rangle$, the only node of L_2 joined with L_1 is u_4; u_4 is joined with u_3 and, in case (1), with u_1. We may assume also that u_7 attaches to L_1, u_4 is joined to u_5 in L_2, and u_2 is joined to u_3 in L_1.
Case (1). Since the diagrams $\langle u_2,u_1,u_4\rangle$ and $\langle u_2,u_3,u_4\rangle$ are elliptic, $[u_2,u_1]$ and $[u_2,u_3]$ do not exceed 5. On the other hand, $\langle u_1,u_2,u_3\rangle=L_1$ is a Lannér diagram, so we may assume that $[u_2,u_1]=5$, and $[u_2,u_3]=4$ or 5. Now attach u_7 to L_1. If u_7 is joined with u_1 or u_2, then the diagram $\langle u_2,u_1,u_4\rangle$ is not elliptic, and if u_7 is joined with u_3, then the diagram $\langle u_2,u_3,u_4\rangle$ is not elliptic, which contradicts Lemma 2.1.
Case (2). It is clear that $[u_2,u_3]=[u_4,u_5]=3$, and u_7 cannot be attached to u_3. Thus, u_7 is joined with u_1 or u_2, which implies that $[u_2,u_1]\le 5$. Therefore, $[u_1,u_3]=3$. So, the diagrams $\langle u_1,u_3,u_4,u_5\rangle$ and $\langle u_2,u_3,u_4,u_5\rangle$ are of the type F_4. Therefore, if u_7 attaches u_1, then the diagram $\langle u_7,u_1,u_3,u_4,u_5\rangle$ is not elliptic, and if u_7 is joined with u_2, then the diagram $\langle u_7,u_2,u_3,u_4,u_5\rangle$ is not elliptic.
Case (3). The signature of Σ_{12} is either $(5,1)$ or $(4,1,1)$. Thus, $\det\Sigma_{12}\le 0$. By Prop. 3.2, $\det(L_1,u_3)\det(L_2,u_4)\le\frac{1}{4}$. We may assume that $|\det(L_1,u_3)|\le|\det(L_2,u_4)|$, in particular, $|\det(L_1,u_3)|\le\frac{1}{2}$. By [E2, Table 2], there are only 6 possibilities for $\langle L_1,u_4\rangle$, we list them in Table 4.3.

Table 4.3: All possibilities for diagram $\langle L_1,u_4\rangle$, see Case (3) of Lemma 4.10



For any of these six diagrams $|\det(L_1,u_3)|\ge\frac{\sqrt{5}-1}{8}$. Thus, $|\det(L_2,u_4)|\le\frac{1}{4}\frac{8}{\sqrt{5}-1}=\frac{2}{\sqrt{5}-1}$. Notice that since the diagrams $\langle u_3,u_4,u_5\rangle$ and $\langle u_3,u_4,u_6\rangle$ are elliptic, $[u_4,u_5]$ and $[u_4,u_6]$ do not exceed 5. Now, since the local determinant is an increasing function of multiplicities of the edges, it is not difficult to list all Lannér diagrams $L_2=\langle u_4,u_5,u_6\rangle$, such that $[u_4,u_5],[u_4,u_6]\le 5$, and $|\det(L_2,u_4)|\le\frac{2}{\sqrt{5}-1}$. This list contains 17 diagrams only.

Then, from $6\cdot 17=102$ pairs (L_1,L_2) we list all pairs with $\det(L_1,u_3)\det(L_2,u_4)\le\frac{1}{4}$. Each of these pairs corresponds to a diagram Σ_{12}. After that, we attach to all diagrams Σ_{12} a node u_7 in the following way: u_7 is joined with L_1 (and may be joined with L_2, too), and it does not produce any new Lannér or parabolic diagram. It occurs that none of obtained diagrams $\langle\Sigma_{12},u_7\rangle$ has zero determinant.

$\square$

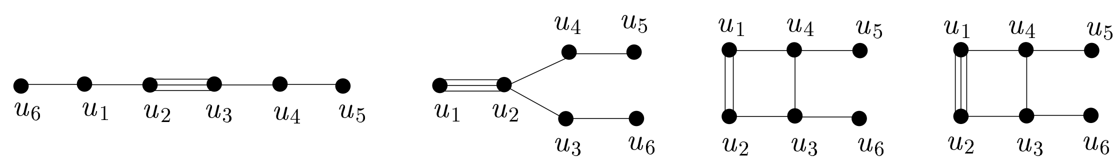
Lemma 4.11. *There are no compact hyperbolic Coxeter polytopes with Gale diagram G_{242}.*

Proof. Suppose that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{242}. The Coxeter diagram Σ of P consists of one Lannér diagram L_1 of order 4, and two Lannér diagrams L_2 and L_3 of order 2. Any two of these Lannér diagrams are joined in Σ, and any subdiagram of Σ not containing one of these three diagrams is elliptic.

We fix the following notation: $L_1 = \langle u_1, u_2, u_3, u_4\rangle$, $L_2 = \langle u_5, u_7\rangle$, $L_3 = \langle u_6, u_8\rangle$, u_5 and u_6 attach to L_1.

Consider the diagram $\Sigma_0 = \langle L_1, u_5, u_6\rangle$. It is connected, has order 6, and contains a unique Lannér diagram which is of order 4. All such diagrams are listed in [E1, Lemma 3.8] (see Tabelle 3, the first two rows of page 27, the case $|\mathcal{N}_F| = 2$, $|\mathcal{L}_F| = 4$). We have reproduced this list in Table 3.4. The list contains five diagrams, but we are interested in four of them: in the fifth one only one of two additional nodes attaches to the Lannér diagram. We list these four possibilities for Σ_0 in Table 4.4.

Table 4.4: All possibilities for diagram Σ_0, see Lemma 4.11



Now consider the diagram $\Sigma' = \langle \Sigma_0, u_7\rangle$. It contains a unique dotted edge u_5u_7. Since the diagram $\langle u_7, u_1, u_2, u_3, u_6\rangle$ is elliptic and the diagram $\langle u_1, u_2, u_3, u_6\rangle$ is of the type H_4 or B_4, u_7 is not joined with $\langle u_1, u_2, u_3\rangle$, and it may attach to u_6 if $[u_1, u_2] = 4$ only. It is easy to see that $[u_7, u_4] = 2$ or 3 in all four cases. We obtain 9 possibilities for attaching u_7 to $\Sigma_0 \setminus u_5$. For each of them we compute the weight of the edge u_5u_7.

By symmetry, we may list all 9 possibilities for the diagram $\Sigma'' = \langle \Sigma_0, u_8\rangle$. Now we are left to compute the weight of the edge u_7u_8 in Σ. Diagrams Σ_0 with $[u_1, u_2] = 5$ produce three possible diagrams Σ each, and the diagram Σ_0 with $[u_1, u_2] = 4$ produces six possible diagrams Σ (we respect symmetry). In all these 15 cases the weight of the edge u_7u_8 is not of the form $\cos\frac{\pi}{m}$ for positive integer m.

$\square$

4.4 Dimension 4

In dimension 4 we must consider four Gale diagrams, namely G_{232}, G_{11311}, G_{21112}, and G_{12121}. Three of them, i.e. G_{232}, G_{11311}, and G_{12121}, give rise to Coxeter polytopes.

Lemma 4.12. *There are exactly three compact hyperbolic Coxeter polytopes with Gale diagram G_{232}. Their Coxeter diagrams are shown in the third row of the second part of Table 4.11.*

Proof. Let P be a compact hyperbolic Coxeter polytope with Gale diagram G_{232}. The Coxeter diagram Σ of P consists of one Lannér diagram L_1 of order 3, and two Lannér diagrams L_2 and L_3 of order 2. Any two of these Lannér diagrams are joined in Σ, and any subdiagram of Σ containing none of these three diagrams is elliptic.

On the first sight, the considerations may repeat ones from the proof of Lemma 4.11. However, there is a small difference: the number of Lannér diagrams of order 3 is infinite. Thus, at first we must bound the multiplicities of the edges of the Lannér diagram of order 3.

We fix the following notation: $L_1 = \langle u_1, u_2, u_3\rangle$, $L_2 = \langle u_5, u_6\rangle$, $L_3 = \langle u_4, u_7\rangle$, u_4 and u_5 attach to L_1. We may also assume that u_4 attaches to u_3.

Since the diagrams $\langle u_1, u_3, u_4\rangle$ and $\langle u_2, u_3, u_4\rangle$ should be elliptic, the edges u_3u_1 and u_3u_2 are not multi-multiple. We consider two cases: u_1 or u_2 is either joined with $\langle u_4, u_5, u_6, u_7\rangle$ or not.

Case 1: u_1 and u_2 are not joined with $\langle u_4, u_5, u_6, u_7\rangle$. In particular, this is true if the edge u_1u_2 is multi-multiple. Then u_5 attaches to u_3. Since the diagrams $\langle u_1, u_3, u_4, u_5\rangle$ and $\langle u_2, u_3, u_4, u_5\rangle$ are elliptic, $[u_3, u_1]$ and $[u_3, u_2]$ do not exceed 3, $[u_3, u_4] = [u_3, u_5] = 3$, and $[u_4, u_5] = 2$. We may assume that $[u_3, u_1] = 3$, and $[u_3, u_2] = 2$ or 3.

Consider the diagram $\Sigma' = \langle L_1, L_2, u_4\rangle = \Sigma \setminus u_7$. We know that u_6 is joined with u_5 by a dotted edge, and u_6 does not attach to u_1 and u_2. Furthermore, since the diagram $\langle u_1, u_3, u_4, u_6\rangle$ is elliptic, $[u_6, u_3] \le 3$ and $[u_6, u_4] \le 4$. By the same reason, either $[u_6, u_3]$ or $[u_6, u_4]$ is equal to 2. Thus, we have four possibilities to attach u_6 to u_3 and u_4.

Denote by d the weight of the dotted edge u_5u_6, and compute the local determinant $\det(\langle u_3, u_4, u_5, u_6\rangle, u_3)$ for all four diagrams $\langle u_3, u_4, u_5, u_6\rangle$ as a function of d.

Case 1.1: $[u_6, u_4] \ne 2$. In this case $\det(\langle u_3, u_4, u_5, u_6\rangle, u_3)$ equals either $\frac{12d^2+4d-5}{4(4d^2-3)}$ (when $[u_6, u_4] = 3$) or $\frac{6d^2+2\sqrt{2}d-1}{4(2d^2-1)}$ (when $[u_6, u_4] = 4$). Both expressions decrease in the ray $[1,\infty)$, so the maximal values are $11/4$ and $(5+2\sqrt{2})/4$ respectively. Now recall that $\det\Sigma' = 0$, so by Prop. 3.1 we have $\det(L_1, u_3) = 1 - \det(\langle u_3, u_4, u_5, u_6\rangle, u_3)$. Therefore, $|\det(L_1, u_3)|$ is bounded from above by $7/4$ or $(1+2\sqrt{2})/4$ if $[u_6, u_4] = 3$ or $[u_6, u_4] = 4$ respectively. Since $|\det(L_1, u_3)|$ is an increasing function on $[u_1, u_2]$, an easy check shows that $[u_1, u_2]$ is bounded by 10 or 8 respectively. So, in both cases we have finitely many possibilities for L_1.

Further considerations follow ones from Lemma 4.11. We list all possible Σ' together with the weight of the dotted edge u_5u_6 (which may be computed from the equation $\det\Sigma' = 0$), then we list all possible diagrams $\Sigma'' = \langle L_1, L_3, u_5\rangle = \Sigma \setminus u_6$ in a similar way. After that for all pairs (Σ', Σ'') (with the same L_1) we compute the weight of the edge u_6u_7. It occurs that in all cases the weight is not of the form $\cos\frac{\pi}{m}$ for positive integer m.

Case 1.2: $[u_6, u_4] = 2$. In this case $\det(\langle u_3, u_4, u_5, u_6\rangle, u_3)$ equals either $\frac{3d^2-2}{4(d^2-1)}$ (when $[u_6, u_3] = 2$) or $\frac{3d-1}{4(d-1)}$ (when $[u_6, u_3] = 3$). These tend to ∞ when d tends to 1, so we do not obtain any bound for $[u_1, u_2]$.

Let $m_{12} = [u_1, u_2]$, $m_{23} = [u_2, u_3]$, and let $m_{36} = [u_3, u_6]$. Notice that $m_{23}, m_{36} = 2$ or 3. Define also $c_{12} = \cos(\pi/m_{12})$. We compute the weight of the edge u_5u_6 as a function $d(m_{12}, m_{23}, m_{36})$ of m_{12}, m_{23} and m_{36}. Solving the equation $\det\Sigma' = 0$, we see that

$$d(m_{12},2,2)=\sqrt{\frac{2c_{12}^2-1}{2c_{12}^2-2}}; \qquad d(m_{12},3,2)=\sqrt{\frac{2c_{12}}{3c_{12}-1}};$$

$$d(m_{12},2,3)=\frac{c_{12}^2}{3c_{12}^2-2}; \qquad d(m_{12},3,3)=\frac{c_{12}+1}{3c_{12}-1}.$$

Consider the diagram Σ. According to Case 1.1, we may assume that $[u_5,u_7]=2$. Since L_2 and L_3 are joined in Σ, $[u_6,u_7]\neq 2$. On the other hand, the diagram $\langle u_3,u_6,u_7\rangle$ is elliptic. Thus, either $[u_3,u_6]$ or $[u_3,u_7]$ equals 2. By symmetry, we may assume that $[u_3,u_7]=2$. We also know how the weight of the edge u_4u_7 depends on m_{12} and m_{23}.

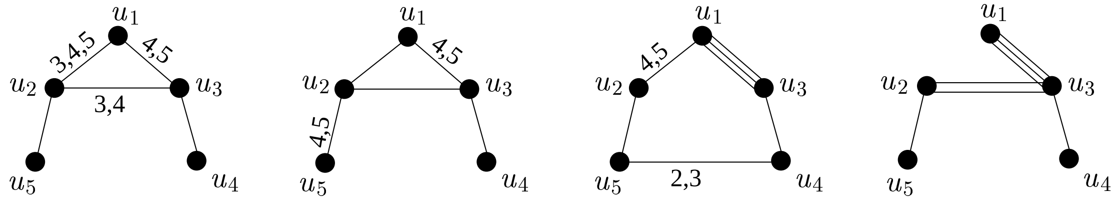
Now we are able to compute the weight $w(m_{12},m_{23},m_{36})$ of the dotted edge u_4u_7 as a function of m_{12},m_{23} and m_{36}. For that we simply solve the equation $\det\Sigma=0$. Notice that since L_1 is a Lannér diagram, $m_{12}\geq 7$ when $m_{23}=2$, and $m_{12}\geq 4$ when $m_{23}=3$. We obtain:

- $w(m_{12},2,2)=\dfrac{1-c_{12}^2}{3c_{12}^2-2}$ is a decreasing function of m_{12} as $m_{12}\geq 7$, and $w(7,2,2)<1/2$;
- $w(m_{12},2,3)=\dfrac{2(1-c_{12}^2)\sqrt{2c_{12}^2}}{(3c_{12}^2-2)^{3/2}}$ is a decreasing function of m_{12} as $m_{12}\geq 7$, $w(9,2,3)<1/2$, and $w(m_{12},2,3)\neq\cos(\pi/m)$ when $m_{12}=7$ or 8;
- $w(m_{12},3,2)=\dfrac{1-c_{12}}{3c_{12}-1}$ is a decreasing function of m_{12} as $m_{12}\geq 4$, and $w(4,2,2)<1/2$;
- $w(m_{12},3,3)=\dfrac{2(1-c_{12})\sqrt{2c_{12}}}{(3c_{12}-1)^{3/2}}$ is a decreasing function of m_{12} as $m_{12}\geq 4$, $w(5,3,3)<1/2$, and $w(4,3,3)\neq\cos(\pi/m)$.

This finishes considerations of Case 1.

Case 2: either u_1 or u_2 is joined with $\langle u_4,u_5,u_6,u_7\rangle$. In particular, this implies that L_1 contains no multi-multiple edges, so we deal with a finite number of possibilities for L_1 only. This list contains 11 Lannér diagrams of order 3. Using that list, it is not too difficult to list all the diagrams $\Sigma_0=\langle L_1,u_4,u_5\rangle$. This list contains 19 diagrams, we present them in Table 4.5. Now we follow the proof of Lemma 4.11. Choose one of 19

Table 4.5: All possibilities for diagram Σ_0, see Case 2 of Lemma 4.12

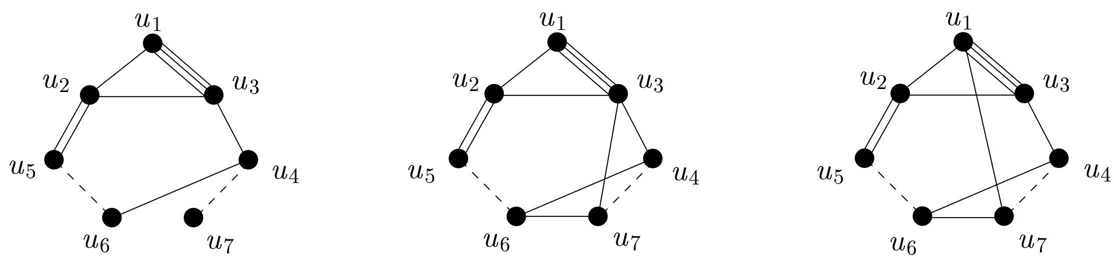


diagrams Σ_0, and consider the diagram $\Sigma'=\langle\Sigma_0,u_6\rangle$. It contains a unique dotted edge

u_5u_6, and that is the only Lannér diagram in Σ' containing u_6. We have a finite number of possibilities to attach u_6 to $\Sigma_0 \setminus u_5$. For each of them we compute the weight of the edge u_5u_6.

Similarly, we list all possibilities for the diagram $\Sigma'' = \langle \Sigma_0, u_7 \rangle$. Now we are left to compute the weight of the edge u_6u_7 in Σ. A computation shows that the weight is of the form $\cos\frac{\pi}{m}$ only for the diagrams listed in Table 4.6. To verify that these diagrams

Table 4.6: Coxeter diagrams of Coxeter polytopes with Gale diagram G_{232}



correspond to polytopes, we need to assign weights to the dotted edges. We assign a weight $\sqrt{2}\frac{\sqrt{5}+1}{4}$ to all edges u_5u_6, and weights $\frac{\sqrt{15(5+\sqrt{5})}}{10}$, $\frac{5+3\sqrt{5}}{10}$ and $\frac{3+\sqrt{5}}{4}$ to the edge u_4u_7 on the left, middle and right diagrams respectively. A direct calculation shows that the diagrams have signature $(4, 1, 2)$.

□

Lemma 4.13. *There are 29 compact hyperbolic Coxeter polytopes with Gale diagram G_{11311}. Their Coxeter diagrams are shown in the first part of Table 4.11 and in the first three rows of the second part of the same table.*

Proof. The proof is identical to one which concerns the diagram G_11411 (see Lemma 4.7). The combinatorial type of polytope defined by Gale diagram G_{11311} is twice truncated 4-simplex. Any such Coxeter polytope may be obtained by gluing one or two prisms to a twice truncated 4-simplex with orthogonality conditions described before Lemma 4.7. Such simplices were classified by Schlettwein in [S], they appear as right ones in rows 1, 2, and 4 of the first part of Table 4.11, and in rows 1 and 2 of the second part. The prisms were classified by Kaplinskaja in [K].

For each twice truncated simplex from the list of Schlettwein we find all the prisms that have "right" base congruent to one of "right" facets of the truncated simplex, and glue these prisms to the truncated simplex. The result is presented in Table 4.11.

The verification of the result above by computations is completely identical to the proof of Lemma 4.7. We only need to replace an arc $J = \lfloor 1, 4, 1 \rfloor_2$ from G_11411 by an arc $J = \lfloor 1, 3, 1 \rfloor_2$, and refer to Item 8 of Table 3.2 instead of Item 2.

□

Lemma 4.14. *There are no compact hyperbolic Coxeter polytopes with Gale diagram G_{21112}.*

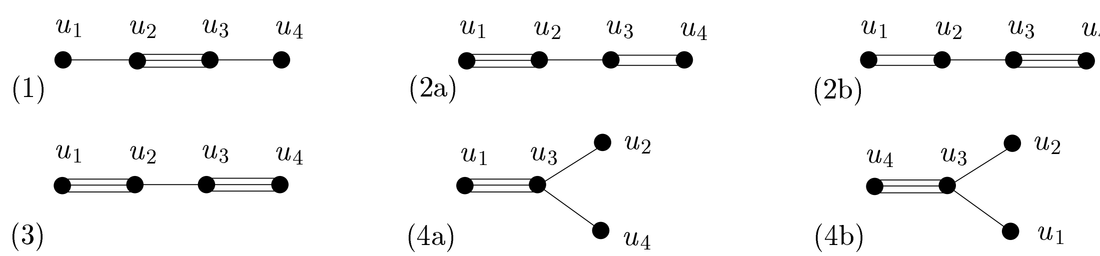
Proof. Suppose that there exists a hyperbolic Coxeter polytope P with Gale diagram G_{21112}. The Coxeter diagram Σ of P consists of one Lannér diagram $L_1 = \langle u_1, u_2, u_3, u_4 \rangle$ of order 4, two Lannér diagrams $L_2 = \langle u_6, u_1, u_2 \rangle$ and $L_3 = \langle u_3, u_4, u_5 \rangle$ of order 3, and two Lannér diagrams $\langle u_6, u_7 \rangle$ and $\langle u_7, u_5 \rangle$ of order 2.

Consider the diagram $\Sigma' = \langle L_1, L_2, L_3 \rangle = \Sigma \setminus u_7$. It is connected, has order 6, and contains no dotted edges. We may also assume that u_5 attaches to u_4. Clearly, any multi-multiple edge that may appear in Σ' belongs to L_2 or L_3 and does not belong to L_1. We consider two cases: either Σ' contains multi-multiple edges or not.

Suppose that Σ' contains no multi-multiple edges. Then we have 9 possibilities for L_2, and 9 possibilities for L_3. For each of 81 pairs (or 45 in view of symmetry) we join nodes of L_2 with nodes of L_3 in all possible ways (9 edges, 4 possibilities for each of them, from empty to triple one). We are looking for diagrams satisfying the following conditions: the determinant should vanish, there are no parabolic subdiagrams, and the diagram contains a unique new Lannér diagram, which has order 4. A computer check (which takes about 10 hours of computer thought) shows that only 39 obtained diagrams have zero determinant, and only 11 of them contain Lannér diagrams of order 4. However, each of them contains some new Lannér diagram of order 3. Therefore, none of them may be considered as Σ'.

Now suppose that Σ' contains at least one multi-multiple edge. We may assume that u_4u_5 is multi-multiple. In this case u_4 must be a *leaf* of L_1, i.e. it should have valency one in L_1. Indeed, if u_4 is joined with two vertices $v, w \in L_1$, then both diagrams $\langle u_5, u_4, v \rangle$ and $\langle u_5, u_4, w \rangle$ are not elliptic, which is impossible. Thus, L_1 is not a cycle, so we have 4 possibilities for L_1 only (see Table 2.2). In Table 4.7 we list all possible diagrams L_1 together with all possible numerations of nodes. A numeration should satisfy the following properties: u_4 is a leaf, and u_3 is a unique neighbor of u_4. We consider numerations up to interchange of u_1 and u_2.

Table 4.7: Numberings of vertices of Lannér diagrams of order 4 without cycles



Consider 6 diagrams case-by-case. For all of them we claim that u_5 and u_4 do not attach to $L_2 = \langle u_1, u_2, u_6 \rangle$: this is because the edge u_4u_5 is multi-multiple.

Diagram (1). Since the diagram $\langle u_1, u_2, u_3, u_5 \rangle$ is elliptic, u_5 is not joined with u_3. Furthermore, since the diagram $\langle u_6, u_2, u_3, u_4 \rangle$ is elliptic, u_6 is not joined with $\langle u_2, u_3 \rangle$. Therefore, $[u_6, u_2] = 2$, so $[u_6, u_1] \geq 7$. Applying Prop. 3.2, we see that $\det(L_2, u_2)\det(L_3, u_3) = \cos^2(\pi/5)$. An easy calculation shows that the inequality $[u_6, u_1] \geq 7$ implies that

$[u_4, u_5] \leq 10$. By symmetry, $[u_6, u_1] \leq 10$, too. We are left with a finite (and very small) number of possibilities for Σ'. For none of them $\det \Sigma' = 0$.

Diagrams (2a), (2b) and (3). Since the diagram $\langle u_1, u_2, u_3, u_5 \rangle$ is elliptic, $[u_3, u_5] \leq 3$. Since the diagram $\langle u_6, u_2, u_3, u_4 \rangle$ is elliptic, u_6 is not joined with u_3, and $[u_6, u_2] \leq 3$, so $[u_6, u_1] \geq 3$. Applying Prop. 3.2, we have $\det(L_2, u_2) \det(L_3, u_3) = 1/4$. By assumption, $[u_4, u_5] \geq 6$, which implies the inequality $|\det(L_3, u_3)| \geq |D\,(2, 4, 6)| = 1$. Thus, $|\det(L_2, u_2)| \leq 1/4$. But since $[u_1, u_2] \geq 4$ and $[u_6, u_2] \geq 3$, either $|\det(L_2, u_2)| \geq |D\,(2, 4, 5)| = 1/\sqrt{5} > 1/4$ or $|\det(L_2, u_2)| \geq |D\,(3, 4, 3)| = \sqrt{2}/3 > 1/4$, so we come to a contradiction.

Diagram (4a). Since the diagram $\langle u_6, u_1, u_3 \rangle$ is elliptic, $[u_6, u_1] \leq 3$. On the other hand, $L_2 = \langle u_1, u_2, u_6 \rangle$ is a Lannér diagram, so $[u_6, u_2] \geq 7$. This implies that $\langle u_6, u_2, u_3 \rangle$ is a Lannér diagram, which is impossible.

Diagram (4b). Since the diagram $\langle u_6, u_2, u_3, u_4 \rangle$ is elliptic, $[u_6, u_2] \leq 3$. Hence, $[u_6, u_1] \geq 7$, and $\langle u_6, u_1, u_3 \rangle$ is a Lannér diagram. This contradiction completes the proof of the lemma.

□

Lemma 4.15. *There are exactly eight compact hyperbolic Coxeter* 4*-polytopes with* 7 *facets with Gale diagram* G_{12121}*. Their Coxeter diagrams are shown in the bottom of the second part of Table 4.11.*

Proof. Let P be a hyperbolic Coxeter polytope with Gale diagram G_{12121}. The Coxeter diagram Σ of P contains two Lannér diagrams $L_1 = \langle u_1, u_2, u_3 \rangle$ and $L_2 = \langle u_3, u_4, u_5 \rangle$ of order 3, a dotted edge $u_6 u_7$, and other two Lannér diagrams $L_3 = \langle u_1, u_2, u_6 \rangle$ and $L_4 = \langle u_7, u_4, u_5 \rangle$ of order 3. Any subdiagram of Σ containing none of these five diagrams is elliptic. Since L_3 and L_4 are connected, we may assume that u_6 attaches to u_2, and u_7 attaches to u_5.

Consider the diagram $\Sigma' = \langle L_3, L_1, L_2 \rangle = \Sigma \setminus u_7$. Clearly, the only multi-multiple edges that may appear in Σ' are $u_1 u_2$, $u_6 u_2$, $u_6 u_1$, and $u_4 u_5$.

At first, suppose that the edge $u_6 u_2$ is multi-multiple. Then $\langle u_6, u_2 \rangle$ is not joined with $\langle u_3, u_4, u_5 \rangle = L_2$. In particular, $[u_2, u_3] = 2$, so $[u_1, u_3] \neq 2$. Thus, $[u_6, u_1]$ is also equal to 2. Furthermore, since diagrams $\langle u_1, u_3, u_4 \rangle$ and $\langle u_1, u_3, u_5 \rangle$ are elliptic, $[u_3, u_4], [u_3, u_5]$ and $[u_1, u_3] \leq 5$. Therefore, since $\langle u_1, u_2, u_3 \rangle = L_1$ is a Lannér diagram, $[u_1, u_2] \geq 4$. Now suppose that $[u_1, u_4] \neq 2$. Then $[u_3, u_4] = 2$, so $[u_4, u_5] \geq 4$, and the diagram $\langle u_2, u_1, u_4, u_5 \rangle$ is not elliptic, which is impossible. The contradiction shows that $[u_1, u_4] = 2$. Similarly, $[u_1, u_5] = 2$. Consequently, the diagram Σ' looks like the diagram shown in Fig. 4.11, where $m_{45} = [u_4, u_5]$. Now we may apply Prop. 3.2: $\det(L_3, u_1) \det(L_2, u_3) = \cos^2(\pi/m_{13})$, where $m_{13} = [u_1, u_3]$. Notice that since $[u_1, u_2] \geq 4$ and $[u_2, u_6] \geq 6$, we have $|\det(L_3, u_1)| \geq |D\,(2, 4, 6)| = 1$.

If $m_{13} = 4$ or 5, we obtain that $[u_3, u_4], [u_3, u_5] \leq 3$, which implies $[u_4, u_5] = 7$ in view of $|\det(L_2, u_3)| \leq \cos^2(\pi/m_{13})$. Thus, $|\det(L_2, u_3)| \geq |D\,(2, 3, 7)|$. This implies that $|\det(L_3, u_1)| \leq \cos^2(\pi/5)/|D\,(2, 3, 7)|$. An easy calculation shows that in this case

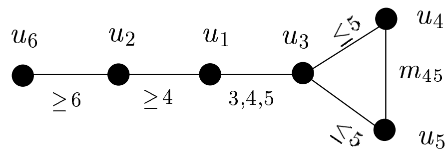


Figure 4.11: A diagram Σ', see Lemma 4.15

$[u_2, u_6] \leq 7$, $[u_2, u_1] \leq 6$. Then we check the finite (small) number of possibilities for Σ' and see that none of them has determinant equal to zero.

If $m_{13} = 3$, then $[u_2, u_1] \leq 7$. Therefore, $|\det(L_3, u_1)| \geq |D(2,6,7)|$. Hence, $|\det(L_2, u_3)| \leq \cos^2(\pi/3)/|D(2,6,7)|$, but such L_2 does not exist.

The contradiction shows that the edge u_6u_2 is not multi-multiple. Similarly, the edges u_6u_1, u_7u_5, and u_7u_4 of Σ are not multi-multiple either. Thus, the only edges that may be multi-multiple in Σ are u_4u_5 and u_1u_2.

Consider again the diagram Σ' and suppose that the diagram $\langle u_4, u_5\rangle$ is not joined with $\langle u_1, u_2, u_6\rangle$. In particular, this holds if at least one of the edges u_4u_5 and u_1u_2 is multi-multiple. We may apply Prop. 3.1:

$$\det(\langle L_3, L_1\rangle, u_3) + \det(L_2, u_3) = 1.$$

By definition,

$$\det(\langle L_3, L_1\rangle, u_3) = \det\langle L_3, L_1\rangle / \det(L_3).$$

We use a very rough bound: $|\det\langle L_3, L_1\rangle| < 16$ since it is a determinant of a 4×4 matrix with entries between -1 and 1, and $|\det(L_3)| \geq |3/4 - \cos^2(\pi/7)| = |\det(\mathcal{L}_{2,3,7})|$, since $\det(\mathcal{L}_{2,3,7})$ is maximal among all determinants of Lannér diagrams of order 3. This bound implies

$$|\det(L_2, u_3)| \leq 1 + |\det(\langle L_3, L_1\rangle, u_3)| \leq 1 + \frac{16}{|3/4 - \cos^2(\pi/7)|} < 261.$$

Now an easy computation shows that $[u_4, u_5] \leq 101$. Considering a diagram $\Sigma'' = \langle L_1, L_2, L_4\rangle = \Sigma \setminus u_6$ in a similar way, we obtain that $[u_1, u_2] \leq 101$, too, and we are left with a finite number of possibilities for Σ' (and for Σ''). We list all diagrams L_2 (less that 1000 possibilities) and all possible diagrams $\langle L_3, L_1\rangle$ (less that 10000 possibilities), and find all pairs such that $\det(\langle L_3, L_1\rangle, u_3) + \det(L_2, u_3) = 1$, there are about 50 such pairs. Therefore, we obtain a complete list of possibilities for Σ' (and for Σ''). Then we look for unordered pairs (Σ', Σ''), such that the diagrams coincide on their intersection, i.e. a subdiagram $\langle L_1, L_2\rangle \subset \Sigma'$ coincides with a subdiagram $\langle L_1, L_2\rangle \subset \Sigma''$. There are only 8 such pairs, all them give rise to Coxeter diagrams of Coxeter polytopes. The diagrams are shown in the bottom of the second part of Table 4.11. The weight of the dotted edge is equal to $\sqrt{2}\cos(\pi/8)$ for the two last diagrams, is equal to $(\sqrt{5}+1)/2$ for the three diagrams in the second row from the bottom, and is equal to $1+\sqrt{2}$ for the three diagrams in the third row from the bottom.

Now suppose that the diagram $\langle u_4, u_5 \rangle$ is joined with $\langle u_1, u_2, u_6 \rangle$. This implies that Σ does not contain multi-multiple edges, so we have a finite number of possibilities for the diagrams Σ' and Σ''. A computation shows that we do not obtain any polytope in this way.

□

The result of the considerations above is presented below. Recall that there are no 7-dimensional polytopes with 10 facets.

Table 4.8: 8-dimensional polytope with 11 facets

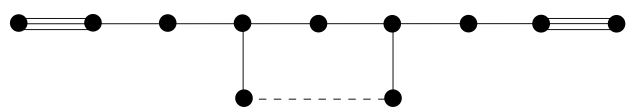


Table 4.9: 6-dimensional polytopes with 9 facets

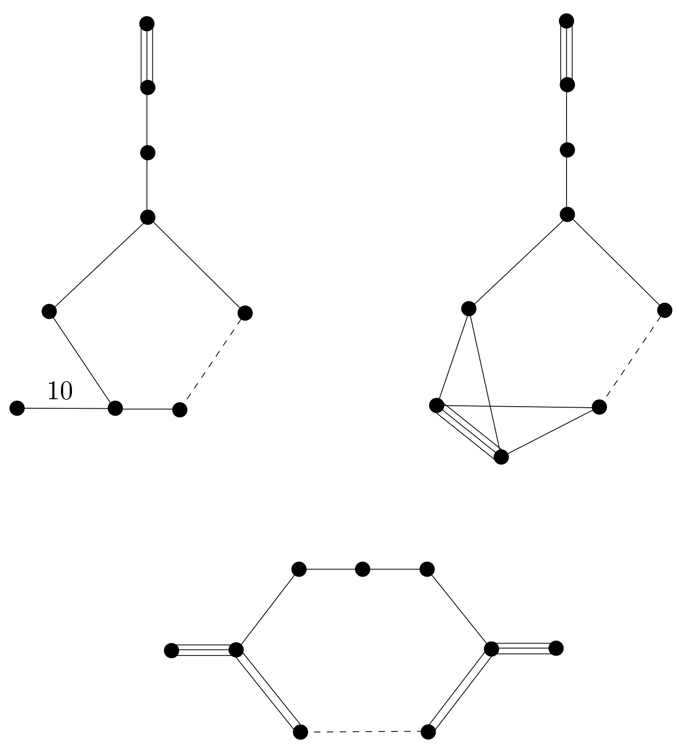


Table 4.10: 5-dimensional polytopes with 8 facets

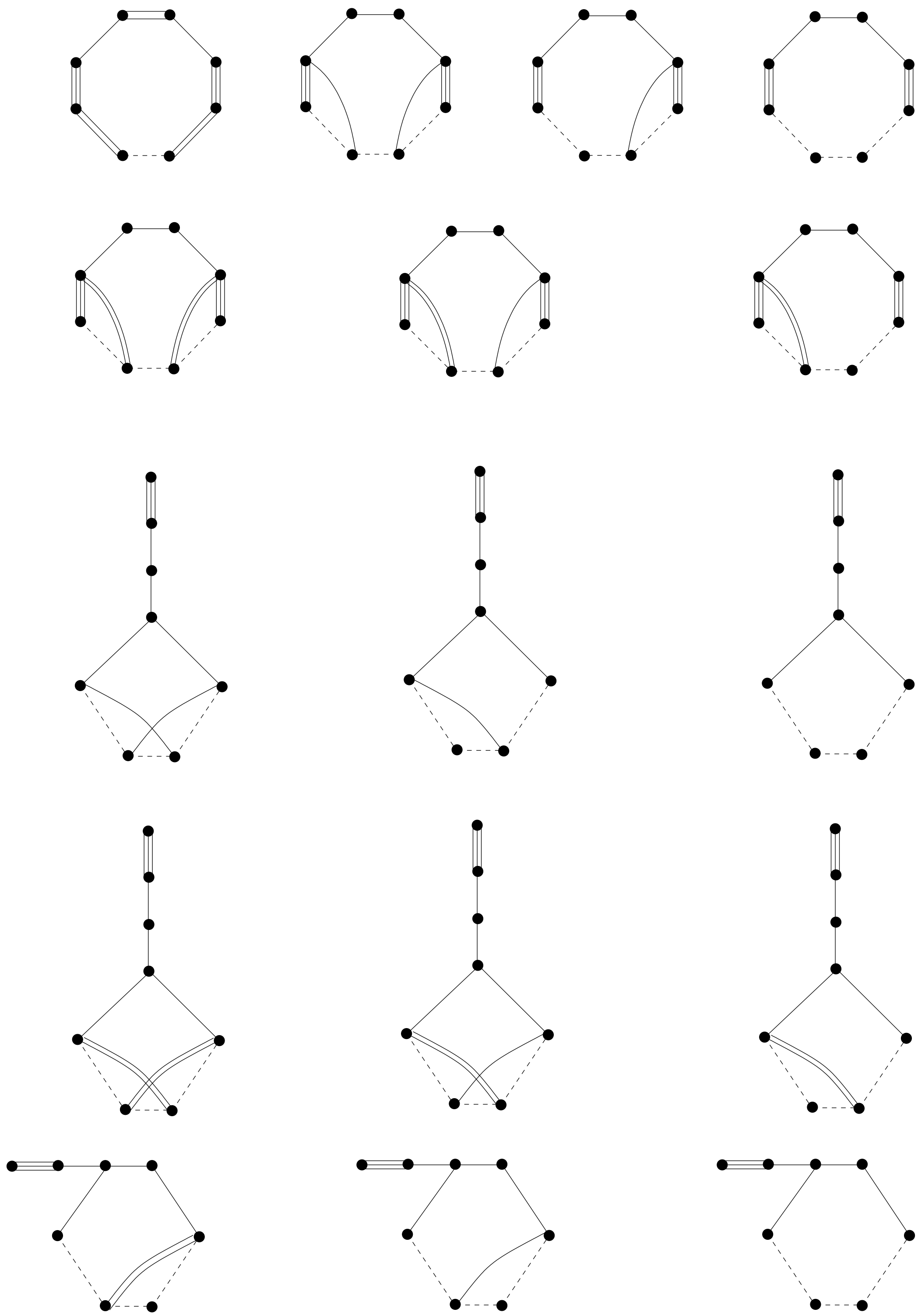


Table 4.11: 4-dimensional polytopes with 7 facets

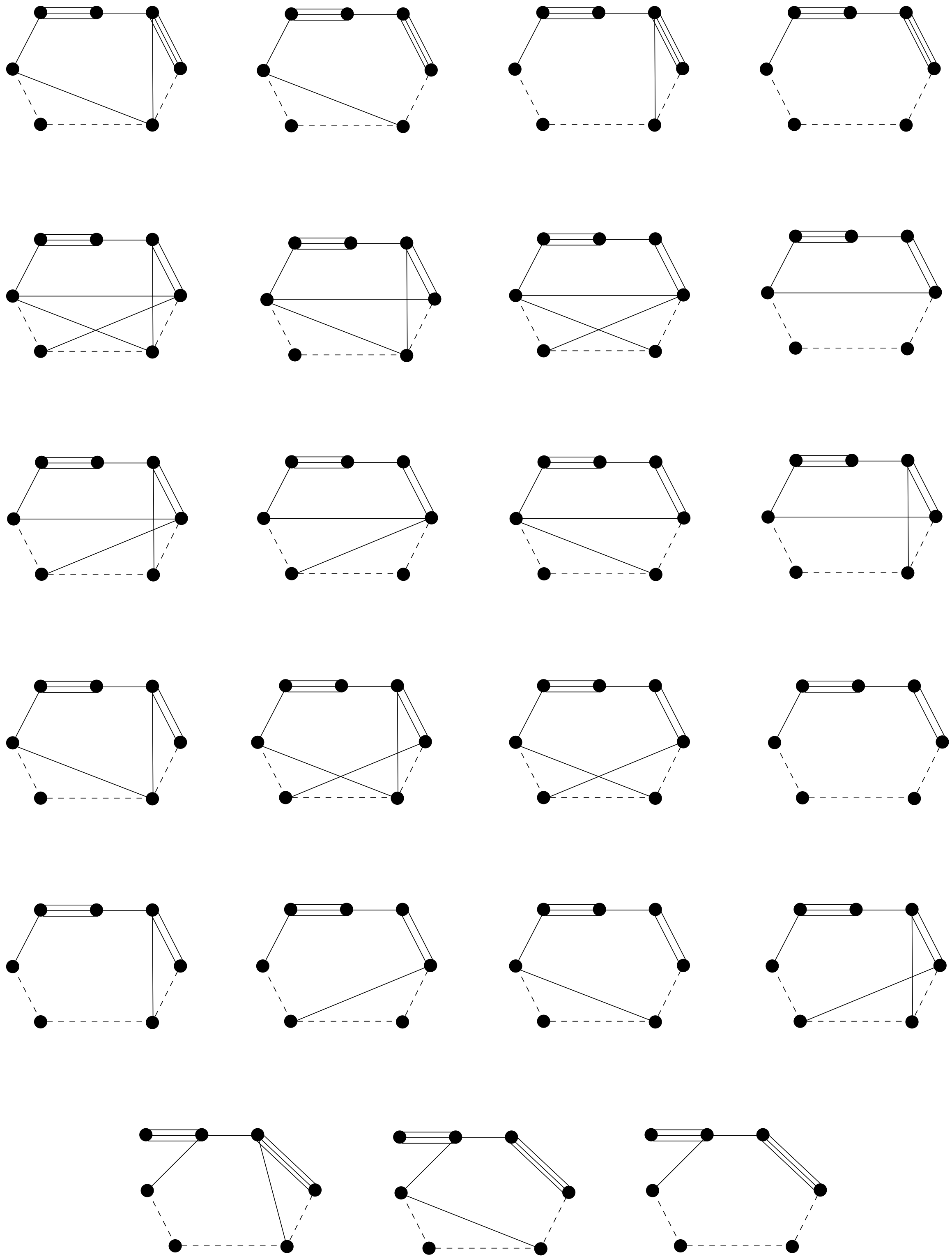
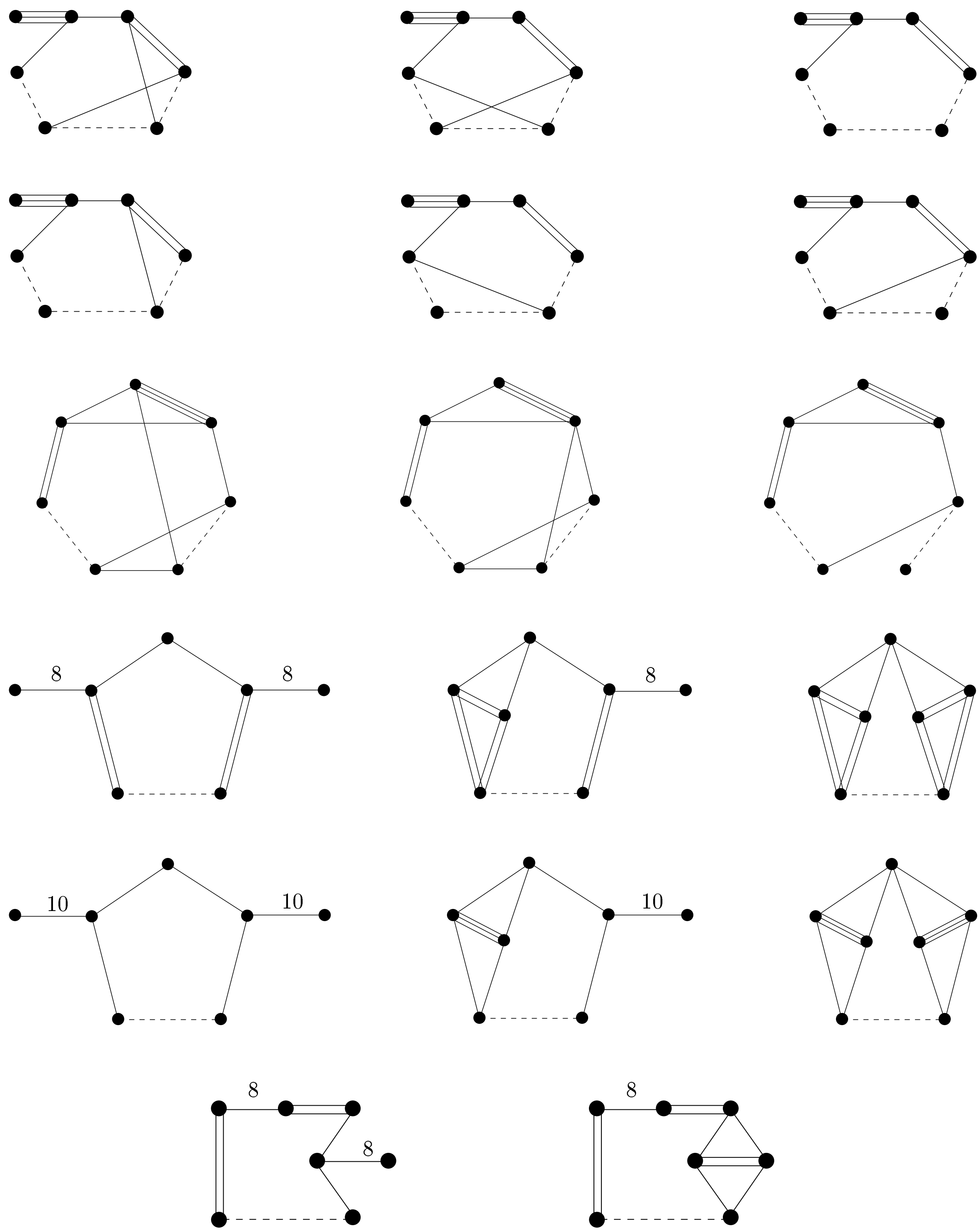


Table 4.11: Cont.



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