Existentially Closed BIBD Block-Intersection Graphs

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Abstract

A graph G with vertex set V is said to be n-existentially closed if, for every $S \subset V$ with $|S| = n$ and every $T \subseteq S$, there exists a vertex $x \in V - S$ such that x is adjacent to each vertex of T but is adjacent to no vertex of $S - T$. Given a combinatorial design $\mathcal{D}$ with block set $\mathcal{B}$, its block-intersection graph $G_{\mathcal{D}}$ is the graph having vertex set $\mathcal{B}$ such that two vertices b_1 and b_2 are adjacent if and only if b_1 and b_2 have non-empty intersection.

In this paper we study balanced incomplete block designs (BIBDs) and when their block-intersection graphs are n-existentially closed. We characterise the BIBDs with block size $k \geqslant 3$ and index $\lambda = 1$ that have 2-e.c. block-intersection graphs and establish bounds on the parameters of BIBDs with index $\lambda = 1$ that are n-e.c. where $n \geqslant 3$. For $\lambda \geqslant 2$ and $n \geqslant 2$, we prove that only simple λ-fold designs can have n-e.c. block-intersection graphs. In the case of λ-fold triple systems we show that $n \geqslant 3$ is impossible, and we determine which 2-fold triple systems (i.e., BIBDs with $k = 3$ and $\lambda = 2$) have 2-e.c. block-intersection graphs.

Keywords: block designs; block-intersection graphs; existentially closed graphs

1 Introduction

Erdős and Rényi first introduced the concept of n-existentially closed graphs when considering random graphs [2]. Specifically, a graph G with vertex set V is *n-existentially*

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closed (or *n-e.c.*) if, for each proper subset S of V with cardinality $|S| = n$ and each subset T of S, there exists some vertex x not in S that is adjacent to each vertex of T but to none of the vertices of $S - T$. Hence a graph G is 1-e.c. if and only if no vertex of G has degree 0 or $(|V| - 1)$. A survey paper on the topic of existentially closed graphs appears in [1].

A *balanced incomplete block design* of order v, having block size k and index λ, or BIBD(v, k, λ), is an ordered pair $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ is a collection of k-subsets of V known as *blocks* such that every pair of points of V occurs in exactly λ blocks of $\mathcal{B}$. The *block-intersection graph* of such a design $\mathcal{D}$ is the graph $G_{\mathcal{D}}$ having vertex set $\mathcal{B}$, and in which two vertices are adjacent if and only if their corresponding blocks share at least one point of V. For further information on combinatorial designs, the reader is referred to [7].

In [3], Forbes, Grannell and Griggs studied the block-interesection graphs of Steiner triple systems; a Steiner triple system of order v, or STS(v), is just a BIBD with parameters $(v, 3, 1)$. In particular, they considered when Steiner triple systems have n-existentially closed block-intersection graphs. For $n = 2$ they concluded that a STS(v) has a 2-e.c. block-intersection graph if and only if $v \geqslant 13$. For $n = 3$ they determined that for a STS(v) to have a 3-e.c. block-intersection graph v must be either 19 or 21. Two STS(19) with 3-e.c. block-intersection graphs were presented, while the existence of a STS(21) with a 3-e.c. block-intersection graph remained unsettled.

In this present paper we expand on the work of Forbes et al. by considering the more general setting of balanced incomplete block designs. We characterise those BIBD$(v, k, 1)$ with $k \geqslant 3$ that have 2-e.c. block-intersection graphs, and for $n \geqslant 3$ we obtain bounds on the design parameters for those BIBD$(v, k, 1)$ for which the block-intersection graph is n-e.c. When $\lambda \geqslant 2$, we obtain similar parameter bounds, and we prove that λ-fold designs with n-e.c. block-intersection graphs are necessarily simple. Further, we determine all BIBD$(v, 3, 2)$ which have 2-e.c. block-intersection graphs, noting that there are no BIBD$(v, 3, 2)$ with n-e.c. block-intersection graphs for any $n \geqslant 3$.

2 A Few Preliminaries

We begin by observing the following result, the proof of which is trivial.

Lemma 1 *If a graph G is n-e.c. and $n > 1$, then G is also $(n-1)$-e.c.*

Another simple result pertaining to block-intersection graphs is:

Lemma 2 *If $\mathcal{D} = (V, \mathcal{B})$ is a BIBD(v, k, λ) such that $G_{\mathcal{D}}$ is n-e.c., then $v \geqslant (n+1)k$.*

Proof. Since $G_{\mathcal{D}}$ is n-e.c., then for each n-set $S \subset \mathcal{B}$ by selecting $T = \emptyset$ we find that there must exist a vertex x that is adjacent to no vertex of S. Inductively, and by using Lemma 1, we conclude that $G_{\mathcal{D}}$ must contain a set of $(n+1)$ independent vertices. Equivalently, the design must contain a set of $(n+1)$ pairwise non-intersecting blocks, each containing k points. Hence $v \geqslant (n+1)k$. □

Throughout the remainder of this paper we adopt the convention that if $\mathcal{D} = (V, \mathcal{B})$ is a BIBD(v, k, λ), then $V = \{1, 2, \ldots, v\}$.

We now show that the existential closure of any BIBD is bounded by the block size of the design:

Theorem 1 *If the block-intersection graph of a BIBD(v, k, λ) is n-e.c., then $n \leqslant k$.*

Proof. Suppose that $\mathcal{D}$ is a BIBD(v, k, λ) for which $G_{\mathcal{D}}$ is n-e.c. Without loss of generality, we may assume that $\mathcal{D}$ contains the block $b = \{1, 2, \ldots, k\}$.

First consider the case in which $k < v$. For each $i = 1, 2, \ldots, k$, let b_i be a block of $\mathcal{D}$ that contains both point i as well as point v, and let $B = \{b_1, b_2, \ldots, b_k\}$ (when $\lambda \geqslant 2$, it may be that $b_1, b_2, \ldots, b_k$ are not all distinct, in which case $|B| < k$). Observe that there is no block x that intersects b but none of $b_1, b_2, \ldots, b_k$. Hence in $G_{\mathcal{D}}$ there can be no vertex x that is adjacent to each vertex of $T = \{b\}$ but no vertex of $S - T$ where $S = T \cup B$. Thus $G_{\mathcal{D}}$ is not $(|B| + 1)$-e.c., and by Lemma 1 it follows that $n \ngeqslant |B| + 1$. Hence $n \leqslant |B| \leqslant k$.

In the case where $k = v$, we find that the design $\mathcal{D}$ consists of λ copies of a single block, so that $G_{\mathcal{D}}$ is just K_{λ}. Hence the hypothesis that $G_{\mathcal{D}}$ is n-e.c. fails to be satisfied, regardless of the value of n. $\square$

3 Designs with Index $\boldsymbol{\lambda = 1}$

In this section we consider designs having index $\lambda = 1$. We begin by considering the case of $n = 2$, for which we obtain a characterisation for BIBDs with n-e.c. block-intersection graphs:

Theorem 2 *The block-intersection graph of a BIBD$(v, k, 1)$ with $k \geqslant 3$ is 2-existentially closed if and only if $v \geqslant k^2 + k - 1$.*

Proof. For any BIBD$(v, k, 1)$ $\mathcal{D} = (V, \mathcal{B})$, the replication number r of any point of V is $r = \frac{v-1}{k-1}$. Hence $r \geqslant k + 2$ if and only if $v \geqslant k^2 + k - 1$.

To prove the forward implication, let $\mathcal{D}$ be a BIBD$(v, k, 1)$ such that $k \geqslant 3$ and $G_{\mathcal{D}}$ is 2-e.c. Then the design must contain a pair of disjoint blocks, so without loss of generality let $b_1 = \{1, 2, \ldots, k\}$ and $b_2 = \{k + 1, k + 2, \ldots, 2k\}$ be disjoint elements of $\mathcal{B}$.

Now consider the 2-subset $S = \{b_1, b_2\}$ of $\mathcal{B}$ and the subset $T = \{b_1\}$ of S. Since $G_{\mathcal{D}}$ is 2-e.c., then there must exist a third block, say b_3, such that $|b_1 \cap b_3| = 1$ but $b_2 \cap b_3 = \emptyset$. Without loss of generality, let $b_3 = \{1, 2k + 1, 2k + 2, \ldots, 3k - 1\}$.

For each $i \in b_2$ consider the unique block, say b_i, that contains the pair $\{1, i\}$. Note that point 1 is now seen to occur in at least $k + 2$ blocks (viz. b_1, b_3, and the blocks $b_{k+1}, \ldots, b_{2k}$) and that point 1 is the only point shared by any pair of these blocks. Hence $r \geqslant k + 2$.

We now prove the converse implication. Let $\mathcal{D}$ be a BIBD$(v, k, 1)$ with $k \geqslant 3$ such that $v \geqslant k^2 + k - 1$. Let b_1 and b_2 be two distinct blocks of $\mathcal{B}$ and let $S = \{b_1, b_2\}$. For each subset T of S, we must show that there exists a vertex x of $\mathcal{B} - S$ that is adjacent in $G_{\mathcal{D}}$ to each vertex of T but to no vertex of $S - T$.

Case 1. $|b_1 \cap b_2| = 1$

Without loss of generality, let $b_1 = \{1, 2, \ldots, k\}$ and $b_2 = \{k, k+1, \ldots, 2k-1\}$.

If $T = S$, then let x be the unique block that contains the pair $\{1, k+1\}$.

If $|T| = 1$, then without loss of generality assume that $T = \{b_1\}$. For each $i \in \{k+1, k+2, \ldots, 2k-1\}$, let b_i be the unique block that contains the pair $\{1, i\}$. The k points of b_2 are therefore paired with point 1 in the k blocks $b_1, b_{k+1}, b_{k+2}, \ldots, b_{2k-1}$. Recall that $r \geqslant k+2$ and so there exists a block x not in S that contains point 1 but none of the points of b_2.

If $T = \emptyset$, then let $\nu(t, k-t)$ denote the number of blocks of $\mathcal{B}$ having exactly t points from the set $W = \{1, 2, \ldots, 2k-1\}$ and exactly $k-t$ points from the set $V - W$. Trivially we obtain $\nu(k, 0) = 2$, $\nu(t, k-t) = 0$ for each $t \in \{k-1, k-2, \ldots, 3\}$, and $\nu(2, k-2) = (k-1)^2$.

Of the $(2k-1)(v-(2k-1))$ pairs of points formed by selecting one point from W and another from $V - W$, each of the $(k-1)^2$ blocks enumerated by $\nu(2, k-2)$ contains $2(k-2)$ of them. Thus $\nu(1, k-1) = \frac{(2k-1)(v-(2k-1))-2(k-2)(k-1)^2}{k-1}$.

Since $|\mathcal{B}| = \frac{v(v-1)}{k(k-1)}$ in any BIBD$(v, k, 1)$, it now follows that $\nu(0, k) = \frac{v(v-1)}{k(k-1)} - \nu(k, 0) - \nu(2, k-2) - \nu(1, k-1) = \frac{v^2-v+k^2+k^4-k^3-2k^2v+kv}{k(k-1)}$. We need to establish that $\nu(0, k) \geqslant 1$, but since $\nu(0, k)$ is an integer and $k \geqslant 3$, it is sufficient to prove that $N = (v^2 - v + k^2 + k^4 - k^3 - 2k^2v + kv)$ is positive. Recalling that $v \geqslant k^2 + k - 1$, let $\epsilon = v - (k^2 + k - 1) \geqslant 0$ so that we now have $N = (2k^2 - 4k + 3k\epsilon + \epsilon^2 - 3\epsilon + 2)$. Clearly $2k^2 > 4k$ and $3k\epsilon \geqslant 3\epsilon$ for all $k \geqslant 3$, and thus $N > 0$.

Case 2. $|b_1 \cap b_2| = 0$

Without loss of generality, let $b_1 = \{1, 2, \ldots, k\}$ and $b_2 = \{k+1, \ldots, 2k\}$.

If $T = S$, then let x be the unique block that contains the pair $\{1, k+1\}$.

If $|T| = 1$, then without loss of generality assume that $T = \{b_1\}$. For each $i \in b_2$, let b_i be the unique block that contains the pair $\{1, i\}$. The k points of b_2 are therefore paired with point 1 in the k blocks $b_{k+1}, b_{k+2}, \ldots, b_{2k}$. Recall that $r \geqslant k+2$ and so there exists a block $x \neq b_1$ that contains point 1 but none of the points of b_2.

If $T = \emptyset$, then let $\nu(t, k-t)$ denote the number of blocks of $\mathcal{B}$ having exactly t points from the set $W = \{1, 2, \ldots, 2k\}$ and exactly $k-t$ points from the set $V - W$. Trivially we obtain $\nu(k, 0) = 2$, $\nu(t, k-t) = 0$ for each $t \in \{k-1, k-2, \ldots, 3\}$, and $\nu(2, k-2) = k^2$. It follows that $\nu(1, k-1) = \frac{2k(v-2k)-2(k-2)(k^2)}{k-1}$ and hence $\nu(0, k) = \frac{v(v-1)}{k(k-1)} - \nu(k, 0) - \nu(2, k-2) - \nu(1, k-1) = \frac{v^2-v-2k^2+2k+k^4+k^3-2k^2v}{k(k-1)}$. Let $N = (v^2 - v - 2k^2 + 2k + k^4 + k^3 - 2k^2v)$ and let $\epsilon = v - (k^2 + k - 1) \geqslant 0$. Then $N = (k^3 - 2k^2 - k + 2k\epsilon + \epsilon^2 - 3\epsilon + 2)$ is positive since $(k^3 - 2k^2 - k) > 0$ and $2k\epsilon \geqslant 3\epsilon$ whenever $k \geqslant 3$. $\square$

We now obtain Theorem 3.1 of [3] as a corollary:

Corollary 1 *The block-intersection graph of a STS*(v) *is* 2*-e.c. if and only if* $v \geqslant 13$.

Proof. Theorem 2 asserts that the block-intersection graph of a STS(v) is 2-e.c. if and only if $v \geqslant 11$. However, a Steiner triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6), thereby eliminating orders 11 and 12 from consideration. $\square$

We now begin to consider those BIBD($v, k, 1$) having n-e.c. block-intersection graphs where n exceeds 2. At this point it is helpful to recall that an *independent set* of points in a design $\mathcal{D} = (V, \mathcal{B})$ is a subset $I \subset V$ such that no block of $\mathcal{B}$ has all of its points contained within I. By $I_k(v)$ we denote the maximum cardinality of an independent set in any BIBD(v, k, λ); i.e., the maximum over all BIBDs with parameters (v, k, λ). For triple systems, the value of $I_3(v)$ was established by Sauer and Schönheim to be at most $\frac{v+1}{2}$ [6]. We now present a modest generalisation of the result by Sauer and Schönheim:

Lemma 3 $I_k(v) \leqslant \dfrac{(v-1)(k-2)}{k-1} + 1.$

Proof. Let $\mathcal{D} = (V, \mathcal{B})$ be a BIBD(v, k, λ), let I be an independent set of points in $\mathcal{D}$, and let $p \in I$. Note that there are $|I| - 1$ pairs of points of the form $\{p, q\}$ where q is a point of I other than p, and that each such pair occurs λ times. Each block of the design can contain at most $(k-2)$ of these pairs without contradicting the status of I as an independent set. Hence there must be at least $\frac{\lambda(|I|-1)}{k-2}$ blocks that contain pairs of points of this form. Necessarily this quantity cannot exceed the replication number, $r = \frac{\lambda(v-1)}{k-1}$, of the design. Therefore $\frac{\lambda(|I|-1)}{k-2} \leqslant \frac{\lambda(v-1)}{k-1}$ and the result follows. $\square$

We now use Lemma 3 to establish upper bounds on the orders of designs having n-e.c. block-intersection graphs.

Theorem 3 *Let $n \geqslant 3$. If $\mathcal{D} = (V, \mathcal{B})$ is a BIBD($v, k, 1$) for which the block-intersection graph is n-e.c., then $v \leqslant k^4 - nk^3 + (2n-2)k^2 - nk + k + 1$.*

Proof. Since $G_{\mathcal{D}}$ is n-e.c., then there must exist a set of $(n+1)$ pairwise disjoint blocks in the design, although we do not require that many. Let D be a set of $(n-1)$ pairwise disjoint blocks in $\mathcal{B}$, and let b_1 and b_2 be two distinct blocks of D.

Let $B \subset \mathcal{B}$ be the set of all k^2 blocks that contain a point from b_1 as well as a point from b_2. Since $\lambda = 1$, each block $b \in B$ contains at most one point from each block of D, and hence b contains at least $\big(k - (n-1)\big)$ points that do not occur as points in any of the blocks of D (recall from Theorem 1 that $n \leqslant k$). So for each block $b \in B$, let P_b be an arbitrary but fixed set of $\big(k - (n-1)\big)$ points of b, none of which occur as a point of any block of D. Let $P = \left(\bigcup_{b \in D} b\right) \cup \left(\bigcup_{b \in B} P_b\right)$ and observe that $|P| \leqslant k(n-1) + (k-n+1)k^2$.

Suppose now that $v > \left(\frac{(v-1)(k-2)}{k-1} + 1\right) + \left(k(n-1) + (k-n+1)k^2\right)$. Then by Lemma 3, $v - |P| > I_k(v)$, and so there must exist a block, say b_n, that uses none of the points of P. Let $S = D \cup \{b_n\}$ and $T = S$. Since $G_{\mathcal{D}}$ is n-e.c., then there must exist a block x that intersects all n blocks of T. Since x intersects each of b_1 and b_2, then necessarily x must be one of the blocks of B. Moreover, since x intersects each block of D, and each pair of blocks of D has empty intersection, then x contains exactly one point from each of the $(n-1)$ blocks of D and so x contains exactly $\big(k-(n-1)\big)$ points not found in any block of D.

Note that x is also adjacent to b_n, which shares no points with any block of D. Hence x and b_n must intersect in one of the $\big(k-(n-1)\big)$ points of x that are not found in any block of D, meaning that x and b_n must intersect in a point of P_x. We therefore have a contradiction, since b_n contains none of the points of P_x. Hence $v \leqslant \big(\frac{(v-1)(k-2)}{k-1}+1\big)+\big(k(n-1)+(k-n+1)k^2\big)$. □

Theorem 3 has the effect of imposing an upper bound on the possible order v for any BIBD$(v,k,1)$ having an n-e.c. block-intersection graph. For instance, when coupled with Theorem 2 and Lemma 1, the only possible orders for a BIBD$(v,4,1)$ with a 3-e.c. block-intersection graph are seen to be those in the interval $19 \leqslant v \leqslant 121$, which can be refined to $25 \leqslant v \leqslant 121$ by noting that the existence of a BIBD$(v,4,1)$ requires that $v \equiv 1$ or 4 (mod 12).

4 $\boldsymbol{\lambda}$-fold Designs

We now consider BIBDs in which the index λ need not be 1. When $\lambda \geqslant 2$, it is possible for designs to contain repeated blocks. A design with no repeated blocks is said to be *simple*, and as is now shown, only for simple designs can the block-intersection graph be n-e.c. when $n \geqslant 2$.

Lemma 4 *If $\lambda \geqslant 2$, $n \geqslant 2$, and $\mathcal{D}=(V,\mathcal{B})$ is a BIBD(v,k,λ) such that $G_{\mathcal{D}}$ is n-e.c., then $\mathcal{D}$ is simple.*

Proof. By way of contradiction, suppose that $\mathcal{B}$ contains repeated blocks, say b_1 and b_1' (so $|b_1 \cap b_1'| = k$). By choosing $S=\{b_1,b_1'\}$ and $T=\{b_1\}$ we find that there is no vertex x of $G_{\mathcal{D}}$ that is adjacent to b_1 but not adjacent to b_1'. Hence $G_{\mathcal{D}}$ cannot be 2-e.c., and by Lemma 1, $G_{\mathcal{D}}$ cannot be n-e.c. for any $n \geqslant 2$. Thus we have the desired contradiction. □

Paralleling the situation with 1-fold designs, upper bounds on the orders of designs having n-e.c. block-intersection graphs can be obtained for λ-fold designs:

Theorem 4 *Let $n \geqslant 3$. If $\mathcal{D}=(V,\mathcal{B})$ is a BIBD(v,k,λ) for which the block-intersection graph is n-e.c., then $v \leqslant \lambda k^4 - \lambda n k^3 + (\lambda+1)(n-1)k^2 - nk + k + 1$.*

Proof. The proof is similar to that of Theorem 3, except that $|B| \leqslant \lambda k^2$, and for each $b \in B$ containing $\big(k-(n-1)\big)$ or more points not in D we define P_b to be a fixed set of $\big(k-(n-1)\big)$ of these points, whereas for those b having fewer than $\big(k-(n-1)\big)$ such points we define P_b to be all of the points of b which do not appear in D. Hence $|P| \leqslant k(n-1)+\lambda k^2(k-n+1)$. □

Moreover, when $\lambda \geqslant 2$ we can improve on the bound established by Theorem 1:

Theorem 5 *If $\lambda \geqslant 2$ and $\mathcal{D} = (V, \mathcal{B})$ is a BIBD(v, k, λ) such that $G_{\mathcal{D}}$ is n-e.c., then $n \leqslant \lfloor \frac{k+1}{2} \rfloor$.*

Proof. Suppose that $\mathcal{D}$ is a BIBD(v, k, λ) for which $G_{\mathcal{D}}$ is n-e.c. Without loss of generality, we may assume that $\mathcal{B}$ contains the block $b = \{1, 2, \ldots, k\}$. Let $T = \{b\}$.

If k is even, then for each $i = 1, 2, \ldots, \frac{k}{2}$, let b_i be a block other than b that contains the two points $(2i-1)$ and $2i$. Let $B = \{b_1, \ldots, b_{\frac{k}{2}}\}$ and let $S = T \cup B$. Possibly $b_i = b_j$ for some $1 \leqslant i < j \leqslant \frac{k}{2}$, in which case $|B| < \frac{k}{2}$.

If k is odd, then for each $i = 1, 2, \ldots, \frac{k-1}{2}$, let b_i be a block other than b that contains the two points $(2i-1)$ and $2i$. Let $b_{\frac{k+1}{2}}$ be a block other than b that contains point k. Let $S = T \cup \{b_1, b_2, \ldots, b_{\frac{k+1}{2}}\}$.

Observe now that in $G_{\mathcal{D}}$ there is no vertex x that is adjacent to b but to no vertex of $S - T$ and hence $G_{\mathcal{D}}$ is not $|S|$-e.c. It follows from Lemma 1 that $n \leqslant |S| - 1 \leqslant \lfloor \frac{k+1}{2} \rfloor$. $\square$

Theorem 5, as it applies to triple systems, states that there are no λ-fold triple systems with 3-e.c. block-intersection graphs. Accordingly, we now focus on the question of determining which simple λ-fold triple systems have 2-e.c. block-intersection graphs. We consider in detail the case of $\lambda = 2$, beginning by showing that every simple BIBD$(v, 3, 2)$ with $v \geqslant 13$ has a 2-e.c. block-intersection graph.

Lemma 5 *If $\mathcal{D} = (V, \mathcal{B})$ is a simple BIBD$(v, 3, 2)$ such that $v \geqslant 13$, then $G_{\mathcal{D}}$ is 2-e.c.*

Proof. For any BIBD(v, k, λ), the replication number r of any point of V is $r = \frac{\lambda(v-1)}{k-1}$, and so for the design $\mathcal{D}$ we have $r = v - 1 \geqslant 12$.

Let b_1 and b_2 be two distinct blocks of $\mathcal{B}$ and let $S = \{b_1, b_2\}$. For each subset T of S, we must show that there exists a vertex $x \in \mathcal{B} - S$ that is adjacent in $G_{\mathcal{D}}$ to each vertex of T but to no vertex of $S - T$.

Case 1. $|b_1 \cap b_2| = 2$

Without loss of generality, let $b_1 = \{1, 2, 3\}$ and $b_2 = \{1, 2, 4\}$.

If $T = S$, then let x be one of the two blocks that contain the pair $\{3, 4\}$.

If $|T| = 1$, then without loss of generality assume that $T = \{b_1\}$. Consider the blocks that contain the pairs of points $\{1, 3\}$, $\{2, 3\}$, and $\{3, 4\}$. Since $\lambda = 2$ there can be at most six such blocks; in fact there can be at most five, since b_1 contains an instance of the pair $\{1, 3\}$ as well as an instance of the pair $\{2, 3\}$ together in a single block. These blocks all contain point 3, and they each intersect both of b_1 and b_2. However, there are at least $(r - 5)$ other blocks that also contain point 3 but none of the points of b_2. Since $r \geqslant 12$, then there exists a vertex x in $G_{\mathcal{D}}$ that is adjacent to b_1 but is not adjacent to b_2.

If $T = \emptyset$, then consider the blocks that contain point 5. Since $|b_1 \cup b_2| = 4$ and $\lambda = 2$, at most eight blocks of the design will contain point 5 as well as some point from $b_1 \cup b_2$. Since $r \geqslant 12$, then there exist other blocks that contain point 5 but none of the points of $b_1 \cup b_2$; any such block is a suitable choice for x.

Case 2. $|b_1 \cap b_2| = 1$
Without loss of generality, let $b_1 = \{1, 2, 3\}$ and $b_2 = \{1, 4, 5\}$.

If $T = S$, then let x be one of the two blocks that contain the pair $\{3, 4\}$.

If $|T| = 1$, then without loss of generality assume that $T = \{b_1\}$. Consider the blocks that contain the pairs of points $\{1, 3\}$, $\{3, 4\}$, and $\{3, 5\}$. Since $\lambda = 2$ there can be at most six such blocks. These blocks all contain point 3, and they each intersect both of b_1 and b_2. However, there are at least $r - 6 \geqslant 6$ other blocks that also contain point 3 but none of the points of b_2; any such block is a suitable choice for x.

If $T = \emptyset$, then consider the blocks that contain point 6. Since $|b_1 \cup b_2| = 5$ and $\lambda = 2$, at most ten blocks of the design will contain point 6 as well as some point from $b_1 \cup b_2$. Since $r \geqslant 12$, then there exist other blocks that contain point 6 but none of the points of $b_1 \cup b_2$; any such block is a suitable choice for x.

Case 3. $|b_1 \cap b_2| = 0$
Without loss of generality, let $b_1 = \{1, 2, 3\}$ and $b_2 = \{4, 5, 6\}$.

If $T = S$, then let x be one of the two blocks that contain the pair $\{3, 4\}$.

If $|T| = 1$, then without loss of generality assume that $T = \{b_1\}$. Consider the blocks that contain the pairs of points $\{3, 4\}$, $\{3, 5\}$, and $\{3, 6\}$. Since $\lambda = 2$ there can be at most six such blocks. These blocks all contain point 3, and they each intersect both of b_1 and b_2. However, there are at least $r - 6 \geqslant 6$ other blocks that also contain point 3 but none of the points of b_2; any such block is a suitable choice for x.

If $T = \emptyset$, then consider the blocks that contain point 7. Since $|b_1 \cup b_2| = 6$ and $\lambda = 2$, at most twelve blocks of the design will contain point 7 as well as some point from $b_1 \cup b_2$. If $r > 12$, then there exist other blocks that contain point 7 but none of the points of $b_1 \cup b_2$; any such block is a suitable choice for x.

However, if $r = 12$ then either $G_{\mathcal{D}}$ is 2-e.c. or else each vertex of $G_{\mathcal{D}}$ is a neighbour of at least one of b_1 and b_2. We suppose that the latter of these two cases is at hand, in which case each block of $\mathcal{B}$ must contain a point from $b_1 \cup b_2$. In particular, each of the $r = 12$ blocks in which point 7 occurs must also contain a point from $b_1 \cup b_2$. Since $\lambda = 2$, point 7 is paired with each point of $b_1 \cup b_2$ exactly twice, yielding a total of 12 pairings throughout the design. Hence each of the $r = 12$ blocks in which point 7 occurs must contain exactly one pair of points of the form $\{p, 7\}$ where $p \in b_1 \cup b_2$. Consider now the subset $W = \{8, 9, 10, 11, 12, 13\}$ of V. There are $\lambda\binom{6}{2} = 30$ pairs of points of the form $\{p, q\}$ where $p, q \in W$, each of which must be in a block with a point of $b_1 \cup b_2$. Hence there are exactly 30 blocks of the form $\{z, p, q\}$ where $z \in b_1 \cup b_2$ and $p, q \in W$. A further 12 blocks are of the form $\{z, 7, p\}$ where $z \in b_1 \cup b_2$ and $p \in W$. None of these 42 blocks contain any of the $\lambda\binom{6}{2} = 30$ pairs of points of the form $\{p, q\}$ where $p, q \in b_1 \cup b_2$. Thus the remaining $|\mathcal{B}| - 42 = 10$ blocks (i.e., b_1, b_2, and eight others) contain only points from $b_1 \cup b_2$ and so form a BIBD(6,3,2). As noted in [4], there is a unique BIBD(6,3,2). It contains no pair of disjoint blocks such as our b_1 and b_2, and so we have the desired contradiction. □

Next we eliminate several small orders from possible consideration. From Lemma 2, it follows that if $\mathcal{D} = (V, \mathcal{B})$ is a simple BIBD$(v, 3, 2)$ such that $G_{\mathcal{D}}$ is 2-e.c., then $v \geqslant 9$, so we only need to consider the admissible orders beginning with $v = 9$:

Lemma 6 *No simple BIBD$(9, 3, 2)$ has a 2-e.c. block-intersection graph.*

Proof. Suppose that $\mathcal{D} = (V, \mathcal{B})$ is a simple BIBD(9,3,2) such that $G_{\mathcal{D}}$ is 2-e.c. and without loss of generality assume that $b_1 = \{1, 2, 3\} \in \mathcal{B}$. Note that $r = 8$ and so there are precisely 18 blocks that intersect b_1, namely the three other blocks containing the pairs $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ plus $3(r - 3)$ blocks that each intersect b_1 in a single point.

Since $|\mathcal{B}| = 24$, then there are exactly five blocks that are disjoint from b_1. Let b_2 be a block that is disjoint from b_1. By selecting $S = \{b_1, b_2\}$ and $T = \emptyset$ then it follows from $G_{\mathcal{D}}$ being 2-e.c. that there must be a third block that is disjoint from both b_1 and b_2. Since $v = 9$, this third block must be $V - (b_1 \cup b_2)$. Thus for each block b_2 that is disjoint from b_1, there is a corresponding block $b_2' = V - (b_1 \cup b_2)$ that is also disjoint from b_1. Hence the five blocks that are disjoint from b_1 can be naturally partitioned into pairs, which is a contradiction since 5 is an odd integer. □

Lemma 7 *No simple BIBD$(10, 3, 2)$ has a 2-e.c. block-intersection graph.*

Proof. Of the 960 non-isomorphic BIBD$(10, 3, 2)$ designs, only 394 are simple. We tested each of these by computer and found that none of them have a 2-e.c. block-intersection graph. □

Östergård [5] reported that there are 88616310 non-isomorphic simple BIBD$(12, 3, 2)$ designs. We generated these designs (thereby independently confirming their enumeration) and subsequently tested each of their block-intersection graphs to determine which ones are 2-e.c. The vast majority are 2-e.c., whereas as only 286962 fail to be 2-e.c.

We now summarise the status of 2-fold triple systems with the following theorem:

Theorem 6 *If $\mathcal{D}$ is a simple BIBD$(v, 3, 2)$ and $G_{\mathcal{D}}$ is 2-e.c., then $v \geqslant 12$. Conversely, the block-intersection graph of any simple BIBD$(v, 3, 2)$ with $v \geqslant 13$ has a 2-e.c. block-intersection graph.*

5 Discussion

Reflecting on Theorem 1, we pose the following question:

Question 1 *For each $k \geqslant 3$, does there exist a BIBD$(v, k, 1)$ that has a k-e.c. block-intersection graph?*

This question is answered in the affirmative for $k = 3$ as Forbes et al. found two STS(19) designs with 3-e.c. block-intersection graphs [3].

For designs with index $\lambda \geqslant 2$, Theorem 5 suggests that the corresponding question should be:

Question 2 *For each $\lambda \geqslant 2$ and $k \geqslant 3$, does there exist a BIBD(v, k, λ) for which the block-intersection graph is $\lfloor \frac{k+1}{2} \rfloor$-e.c.?*

In the case where $\lambda = 2$ and $k = 3$, Theorem 6 establishes an affirmative answer to this question.

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