Directed animals and gas models revisited

Yvan Le Borgne and Jean-François Marckert

CNRS, LaBRI
Université Bordeaux 1
351 cours de la Libération
33405 Talence cedex, France

email: {borgne,marckert}@labri.fr

Submitted: Jul 10, 2007; Accepted: Oct 29, 2007; Published: Nov 5, 2007
Mathematics Subject Classification: 82B41, 82B43,76M28

Abstract

In this paper, we revisit the enumeration of directed animals using gas models. We show that there exists a natural construction of random directed animals on any directed graph together with a particles system – a gas model with nearest exclusion – that explains combinatorialy the formal link known between the density of the gas model and the generating function of directed animals counted according to the area. This provides some new methods to compute the generating function of directed animals counted according to area, and leads in the particular case of the square lattice to new combinatorial results and questions. A model of gas related to directed animals counted according to area and perimeter on any directed graph is also exhibited.

1 Introduction

1.1 Directed animals on a directed graph

Let $G = (V, E)$ be a connected graph with set of vertices V and set of edges E. An animal A in G is a subset of V such that between any two vertices u and v in A, there is a path in G having all its vertices in A. The vertices of A are called cells and the number of cells is denoted $|A|$ and called the area of A. A neighbor u of A is a vertex of G which is not in A and such that there exists $e \in E$ between u and a vertex $v(u)$ of A. The perimeter of A, denoted by $\mathcal{P}(A)$, is the set of neighbors of A and its cardinality is denoted by $|\mathcal{P}(A)|$.

Directed animals (DA) are animals built on a directed graph G:

Definition 1.1 *Let A and S be two subsets of V, with* **finite or infinite** *cardinalities. We say that A is a DA with source S, if S is a subset of A such that any vertex of A can be reached from an element of S through a directed path having all its vertices in A.*

In the setting of DA, the definition of cells and area are the same as in the case of animals, but the notion of neighbor is changed since the edge $(v(u), u)$ is required to be a directed edge of G. If there is a directed edge from v to u in G, then u is said to be a child of v, and v to be the father of u; this induces a notion of descendant and ancestor. Each node of $\mathcal{P}(A)$ has at least one father in A. In this paper, we deal only with DA built on graphs having some suitable properties:

Definition 1.2 *A directed graph $G = (V, E)$ is said to be agreeable if*
(A) G does not contain multiple edges,
(B) the graph G has no directed cycles,
(C) the number of children of each node is finite.

First, if G and G' are two graphs with the same set of edges up to their multiplicities, then the set of DA of G and G' coincide. Hence, Condition (A) is mainly set to avoid complications in Section 2.4. Condition (C) is needed to have a finite number of DA with a given area, for all sources. Notice that an agreeable graph is not necessarily connected, and is not necessarily locally finite (some nodes may have an infinite incoming degree). Even if never recalled in the statements of the results, V is supposed to be finite or countable.

As examples, (finite or infinite) trees and forests are agreeable graphs, the square lattice $\mathsf{Sq} = \mathbb{Z}^2$ directed in such a way that the vertex (x, y) has as children $(x, y+1)$ and $(x+1, y+1)$ is agreeable (as well as all usual directed lattices).

A subset S of V is said to be free if for any $x, y \in S$, $x \neq y$, x is not an ancestor of y. For any DA A, the set

$$\mathcal{S}(A) := \{x, x \in A, x \text{ has no father in } A\}$$

is a free subset of V and is the unique minimal source of A according to the inclusion partial order (it is also the intersection of all possibles sources of A).

We denote by $\mathcal{A}(S, G)$ (or more simply $\mathcal{A}(S)$ when no confusion on G is possible), the set of finite or infinite DA on G with source S, and by $\mathbf{G}_S^G$ (or $\mathbf{G}_S$) the generating function (GF) of finite DA counted according to the area:

$$\mathbf{G}_S^G(x) := x^{|S|} + \dots$$

where $\dots$ stands for a sum of monomials whose degree are at least $|S| + 1$. We will need also sometimes to consider DA having their sources included (or equal) in a given set S. For such a DA, S is called an *over-source* . We denote by $\overline{\mathcal{A}(S, G)}$ (or $\overline{\mathcal{A}(S)}$) the set of DA having S as over-source, and by $\overline{\mathbf{G}}_S^G$ (or $\overline{\mathbf{G}}_S$) its GF ($\overline{\mathbf{G}}_S^G(x) = 1 + |S|x + \dots$). Finally, we set $\mathbf{G}_\varnothing^G = \overline{\mathbf{G}}_\varnothing^G = 1$.

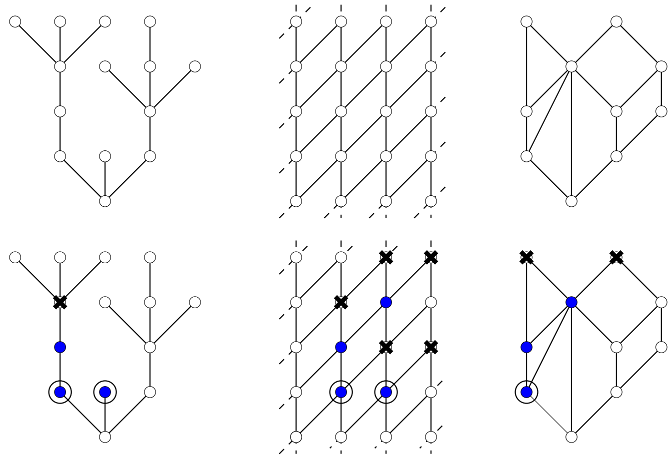

Figure 1: On the first line, three examples of agreeable graphs on which the directed edges are directed upwards. The first example is a tree, the second is the square lattice and the third a "non-layered" agreeable graph. On the second line are represented some DA on these graphs; filled points are the cells, the crosses are perimeter sites, and the surrounded points are the minimal sources of these DA.

When dealing with elements A of $\overline{\mathcal{A}(S,G)}$, the set $S \setminus A$ are also considered as (special) perimeter sites, and we set $\overline{\mathcal{P}_S}(A) = \mathcal{P}(A) \cup (S \setminus A)$.

We introduce a notion of particle systems, or gas occupation, on a graph:

Definition 1.3 *Let $G = (V, E)$ be a graph. A particle system on G, or gas occupation on G, is a map X from V to $\{0, 1\}$. The vertices v such that $X(v) = 1$ are said to be occupied, the others are said to be empty.*

In the physic literature, a hard particle model on a graph (or a gas model with nearest neighbor exclusion) is a (probability model of) gas occupation X with the additional constraint that the occupied sites are not neighbors in G. In the following, we will construct some random model of gas: $(X(v))_{v\in V}$ will be seen as a family of random variables indexed by the set of vertices V.

Definition 1.4 *When a random gas model X is defined on a probability space $(\Omega, \mathbb{P})$, we call density of the gas model at a vertex x, the probability $\mathbb{P}(X(x) = 1)$.*

1.2 Contents

The equivalence between the enumeration of DA on lattices according to the area and the computation of the density of a hard particle model has been a key point in the study of DA from its very beginning with the works of Dhar [8], [9]. Dhar [9], using statistical mechanics techniques, explains that in special cases there exists an exact equivalence between the enumeration of DA in d dimensions and the computation of the free-energy of a $(d-1)$-dimensional lattice gas, with nearest neighbor exclusion. He then computes the density of this gas on the square lattice and finds the GF of DA on the square lattice. This work takes place after some investigations of Nadal & al. [14] and Hakim and Nadal [12]. In order to prove some conjectures of Dhar & al. [10] on the enumeration of DA on the square lattice, [14] and [12] studied the (easier) problem of enumeration of DA on a "cylinder" of the square lattice (a cylinder is a strip of the lattice in which are identified the two infinite borders). The problem of computing the generating function of DA with a given source on this cylinder is totally solved by Hakim and Nadal [12]. As said later on in the Appendix, knowing exactly the GF of DA on the cylinder allows in principle to get the GF of DA on the whole lattice by a formal passage to the limit, but the expressions obtained by [12] seems to be not really suitable for that. It is worth mentioning that Nadal & al. [12, appendix B] contains the main idea allowing to pass from a model of gas with nearest neighbor exclusion in the square lattice to the enumeration of DA; they do not use the notion"gas", but they indeed construct an object which is equivalent to the gas with nearest neighbor exclusion, via an exclusion-inclusion principle.

The work of Dhar, and later on of Bousquet-Mélou [4] and Bousquet-Mélou & Conway [3] raise on the fact that the GF of DA on a lattice, and a gas with nearest neighbor exclusion on the same lattice, have the same "recursive decomposition" (this is, up to an inclusion-exclusion procedure, what is made by [14, 12]). This recursive decomposition is done using a decomposition of the lattice itself by layers (see Section 5.3, where the approach used by Bousquet-Mélou [4] is detailed).

To solve the equation involving the GF of DA obtained by this recursive decomposition some properties of the gas model are used. The gas model is a stochastic process indexed by the lattice having in the tractable cases some nice Markovian-type properties on the layers. The arguments given in [4, 3] avoid the construction of the gas model on the whole lattice as done by Dhar: the gas model is defined on the layers of a cylinder, and the transition allowing to pass from a layer to the following one are of Markovian type. Bousquet-Mélou [4] finds an explicit solution of the gas process on a layer (in the square lattice case, in the triangular lattice case, and in other lattices in the joint work with Conway). Then, the computation of the density of the gas distribution is explicitly solved using that the number of configurations on such layers is finite, leading to rational GF. The GF of DA on the entire lattice (without the cyclical condition) is then obtained by a formal passage to the limit (see the Appendix).

In this paper we revisit the relation between the enumeration of DA on a lattice, and more generally on any agreeable graph, and the computation of the density of a model of gas. Our construction coincides with that used by Dhar or Bousquet-Mélou in their works. In Section 2, we explain how the usual construction of random DA on a graph G, using a Bernoulli coloring of the vertices of G, allows to define in the same time a random model of gas (that we qualified to be of type 1, and which is a model of gas with nearest neighbor exclusion). Here the construction is not done in "parallel" as in the works previously cited but on the same probability space; this provides a coupling of these objects. Using this coupling, we provide a general explanation of the fact that the GF of DA with source $\{x\}$ counted according to the area on any agreeable graph equals the density of the associated gas at vertex x, up to a simple change of variables (Theorem 2.7). This explanation is not of the same nature than in the previously cited works: the link between the density and the GF is not only formal but is explained combinatorially at the level of the DA (Section 2). Moreover, the construction of the gas model is possible not only on finite or regular graphs as lattices but on any agreeable graph, in a rigorous manner. The link between the density of the gas and the generating function of DA is then given directly on the whole graph. This allows to avoid the passage to the limit used by Bousquet-Mélou.

Dhar [9] also works on the whole lattice using a measure coming from the statistical mechanics. Even if morally our construction and that of Dhar should be the same, it is quite difficult to pass from a construction to the other. The reason is that the measure used by Dhar is in some sense a formal measure. The status of this measure appears clearly in Verhagen [15]: the weight $w(C)$ of a gas configuration C on the plane is given by the exponential of a simple function of two sums of non-zero integers depending of C (then the weight of C is non defined, or at least in a usual sense).

In Section 3, we revisit the study of DA on the square lattice; the new description of the gas model on the whole lattice allows us to provide a description of the gas model on a line (Theorem 3.3). On this line the gas model is a Markov chain which is identified. We provide then a new way to compute the GF of DA counted according to the area (Theorem 3.3). This extends to the enumeration of DA with any source on a line (Proposition 3.6): this was obtained on the cylinder by Nadal & al. [14] and Hakim and Nadal [12]. From there, a passage to the limit was also possible. We explain also how to compute the GF of DA with sources that are not contained in a line (Remark 3.8) and provide an example (Proposition 3.7).

In Section 4, we present an other model of gas, that we qualify to be of type 2 (this is not a gas model with nearest neighbor exclusion). The density of this gas model is related to the GF of DA counted according to the area and perimeter (Theorem 4.3). This construction explained once again at the level of object on any agreeable graph a relation used by Bousquet-Mélou [4] in a formal way on the square lattice. Even if we haven't find any deep application to this construction, we think that it provides an interesting generic approach to the computation of the GF of DA according to the area and perimeter, and it should lead to new results in the future.

Some other references concerning DA on lattices

One finds in the literature numerous works concerning the enumeration of DA on lattices, most of them avoids gas model considerations. We don't want to be exhaustive here (we send the reader to Bousquet-Mélou [4], Viennot [16, 17] and references therein), but we would like to indicate some combinatorial works directly related to this paper. It is interesting to notice that an important part of the papers cited below are combinatorial proofs of results found before using gas techniques.

First we refer to Viennot [17] and Bétréma & Penaud [6] for an algebraico-combinatorial relation between DA and heaps of pieces. This powerful point of view having some applications everywhere in the combinatorics, allows to compute the GF of DA on the triangular lattice, and by a change of variables on the square lattice (see also Dhar [8] for an other approach). A direct combinatorial enumeration of DA on the square lattice has been done by Bétréma & Penaud [5]; they found a bijection with a family of trees:"les arbres guingois".

Heap of pieces techniques have been used by Corteel, Denise & Gouyou-Beauchamps [7] to give a combinatorial enumeration of DA on some lattices, first counted by Bousquet-Mélou & Conway [3] using gas model (of type 1). Viennot and Gouyou-Beauchamps [11] provide a bijection between DA with compact sources on the square lattice and certain paths in the plane; they are able to enumerate these DA. Barcucci & al. [2] studied DA on the square and triangular lattices with the help of the ECO method. They found some relations with permutations with some forbidden subsequences and a family of trees.

2 Simultaneous construction of DA and gas model of type 1

In this part, we construct on any agreeable graph G a probability space on which are well-defined a model of gas – that we qualified to be "of type 1" – and a notion of random DA. This space is simply the space of the random colorings of the sites of G by independent Bernoulli random variables. Given a vertex $s \in G$, the coloring is first used to build a random animal $\mathbf{A}^s$ with over-source $\{s\}$, on the other hand, to compute the occupation of the gas at s. The relation between the density of this gas at s, and the GF of DA with source $\{s\}$, counted according to the area is then explained to be a simple combinatorial relation between two functionals of $\mathbf{A}^s$.

2.1 Construction of DA

Let $G = (V, E)$ be an agreeable graph. We introduce a random coloring of V by the two colors a and b. We need to be a little bit formal here since when G is infinite the existence of a probability space where such a construction is possible is not so obvious, and the measurability of our functions are not necessarily clear as one may see in the following Proposition (we recall that a random variable is a measurable function). We consider the probability space $\Omega = \{a, b\}^V$ and we let C be the identity mapping on Ω: for any $\omega \in \Omega$, $\omega = (C_x(\omega), x \in V)$, and then $C_x(\omega)$ gives the color of x for a global coloring ω. We equip Ω with the σ-algebra $\mathcal{F}$ generated by the cylinders: the cylinders are the subsets $\{\omega, C_x(\omega) = c_x, x \in I\}$ of Ω, with I finite subset of V and $(c_x)_{x\in I}$ a coloring of the points of I. In other words, they correspond to a specification of a coloring on a finite subset of V. We endow the space $(\Omega, \mathcal{F})$ with the measure product $\mathbb{P}_p = (p\delta_a + (1-p)\delta_b)^{\otimes V}$, where δ_a is the standard Dirac measure on $\{a\}$, and we denote by $\mathbb{E}_p$ the expectation under $\mathbb{P}_p$.

Hence, under $\mathbb{P}_p$, C is a random coloring of V, and the random variables $(C_x)_{x\in V}$ giving the color of the vertices of V are independent and take the value a and b with probability p and $1-p$.

Let $\omega \in \Omega$ and S be a subset of V. We denote by $S_\bullet(\omega) = \{x, x \in S, C_x(\omega) = a\}$ the subset of S having color a. We denote by $\mathbf{A}^S(\omega)$ the maximal DA for the inclusion partial order with source $S_\bullet(\omega)$ and whose cells are the vertices x such that $C_x(\omega) = a$ that can be reached from $S_\bullet(\omega)$ by an a-colored path. By construction the perimeter sites of $\mathbf{A}^S(\omega)$ are b-colored (see Fig. 2).

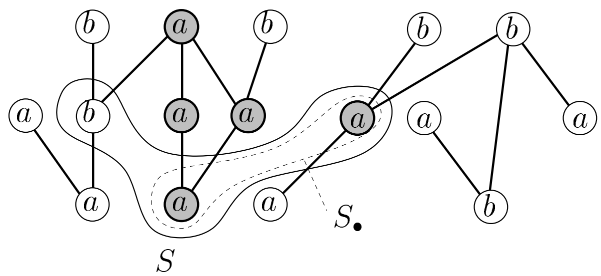

Figure 2: The DA $\mathbf{A}^S(\omega)$ is the set of gray cells.

In the following we equip $\mathcal{A}(S,G)$ with the σ-algebra $\mathcal{F}_S$, where $\mathcal{F}_S$ is the set of the subsets of $\mathcal{A}(S,G)$.

Proposition 2.1 *Let $G = (V,E)$ be an agreeable graph and S a free subset of V.*
(i) $\mathbf{A}^S$ is a measurable function from $(\Omega, \mathcal{F})$ onto $(\mathcal{A}(S,G), \mathcal{F}_S)$; in other words $\mathbf{A}^S$ is a random variable.
(ii) For any finite DA B in $\mathcal{A}(S,G)$ with source S,

$$\mathbb{P}_p(\mathbf{A}^S = B) = p^{|B|}(1-p)^{|\mathcal{P}(B)|}.$$

(iii) For any finite DA B in $\overline{\mathcal{A}(S,G)}$ with over-source S,

$$\mathbb{P}_p(\mathbf{A}^S = B) = p^{|B|}(1-p)^{|\overline{\mathcal{P}_S}(B)|}.$$

Proof (i) For any DA A with source S, we let $\Phi_h(A)$ be the set of cells of A having a graph distance to S smaller than h. We extend this definition to the sets E of DA with source S: if $E = \{A_i, i \in I\}$, $\Phi_h(E) = \{\Phi_h(A_i), i \in I\}$. For any fixed A with source S, $\{\omega, \mathbf{A}^S(\omega) = A) = \cap_h\{\omega, \Phi_h(\mathbf{A}^S(\omega)) = \Phi_h(A)\}$. But $\Phi_h(\mathbf{A}^S) = \Phi_h(A)$ is clearly a condition involving a finite number of cells, since S is finite and that the outdegrees of the nodes of G are finite. Hence the function $B \mapsto 1_A(B)$ is measurable. Now let $\mathcal{B} \in \mathcal{F}_S$.

We have to show that $(\mathbf{A}^S)^{-1}(\mathcal{B})$ belongs to $\mathcal{F}$. Write

$$\begin{aligned}(\mathbf{A}^S)^{-1}(\mathcal{B}) &= \{\omega, \mathbf{A}^S(\omega) \in \mathcal{B}\} \\ &= \cap_h \{\omega, \Phi_h(\mathbf{A}^S(\omega)) \in \Phi_h(\mathcal{B})\}.\end{aligned}$$

Again, since S is finite as well as the outdegrees of the nodes of G, the set $\Phi_h(\mathcal{B})$ contains a finite number of finite animals. Hence for any h, $\{\omega, \Phi_h(\mathbf{A}^S(\omega)) \in \Phi_h(\mathcal{B})\}$ is measurable, and then the measurability of $\mathbf{A}^S$ follows.
The proof of (ii) is immediate. For (iii) use moreover that $\overline{\mathcal{P}_S}(A) := \mathcal{P}(A) \cup (S \setminus A)$. $\square$

2.2 Directed animals and percolation

When G is an infinite graph and $|S| \geq 1$, under $\mathbb{P}_p$ the random DA $\mathbf{A}^S$ may be infinite with positive probability. The probability to have an infinite DA with source S is also that of the directed sites percolation starting from S where the cells of the percolation cluster are the vertices with color a reachable from S by an a-colored directed path.

Denote by p^S_{crit} the threshold for the existence of an infinite DA with positive probability:

$$p^S_{crit} = \sup\{\, p,\ \mathbb{P}_p(|\mathbf{A}^S| < +\infty) = 1\}.$$

Most of the results of the present paper are valid only when $p < p^S_{crit}$. The threshold p^S_{crit} is in general difficult to compute, but here is a simple sufficient condition on G for which $p^S_{crit} > 0$.

Proposition 2.2 *Let G be a agreeable graph such that the maximum number of children of its vertices is bounded by K. Then for any finite subset S of G, $p^S_{crit} \geq 1/K$.*

A proof of that result is given in the Appendix. Also in the Appendix, Comment 5.1 provides a graph in which $p_{crit} = 0$.

We recall two results giving some insight on the percolation probabilities (and easy to prove).
• Let S_1 and S_2 be two subsets of V. We have $p^{S_1 \cup S_2}_{crit} = \min\left\{ p^{S_1}_{crit}, p^{S_2}_{crit} \right\}$.
• Let $\mathbf{p}_{crit} = \inf_{v \in V} p^{\{v\}}_{crit}$. For any $p \in [0, \mathbf{p}_{crit})$, under $\mathbb{P}_p$, almost surely (a.s.) all DA in G having a finite source are finite (simultaneously). This is a consequence of the fact that a countable intersection of events having probability one has also probability one, and that the set of finite sources is at most countable.

2.3 Construction of the gas model of type 1

The construction of the gas model of type 1, Proposition 2.4 and the Nim game construction presented in this section are generalizations and formalization of the work of the first author [13, section 1.4].

Let us build a gas model X on an agreeable graph $G = (V, E)$ (see Definition 1.3). This construction takes place on the probability space Ω introduced in Section 2.1, and X is defined thanks to the random coloring C.

For any $x \in V$ and $\omega \in \Omega$, denote by

$$X_x(\omega) := \begin{cases} 0 & \text{if } C_x(\omega) = b \\ \displaystyle\prod_{c:\text{ children of } x} (1 - X_c(\omega)) & \text{if } C_x(\omega) = a \end{cases} \tag{1}$$

$$= \mathbb{1}_{C_x(\omega)=a} \prod_{c:\text{ children of } x} (1 - X_c(\omega)). \tag{2}$$

If x has no children the product in (1) is empty, and as usual, we set its value to 1.

For any ω and any $x \in V$, $X_x(\omega)$ is to be interpreted as the gas occupation in the vertex x.

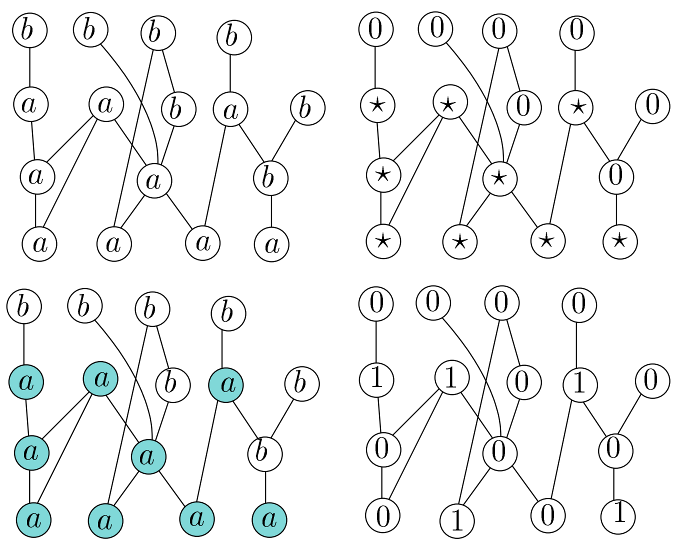

Figure 3: On the first column on top, a random coloring. Below, the family of DA derived from it. On the second column on top, the beginning of the computation of the gas occupation. "$\star$" stands for the places where the calculus $X_x \leftarrow \prod_{c:\text{ children of } x}(1 - X_c)$ must be done. Below, the gas occupation has been computed.

We have to investigate when the recursive definition giving $X_x(\omega)$ is correct, that is when it allows to indeed compute a value $X_x(\omega)$ (see Fig. 3).

- When $C_x(\omega) = b$ then $X_x(\omega) = 0$: there is no problem to define $X_x(\omega)$.
- When $C_x(\omega) = a$, to compute $X_x(\omega)$ it is sufficient to know all the values $X_y(\omega)$ for y child of x; their values are given by the same rule. By successive iterations, one can see that in each cells of $\mathbf{A}^{\{x\}}(\omega)$ – whose color are a by construction – the following computation is done

$$X_x(\omega) \leftarrow \prod_{c:\text{ children of } x} (1 - X_c(\omega));$$

since $C_y(\omega) = b$ for any perimeter sites of $\mathbf{A}^{\{x\}}(\omega)$, $X_y(\omega) = 0$ on $\mathcal{P}(\mathbf{A}^{\{x\}})(\omega))$. Then, one sees if $\mathbf{A}^{\{x\}}(\omega)$ is finite then $X_x(\omega)$ is well defined because this recursive computation of $X_x(\omega)$ ends. In this case, if $\mathbf{A}^{\{x\}}(\omega) = A$ the value $X_x(\omega)$ is a deterministic function of A that we denote by $\chi_x(A)$ (the map χ_x is defined only on finite DA with source x). The maps $(\chi_x)_{x\in V}$ satisfies then for simple reasons the following decomposition. Let v be a vertex in G, $A^{\{v\}}$ a finite DA with source v, and denote by $v_1, \dots, v_d$ the children of v in G, and $A^{\{v_1\}}, \dots, A^{\{v_d\}}$ be the maximal DA included in $A^{\{v\}}$ with over-source $\{v_1\}, \dots, \{v_d\}$ respectively. Then

$$\chi_v\left(A^{\{v\}}\right) = \mathbb{1}_{|A^{\{v\}}|>0} \prod_{i=1}^{d} \left(1 - \chi_{v_i}(A^{(v_i)})\right). \tag{3}$$

In the same vein, assume that a finite free subset S of G is given. The vector $(X_x(\omega))_{x\in S}$ giving the gas occupation on S is also a deterministic function of the DA $\mathbf{A}^S(\omega)$.

Remark 2.3 *In the case where $C_x(\omega) = a$ but $|\mathbf{A}^{\{x\}}(\omega)|$ is infinite, the computation of $X_x(\omega)$ may also ends within a finite number of steps since the product $\prod_{c:\ children\ of\ x}(1 - X_c(\omega))$ is known to be 0 when one of its terms is null, which can be the case on a finite sub animal of $\mathbf{A}^S(\omega)$. The value returned by this procedure in this case does not really matter, if the set of infinite DA has probability 0. Notice also that X_x may be undefined, the calculus may never end (this is the case on a lattice for the largest animal). In this case set $X_x = u$. In the following considerations we will restrict ourselves to $p < p_{crit}$ in order to avoid with probability 1 these infinite DA. The a.s. finiteness of $\mathbf{A}^S$ is of course crucial in the following combinatorial proofs.*

By the previous consideration we may conclude by the following proposition.

Proposition 2.4 *Let $G = (V, E)$ be an agreeable graph, $x \in V$ and $p \in [0, p_{crit}^{\{x\}})$. Under $\mathbb{P}_p$ the random variable X_x is a.s. well defined by (1), and*

$$\mathbb{E}_p(X_x) = \mathbb{P}_p(X_x = 1) = \mathbb{E}_p(\chi_x(\mathbf{A}^{\{x\}})) = \sum_{A\in\mathcal{A}(\{x\},G),|A|<+\infty,\chi_x(A)=1} p^{|A|}(1-p)^{|\mathcal{P}(A)|}$$

where $\mathcal{A}(\{x\}, G)$ has been defined in Section 1.1.

One may check from (1) that the random gas X hence defined is a gas model with nearest neighbor exclusion.

Proof. First, we check that $X_x : \Omega \to \{0, 1, u\}$ is measurable, the letter u stands for "undefined". This is trivial, since $\mathbf{A}^S$ is measurable: for any $i \in \{0, 1, u\}$, $X_x^{-1}(i) = (\mathbf{A}^{\{x\}})^{-1}(B_i)$ where B_i is the set of DA A such that $\chi(A) = i$, and this is measurable by the proof of Proposition 2.1.

For $p \in [0, p_{crit}^{\{x\}})$, the DA $\mathbf{A}^{\{x\}}$ is a.s. finite, and then a.s. $\chi_x(\mathbf{A}^{\{x\}}) = X_x$. The three equalities are straightforward. □

Remark 2.5 *In the following we will sometimes work with a sequence* $\mathbf{X} = (X_x)_{x\in J}$ *indexed by a countable subset* J *of* V. *This process* $\mathbf{X}$ *is measurable when equipped with the* σ*-algebra* $\mathcal{F}_J$ *of the cylinders (that is the sets* $\{\omega, X_x(\omega) = c_x, x \in I\}$ *of* Ω*, for* I *finite subsets of* J*, and* $x \in \{0,1\}$*, for* $p < p_{crit}$*), since each of the variables* X_x *are measurable. The distribution of the process* $\mathbf{X}$ *on* $\left(\{0,1\}^J, \mathcal{F}_J\right)$ *is characterized by the finite dimensional distributions.*

Remark 2.6 *In the case where* $\mathbf{A}^{\{x\}}(\omega) = A$ *is finite, it may be convenient for the reader to see the computation of* $X_x(\omega)$ *as the result of a Nim type game with two players, player(0) and player(1) on* A *according to the following rules:*
– the first player that can not play is the looser.
– at time 0, player(0) places a token on x *(if* A *is empty,* $\text{player}(0)$ *is the looser).*
– then, the players move in turn the token upward in A*. At time* i*,* $\text{player}(i \mod 2)$ *takes the token from where it is, say in* v*, and can move it in* u*, if* u *is in* A *and if* (v,u) *is a directed edge of* G*.*

The fact is that $\chi_x(A) = 1$ *iff player(0) has a strategy to win against all defense of player(1). We let the reader proves this property as an exercise. Indication: proceed to the computation of* $\chi_x(A)$ *by the last moves of the players.*

One of the main aim of this paper is to provide an explication at the level of objects of the following theorem

Theorem 2.7 *Let* $G = (V,E)$ *be an agreeable graph,* x *be a vertex of* V*,* R_x *be the radius of convergence of* $\mathbf{G}^G_{\{x\}}$ *and* $p \in [0, R_x)$*. We have*

$$\mathbb{E}_p(X_x) = -\mathbf{G}^G_{\{x\}}(-p). \tag{4}$$

We will see in Lemma 2.10 that in any graph $R_x \le p^{\{x\}}_{crit}$.

This relation between the density of a gas model and the GF of DA in the case of lattice graph is first discovered by Dhar [9] in the square lattice case; it is then generalized by Bousquet-Mélou [4, 3]. In each case, a formula similar to (4) is obtained, but there the equality is only formal: $\mathbb{E}_p(X_v)$ and $-\mathbf{G}^G_{\{v\}}(-p)$ are shown to be formal series satisfying the same recursive decomposition. We want also to point out that in [4, 3], the gas model is studied on a cylinder and some arguments using the finiteness of the number of states, and the convergence of some Markov chains to their stationary regimes are used (see Section 5.3). Here these steps are not needed since the construction of the gas model is done "on the right graph at once". We want also to stress on the fact that Theorem 2.7 holds on any agreeable graph and not only on lattices.

In order to express our relation in the level of objects, we will rewrite the right hand side of (4) under the form of an expectation, in order to make more apparent that $\mathbb{E}_p(X_x)$ and $-\mathbf{G}^G_{\{x\}}(-p)$ are both sum on weighted DA (the weight of a DA A being simply $p^A(1-p)^{|\mathcal{P}(A)|}$ its probability on Ω). But first, we need to introduce the notion of sub animal.

Definition 2.8 *Let A and A' be two DA. We say that A is a sub animal (sub-DA) of A' (we write $A \prec A'$), if A is included in A' and if $\mathcal{S}(A)$ is included in $\mathcal{S}(A')$.*

For any DA A with minimal source $\mathcal{S}(A) = S$, and $p \in (0, p^S_{crit})$, we have

$$p^{|A|} = \sum_{A':\ A \prec A', \mathcal{S}(A')=S} \mathbb{P}_p(\mathbf{A}^S = A') = \sum_{A':\ A \prec A', \mathcal{S}(A')=S} p^{|A'|}(1-p)^{|\mathcal{P}(A')|}.$$

Indeed, $p^{|A|}$ is the probability of the event $\{\omega, X_x(\omega) = a \text{ for any } x \in A\}$, and this latter equals $\{\omega, A \text{ is a sub-DA of } A^S(\omega)\}$. We have

Proposition 2.9 *Let $G = (V, E)$ be an agreeable graph. For any $x \in V$ and any $p \in [0, R_x)$,*

$$-\mathbf{G}^G_{\{x\}}(-p) = \mathbb{E}_p\left(D_x(\mathbf{A}^{\{x\}})\right)$$

where for any A with source $\{x\}$,

$$D_x(A) = \sum_{B:\ B \prec A, \mathcal{S}(B)=\{x\}} (-1)^{|B|+1} \tag{5}$$

is the difference between the number of sub-DA of A having an odd number of cells and those having an (non zero) even number of cells.

Before proving this proposition, we establish the following Lemma

Lemma 2.10 *For any agreeable graph $G = (V, E)$ and x in V,*

$$R_x \leq p^{\{x\}}_{crit}. \tag{6}$$

Proof. Since for any A with source $\{x\}$, $p^{|A|} = \mathbb{P}(A \text{ is a sub-DA of } \mathbf{A}^{\{x\}})$ (where $\mathbf{A}^{\{x\}}$ may be infinite),

$$\sum_A p^{|A|} = \sum_A \mathbb{P}(A \prec \mathbf{A}^{\{x\}}) = \sum_A \mathbb{E}_p(1_{A \prec \mathbf{A}^{\{x\}}}) = \mathbb{E}_p(\text{ Number of sub-DA of } \mathbf{A}^{\{x\}}). \tag{7}$$

If $p \in [0, R_x)$ then these quantities are finite. This implies that $\mathbb{P}_p$ a.s. the number of sub-DA of $\mathbf{A}^{\{x\}}$ is finite, which in turn implies that $\mathbf{A}^{\{x\}}$ is $\mathbb{P}_p$ a.s. finite, which finally yields $p \leq p^{\{x\}}_{crit}$. $\square$

Proof of Proposition 2.9. It relies on a permutation of sum symbols. For any $p \in [0, R_x)$,

$$-\mathbf{G}^G_{\{x\}}(-p) \ = \sum_{A, \mathcal{S}(A)=\{x\}} p^{|A|}(-1)^{|A|+1}.$$

Since by Lemma 2.10 we have $p < p_{crit}^{\{x\}}$, this is equal to

$$\sum_{A\colon\ \mathcal{S}(A)=\{x\}} \left(\sum_{A'\colon\ A \prec A', \mathcal{S}(A')=\{x\}} p^{|A'|}(1-p)^{|\mathcal{P}(A')|}(-1)^{|A|+1} \right). \tag{8}$$

This sum converges absolutely when $p < R_x$ since with an absolute value it is the mean of the number of Sub DA of $\mathbf{A}^{\{x\}}$ (which is finite when $p < R_x$, see the proof of Lemma 2.10). By the Fubini's theorem the double sum in (8) is

$$\sum_{A'\colon\ \mathcal{S}(A')=\{x\}} p^{|A'|}(1-p)^{|\mathcal{P}(A')|} D_x(A') = \mathbb{E}_p\left(D_x(\mathbf{A}^{\{x\}})\right). \ \square$$

Remark 2.11 • *In general the inequality (6) is strict, since*

$$R_x = \sup\{p, \mathbb{E}_p(\text{Number of sub-DA of } \mathbf{A}^{\{x\}}) < +\infty\}$$

when

$$p_{crit} = \sup\{p, \mathbb{P}_p(|\mathbf{A}^{\{x\}}| < +\infty) = 1\}.$$

In the case of the square lattice $R_x = 1/3$ when p_{crit} is larger than $1/2$ (by Proposition 2.2) and is expected to be around 0.6.

• *Proposition 2.9 is a relation between two series in p that are not defined in general for the same p in $\mathbb{R}$. Viewed as formal series in p they are equal. This hiatus comes from the classical fact that a sum which is not absolutely convergent is well defined only if an order of summation of its terms is provided. In the proof of Proposition 2.9, we reorder the terms of the serie $\mathbf{G}_{\{x\}}^G$; this serie is not absolutely convergent for $p > R_x$ and then this reordering is not legal. In some sense, the series $\sum_{A\colon\ \mathcal{S}(A)=\{x\}} p^{|A|}(1-p)^{|\mathcal{P}(A)|} D_x(A)$ contains the monomials of $\mathbf{G}_{\{x\}}^G$ which have been packed in a more efficient way as regards the convergence of series. The limitation that appears here is interesting since in the combinatorial literature, the habit is to work formally, and then no such limitation appears (or say, at the end of the computation, when the radius of convergence of the series appears clearly).*

The problem of convergence appears here clearly since we make important use of probabilistic results. When the approach is more formal these problems may not appear at all (as they should). The formal study of the gas configurations on a cylinder (as done in [4, 3]) by the use of finite states Markov chain does not address any problem since the computation on finite states Markov chain are simply linear algebra.

We saw that $-\mathbf{G}_{\{x\}}^G(-p) = \mathbb{E}_p\left(D_x(\mathbf{A}^{\{x\}})\right)$ for $p < R_x \leq p_{crit}^{\{x\}}$ and the density of the gas $\mathbb{E}_p(X_x)$ is equal to $\mathbb{E}_p(\chi_x(\mathbf{A}^{\{x\}}))$, for $p < p_{crit}^{\{x\}}$. Here is the explication of Theorem 2.7 at the level of object:

Theorem 2.12 *For any finite DA A with source v, we have $D_v(A) = \chi_v(A)$.*

This allows to deduce that Theorem 2.7 holds true, since a.s. when $p < p_{crit}^{\{v\}}$, the random DA $\mathbf{A}^{\{v\}}$ is a.s. finite, and then $X_v(\mathbf{A}^{\{v\}}) = \chi_v(\mathbf{A}^{\{v\}}) = D_v(\mathbf{A}^{\{v\}})$ a.s., and then these variables have the same expectation (notice that this also implies that $D_v(A)$ takes its values in $\{0,1\}$, which is not necessarily obvious).

In order to prove Theorem 2.12, we will show that D_v owns the same recursive decomposition as χ_v given in (3); this is done via the introduction of a notion of embedding of trees in DA. An heuristic is given in Section 2.5.

2.4 Embedded trees

Let $G = (V, E)$ be an agreeable graph. The set V being at most countable we assume from now on that an order denoted by $\underset{V}{<}$ is given on V. This order induces an order among the children of a given vertex in G. Since we are to "canonically" embed some ordered trees in G we need also to define a suitable ordering of the nodes of those trees; this is inspired from the Neveu's definition of trees.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of non-negative integers and $\mathcal{W} = \{\varnothing\} \cup \bigcup_{i=1}^{+\infty} \mathbb{N}^i$ the set of finite words on the alphabet $\mathbb{N}$, where $\varnothing$ denotes the empty word. We define the concatenation product of two words $u = u_1 \dots u_k$ and $v = v_1 \dots v_l$ of $\mathcal{W}$ by $uv := u_1 \dots u_k v_1 \dots v_l$; the empty word $\varnothing$ is the neutral element for this operation: $\varnothing u = u \varnothing$ for any $u \in \mathcal{W}$.

Definition 2.13 *We denote by tree a subset* $\mathbf{t}$ *of* $\mathcal{W}$ *such that* $\varnothing \in \mathbf{t}$ *and if* $u = u_1 \dots u_k \in \mathbf{t}$ *for some* $k \geq 1$ *then* $u_1 \dots u_{k-1} \in \mathbf{t}$*. In other word if a word* u *is in* $\mathbf{t}$*, its prefixes are also in* $\mathbf{t}$ *(see Fig. 4).*

The set of trees is denoted by $\mathcal{T}$.

In the combinatorial literature, what is called tree here is sometimes viewed as a depth first traversal encoding of trees.

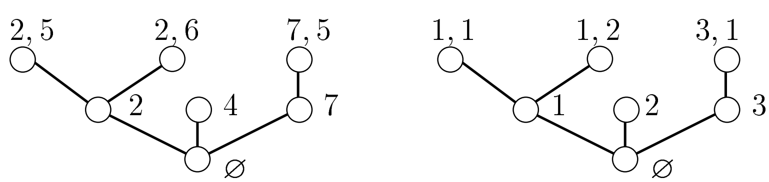

Figure 4: The trees admit some usual representation as embedded figure in the plane, mapping the order between brothers into the usual order between the abscissas. In the usual Neveu's convention, there is an additional axiom (if $v = u_1 \dots u_k j \in \mathbf{t}$ for some $j \geq 2$ then $v = u_1 \dots u_k (j-1) \in \mathbf{t}$) which ensures that the "branching structure of the tree" characterizes the tree (for Neveu, only the second drawing represents a tree). This is not the case, here.

The vocabulary attached to trees are as usual: $\varnothing$ is called the root, the strict prefixes of $u \in \mathbf{t}$ are called ancestors of u, and u is a descendant of its ancestors. The elements of

$\mathbf{t}$ are called nodes, and for $u \in \mathbf{t}$, $|u|$ stands for the number of letters in u (by convention $|\varnothing| = 0$) and is called the depth of u. If $u = u_1 \dots u_k i$ then $u_1 \dots u_k$ is the father of u, and if $v = u_1 \dots u_k j \in \mathbf{t}$ for $i \neq j$, we say that u and v are brothers. For any $u = u_1 \dots u_k \in \mathbf{t}$, we denote by $\mathcal{C}_u(\mathbf{t}) = \{ui, i \in \mathbb{N}\} \cap \mathbf{t}$ the set of children of u in $\mathbf{t}$. The size of $\mathbf{t}$ denoted by $|\mathbf{t}|$ is the cardinality of $\mathbf{t}$.

Let $\mathbf{t}$ be a tree and $i \in \mathcal{C}_\varnothing(\mathbf{t})$ a child of the root in $\mathbf{t}$. For any $i \in \mathbb{N}$, we denote by $\mathbf{t}^i$ the set:

$$\mathbf{t}^i := \{v, u := iv \in \mathbf{t}\}.$$

It is easy to see that $\mathbf{t}^i$ is a tree, and that it is obtained from $\mathbf{t}$ by deleting all the nodes that are not descendant of i, and "rerooted" in i: i and the descendants of i are kept but they lose they first letter in order to form a tree (see Fig. 5). The tree $\mathbf{t}^i$ is called *fringe subtree* in the literature.

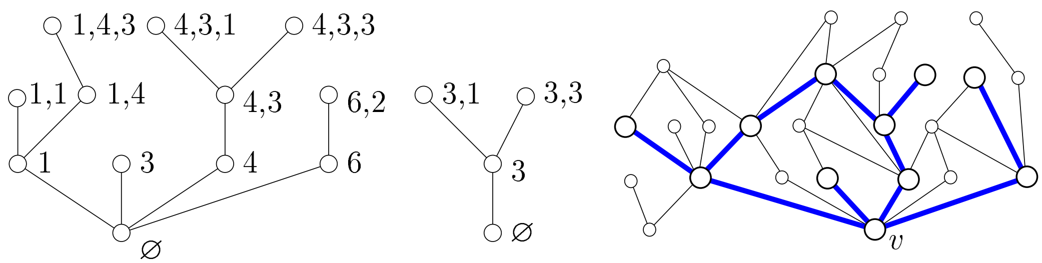

Figure 5: The tree $\mathbf{t} = \{\varnothing, 1, 3, 4, 6, (1,1), (1,4), (4,3), (6,2), (1,4,3), (4,3,1), (4,3,3)\}$. The second tree is $\mathbf{t}^4$. We have $\mathcal{C}_\varnothing(\mathbf{t}) = \{1,3,4,6\}$. On the third figure is drawn the embedding $\pi_v(\mathbf{t})$ on an agreeable graph, where the edges are directed upward.

Definition 2.14 *Let $G = (V, E)$ be an agreeable graph, $v \in V$ and $\mathbf{t}$ a tree. Let $v_1, \dots, v_d$ be the $d = d_v$ children of v in G sorted according to $\underset{V}{<}$. We say that $\mathbf{t}$ is embeddable in G at v if $\mathbf{t} = \{\varnothing\}$ or if for any $i \in \mathcal{C}_\varnothing(\mathbf{t})$, $\mathbf{t}^i$ is embeddable at v_i. If $\mathbf{t}$ is embeddable in G at v, we denote by $\pi_v(\mathbf{t})$ its embedding in G at v defined by:*

$$\pi_v(\mathbf{t}) = \{v\} \cup \bigcup_{i \in \mathcal{C}_\varnothing(\mathbf{t})} \pi_{v_i}(\mathbf{t}^i).$$

An illustration of this embedding is given in Fig. 5.

In other words, $\pi_v(\mathbf{t})$ is the subset of G obtained by drawing $\mathbf{t}$ on G according to the following rules: first draw the root of $\mathbf{t}$ on v (in other word $\pi_v(\{\varnothing\}) = v$). Each branch of $\mathbf{t}$ is a succession of nodes $a_k = u_1 \dots u_k$ where a_k is a prefix of a_{k+1} (and $a_0 = \varnothing$). Then draw a_{k+1} in G in such a way that a_{k+1} is the u_{k+1} child of a_k (the edge $(\pi_v(a_k), \pi_v(a_{k+1}))$ of G is the u_{k+1}th edge starting from $\pi_v(a_k)$) (see Fig. 5). Hence each branch of $\mathbf{t}$ is finally embedded in a simple path of G issued from v.

For any tree $\mathbf{t}$ embeddable in G at v, $\pi_v(\mathbf{t})$ is a DA on G with source $\{v\}$. For any finite DA $A^{\{v\}}$ with source $\{v\}$ in G, we set

$$\Delta_v(A^{\{v\}}) := \sum_{\mathbf{t}:\ \pi_v(\mathbf{t}) \prec A^{\{v\}}} (-1)^{|\mathbf{t}|+1}, \tag{9}$$

where the sum is taken on the (necessarily finite) set of trees embeddable in G at v. Our proof of Theorem 2.12 consists of proving the two following equalities valid for any finite DA A with source $\{v\}$:

$$D_v(A) = \Delta_v(A) \text{ and } \Delta_v(A) = \chi_v(A). \tag{10}$$

The key lemma in the proof of the first equality is

Lemma 2.15 *For any finite DA $A^{\{v\}}$ with source $\{v\}$, we have*

$$\sum_{\mathbf{t}:\ \pi_v(\mathbf{t}) = A^{\{v\}}} (-1)^{|\mathbf{t}|} = (-1)^{|A^{\{v\}}|}.$$

Proof: This is true if $A^{\{v\}} = \{v\}$. Assume that this is true for all DA having k cells or less than k cells. Take a DA $A^{\{v\}}$ with $k+1$ cells. Among those cells, there exists w such that w has no child in $A^{\{v\}}$. Consider the DA $A' = A^{\{v\}} \setminus \{w\}$ and compare the set of trees ξ whose embedding is $A^{\{v\}}$ with ξ' the set of trees whose embedding is A'. The trees in ξ are obtained from those of ξ' by addition of leaves which embedding is w. Hence by the natural projection from ξ into ξ', (see Fig. 6)

$$\xi = \bigcup_{t \in \xi'} K_t \tag{11}$$

where K_t is the subset of ξ containing the trees obtained from t by the addition of some leaves. The K_t's form a partition of ξ. One has, for any $t \in \xi'$,

$$\sum_{\tau \in K_t} (-1)^{|\tau|} = \sum_{\tau \in K_t} (-1)^{|t|+|\tau \setminus t|}.$$

Given t, there is a maximal tree in K_t (for the inclusion) obtained from t by the addition of a set R_t of leaves. The others trees of K_t consist of t together with a non-empty part of R_t. Hence

$$\sum_{\tau \in K_t} (-1)^{|\tau|} = \sum_{B, B \subset R_t, |B| \geq 0} (-1)^{|t|+|B|} = (-1)^{|t|}((1-1)^{|R_t|} - 1) = (-1)^{|t|+1}.$$

By (11), $\sum_{\tau \in \xi} (-1)^{|\tau|} = -\sum_{t \in \xi'} (-1)^{|t|}$ and we conclude by recurrence on the number of cells of A. $\square$

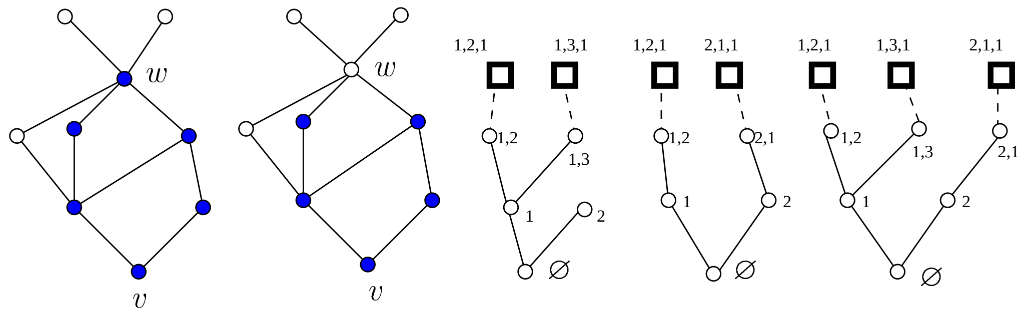

Figure 6: On the first picture, a DA $A^{\{v\}}$ (whose cells are colored). On the second one, a DA A' obtained from $A^{\{v\}}$ by the suppression of a cell having no child. The three trees with straight lines represent trees embeddable in A' at v. The trees with in addition some of the doted lines are the trees embeddable in $A^{\{v\}}$ at v. The square nodes are the nodes whose embedding are w

Now, we conclude the proof of Theorem 2.12.

Proof of Theorem 2.12. On one hand, by Lemma 2.15, $D_v(A)$ (defined in (5)) satisfies

$$D_v(A) = \sum_{B: B \prec A, \mathcal{S}(B)=\{v\}} \left(\sum_{\mathbf{t}:\ \pi_v(\mathbf{t})=B} (-1)^{|\mathbf{t}|+1} \right) = \sum_{\mathbf{t}:\ \pi_v(\mathbf{t}) \prec A} (-1)^{|\mathbf{t}|+1} = \Delta_v(A).$$

We now establish that Δ_v owns the same recursive decomposition (3) as χ_v. If $|A| = 1$ then $\Delta_v(A) = 1 = \chi_v(A)$. Assume that the source v of A has d children $v_1, \ldots, v_d$. For any $i \in \{1, \ldots, d\}$, we have

$$\Delta(A^{v_i}) := -\sum_{\mathbf{t}:\ \pi_{v_i}(\mathbf{t}) \prec A^{v_i}} (-1)^{|\mathbf{t}|},$$

where A^{v_i} is the maximal DA included in A with source v_i. Decomposing the trees $\mathbf{t}$ embeddable in A according to their embeddings in the A^{v_i}, we get

$$\begin{aligned} \Delta_v(A^v) &= 1_{|A^v|>0} \left(\sum_{B \subset \{v_1,\ldots,v_d\}} \prod_{i \in B} (-1)^{|\mathbf{t}^i|} \right) \\ &= 1_{|A^v|>0} \left(\sum_{B \subset \{v_1,\ldots,v_d\}} (-1)^{|B|} \prod_{i \in B} \Delta_{v^i}(A^{v_i}) \right) = 1_{|A^v|>0} \prod_{i=1}^{d} (1 - \Delta_{v^i}(A^{v_i})). \end{aligned}$$

Hence Δ_v owns the same recursive decomposition (3) as χ_v, and finally Δ_v and χ_v coincide on the set of finite DA with source v. $\square$

2.5 Why is the embedding of trees in DA "natural"?

According to (1), we have

$$\mathbb{E}_p(X_x) = \mathbb{E}_p\left(B^x(p) \prod_{c:\text{ children of } x} (1 - X_c)\right) \tag{12}$$

where $B^x(p)$ is the Bernoulli(p) random variable $\mathbb{1}_{C_x=a}$ (recall that all the random variables $B^x(p)$ are independent). We now expand the right hand side of (12), conditioning by $\mathbf{A}^{\{x\}}$:

$$\mathbb{E}_p(X_x) = \mathbb{E}_p\left(\mathbb{E}_p\left(B^x(p) \prod_{c:\text{ children of } x} (1 - X_c)\middle|\mathbf{A}^{\{x\}}\right)\right).$$

Assume now that $\mathbf{A}^{\{x\}}$ is known (and finite). By construction, $B^u(p) = 1$ for any u in $\mathbf{A}^{\{x\}}$ and $B^u(p) = 0$ for any $u \in \mathcal{P}(\mathbf{A}^{\{x\}})$. We now replace each of the X_c by a product involving its children $B^c(p) \prod_{c':\text{ children of } c}(1 - X_{c'})$, recursively; we stop this expansion when the children of a node are all perimeter sites, since in this case all the $X_{c'}$ equal 0.

When one expands this formula (there are no more variables X_c since they have been replaced by those concerning their children, until the perimeter of $\mathbf{A}^{\{x\}}$ has been reached), one builds a tree: we have $\prod_{c':c'\text{ children of } c}(1 - X_{c'}) = \sum_{C\subset\{\text{ children of } c\}} \prod_{c'\in C} -X_{c'}$ corresponding in the tree like decomposition to the choice of some 1 and some $-X_{c'}$. The "1" in the decomposition are the leaves of this tree, and when C is chosen, $|C|$ is the degree of an internal node in this tree. One then sees that the final value of X_x knowing $\mathbf{A}^{\{x\}}$, is $\sum_{\mathbf{t}:\pi(\mathbf{t})\prec A^{\{x\}}}(-1)^{|\mathbf{t}|+1}$ in other words $\Delta_x(\mathbf{A}^{\{x\}})$. Indeed, the terms $(-1)^{|\mathbf{t}|+1}$ counts the number of times $-X_{c'}$ has been chosen in the expansion: this number equals the total number of children in the tree that is $|\mathbf{t}| - 1$. Hence, for $p < p_{crit}^{(x)}$,

$$\mathbb{E}_p(X_x) = \sum_A p^A(1-p)^{|\mathcal{P}(A)|} \sum_{\mathbf{t}:\pi(\mathbf{t})\prec A} (-1)^{|\mathbf{t}|+1} = \mathbb{E}_p(\Delta_x(\mathbf{A}^{\{x\}})).$$

On the other hand for $p < R_x \leq p_{crit}^{(x)}$,

$$-\mathbf{G}_x(-p) = \sum_A p^A(1-p)^{|\mathcal{P}(A)|} \sum_{A'\prec A} (-1)^{|A|+1} = \mathbb{E}_p(D_x(\mathbf{A}^{\{x\}})).$$

Then, $D(A) = \Delta(A) = \chi(A)$ is indeed an explication at the level of objects of $-\mathbf{G}_x(-p) = \mathbb{E}_p(X_x)$.

2.6 Directed animal with compact sources

The previous subsections deal almost only with DA with sources of cardinality 1. But most of the results stated there admit some generalizations to DA with larger sources. The main results that we want to state is the following generalization of Theorem 2.7.

Theorem 2.16 *Let $G = (V, E)$ be an agreeable graph, k be a positive integer, $S = (s_1, \ldots, s_k)$ be a free finite subset of V and p smaller than the radius of convergence of $\mathbf{G}_S^G$. Under $\mathbb{P}_p$, the random variables $(X_x)_{x\in S}$ are a.s. well defined by (1), and we have*

$$\mathbb{E}_p\left(\prod_{x\in S} X_x\right) = \mathbb{P}_p(X_x = 1, x \in S) = (-1)^{|S|}\mathbf{G}_S^G(-p). \tag{13}$$

Observe that

$$\mathbb{P}_p(X_x = 1, x \in S) = \sum_{A, A\in\mathcal{A}(S,G), |A|<+\infty, X_x(A)=1, x\in S} p^{|A|}(1-p)^{|\mathcal{P}(A)|} \tag{14}$$

for any $p \in [0, p_{crit}^S)$ is the sum on DA occupied on their source S.

The proof of this Theorem is postponed in the Appendix.

3 Gas model of type 1 and DA on $\mathbb{Z}^2$

3.1 New derivation of the GF of DA counted according to the area

In this part we work on the square lattice $\mathsf{Sq} = \mathbb{Z}^2$ where as said in the Introduction, Sq is viewed as a directed graph where each vertices (x, y) has children $(x, y+1)$ and $(x+1, y+1)$. We represent this lattice as on Fig. 1. The real number $p_{crit}^{\mathsf{Sq}} := p_{crit}^{\{x\}}$ does not depend on x by symmetry, and for any finite source S, $p_{crit}^S = p_{crit}^{\mathsf{Sq}}$. According to Proposition 2.2, $p_{crit}^{\mathsf{Sq}} \geq 1/2$ and by Proposition 2.4, for $p < p_{crit}^{\mathsf{Sq}}$ the gas occupation is $\mathbb{P}_p$ a.s. defined everywhere. The radius R_x can also be shown easily to be positive using a simple injection from the set of DA with n cells in Sq into the set of trees having only internal nodes with total degree 2 and 3 and n nodes (keep the left most paths linking the cells to the source).

The set $L_i := \{(x, i-x), x \in \mathbb{Z}\}$ is called the ith line and we denote by Z_i its gas occupation:

$$Z_i(x) = X_{(x,i-x)}, \qquad \text{for any } x \in \mathbb{Z}.$$

By construction Z_i is a process indexed by $\mathbb{Z}$ taking its values in $\{0, 1\}$. It is invariant in the strong sense: for any $k \in \mathbb{Z}$ the process $(Z_i(x))_{x\in\mathbb{Z}}$ and $(Z_i(x+k))_{x\in\mathbb{Z}}$ have the same distribution. By construction also the processes Z_i and Z_{i+1} have the same distribution. Since Z_{i+1} is built from Z_i via the gas evolution (1), for any $i \in \mathbb{Z}$, Z_i and Z_{i+1} are related by:

$$Z_i(k) = B_k^i(p)\,(1 - Z_{i+1}(k))\,(1 - Z_{i+1}(k+1)) \tag{15}$$

where the $B_k^i(p)$ are independent and identically distributed (i.i.d.) Bernoulli(p) random variables, independent of Z_{i+1}.

The existence (and the construction) of a solution is guaranteed when $p \in [0, p^{\mathsf{Sq}}_{crit})$ by Proposition 2.4, and a solution, say $Z^\star$ is characterized by the equation (14): let $S = \{(x, -x), x \in I\}$ be included in L_0, with I finite. We have

$$\mathbb{P}(X_s = 1, s \in S) = \mathbb{P}(Z^\star(l) = 1, l \in I) = \sum_{A, A \in \mathcal{A}(S), \chi_s(A)=1, s \in S} p^{|A|}(1-p)^{|\mathcal{P}(A)|}.$$

By inclusion-exclusion, this determines the finite dimensional distribution of $Z^\star$ (recall Remark 2.5).

Proposition 3.1 *For all $p \in [0, p^{\mathsf{Sq}}_{crit})$, there exists a unique law μ of processes taking their values in $\{0,1\}$ solution of (15) (in other words, there exists a unique law μ such that if the process Z_{i+1} is μ distributed then so do Z_i).*

Proof. The uniqueness of the solution will be proved to be a consequence of Theorem 2.16: assume that Z'_0 has a certain law μ' which is a solution of (15). One then builds an infinite sequence $(Z'_i)_{i \leq -1}$ such that for any Z'_i is obtained from Z'_{i+1} by

$$Z'_i(k) := B^{(i)}_k(p)\,(1 - Z'_{i+1}(k))\,(1 - Z'_{i+1}(k+1)) \text{ for } i \leq -1 \tag{16}$$

where $B^{(i)}_k(p), i \leq -1, k \in \mathbb{Z}$ is an array of i.d.d. Bernoulli(p) random variables.
Consider I a finite subset of $\mathbb{Z}$. We will examine the probability $\mathbb{P}(Z'_i(l) = 1, l \in I)$ and show that it converges when $i \to +\infty$ to the distribution of $\mathbb{P}(Z^\star_i(l) = 1, l \in I)$. Since $\mathbb{P}(Z'_i(l) = 1, l \in I)$ does not depend on i this entails that the finite distribution of Z' and those of $Z^\star$ are equals. This result being valid for any finite set I, this implies that Z' has the same law as $Z^\star$.

Let I be fixed, and consider $S(i) = \{(x, i-x), x \in I\}$ the subset of the ith line L_i, which is simply obtained by the translation of $S := S(0)$. Under $\mathbb{P}^{\mathsf{Sq}}_p$, $\mathbf{A}^S$ is a.s. finite; this implies that for any $\varepsilon > 0$ there exists n_ε such that $\mathbb{P}_p(|\mathbf{A}^S| \geq n_\varepsilon) < \varepsilon$.

Consider now $S(-n_\varepsilon)$, the subset of L_{-n_ε}. We have $\mathbb{P}_p(|\mathbf{A}^{S(-n_\varepsilon)|} \geq n_\varepsilon) < \varepsilon$, by invariance by translation. When $|\mathbf{A}^{S(-n_\varepsilon)}| < n_\varepsilon$, $\left(\mathbf{A}^{S(-n_\varepsilon)} \cup \mathcal{P}(\mathbf{A}^{S(-n_\varepsilon)})\right) \cap L_0 = \emptyset$. In this case, the values of $(X_s)_{s \in S(-n_\varepsilon)}$ does not depends on the value of X on L_0 (and then neither on its distribution on this line), since as said in Section 2.3, the gas occupation on a subset S is a deterministic function of $\mathbf{A}^S$ and its perimeter sites (and then does not depend on the other sites).

The formula (16) defining Z'_{-n} is the same as that defining the gas model of type 1, and then, if $\mathbf{A}^{S(-n_\varepsilon)}$ satisfies $|\mathbf{A}^{S(-n_\varepsilon)}| < n_\varepsilon$, then $(Z'_{-n_\varepsilon}(l), l \in I)$ does not depend on Z'_0. Therefore, for any $n \geq n_\varepsilon$, we have

$$\left|\mathbb{P}_p(Z'_{-n}(l) = 1, l \in I) - \mathbb{P}_p(Z^\star(l) = 1, l \in I)\right| < \varepsilon$$

and then $\mathbb{P}_p(Z'_{-n}(l) = 1, l \in I) \to \mathbb{P}_p(Z^\star(l) = 1, l \in I)$. $\square$

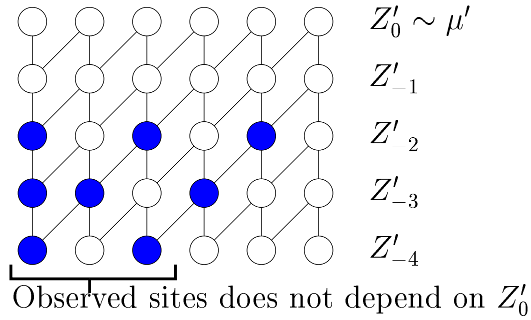

Figure 7: The colored sites are the sites where the Bernoulli(p) coloring $= 1$. The gas occupation on the three first cells on the last row depends only on the DA (with these three cells as over-sources). The height of this DA is so small that the gas occupation of these cells does no depend on Z'_0.

Comments 3.2 *The sequence (Z_i) forms ("a vertical") Markov chain with infinite number of states. If one wants to determine the value of $Z_i(x)$ using the Bernoulli tossing allowing to realize this Markov chain, only the DA of calculus $\mathbf{A}^{(x,i-x)}$ is needed. This is reminiscent of the Propp & Wilson coupling from the past technique to simulate a Markov chain under the stationary distribution: only the last transitions of the Markov chain sufficient to determine the current state are needed.*

We now determine the distribution μ of the processes Z_i and the density of the gas model of type 1.

Let $Z := Z_0 = (Z_0(x))_{x\in\mathbb{Z}}$ be the gas process on the line. We denote by $(B_i^{\bullet})_{i\geq 1}$ (resp. $(B_i^{\circ})_{i\geq 1}$) the successive sizes of the blocks of consecutive occupied (empty) positions at the right of zero. If a block contains some negative positions, only the part at the right of zero is counted as on Fig. 8.

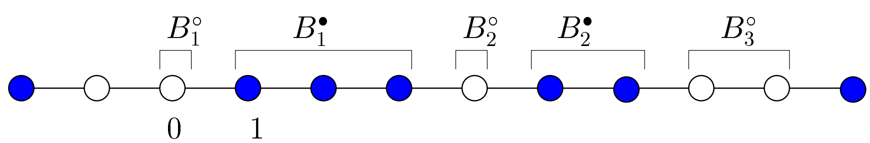

Figure 8: On this example, the black cells are occupied and corresponds to the places where $Z = 1$, and the whites one are empty.

A random variable G is said to follow the geometric distribution with parameter α, we denote $G \sim \mathcal{G}(\alpha)$, when $\mathbb{P}(G = k) = \alpha(1-\alpha)^{k-1}$ for any $k \geq 1$.

Theorem 3.3 *Let $p \in (0, p_{crit})$. The density of the gas model is given by*

$$\mathbb{P}_p(Z(0) = 1) = \frac{\alpha_{\bullet}^{-1}}{\alpha_{\circ}^{-1} + \alpha_{\bullet}^{-1}},$$

where

$$\alpha_\bullet = \frac{-1+p+\sqrt{1+2p-3p^2}}{2p} \qquad \text{and} \qquad \alpha_\circ = \frac{-1+p+\sqrt{1+2p-3p^2}}{1+p+\sqrt{1+2p-3p^2}}.$$

Under $\mathbb{P}_p$ the random variables $B_i^\bullet$ and B_i° for $i \geq 1$ are independent and independent of $Z(0)$, and $B_i^\bullet \sim \mathcal{G}(\alpha_\bullet)$ and $B_i^\circ \sim \mathcal{G}(\alpha_\circ)$.

Hence, for any $b_i^\circ, b_i^\bullet \geq 1$

$$\mathbb{P}(Z_0(0) = x, B_i^\circ = b_i^\circ, B_i^\bullet = b_i^\bullet, i \in \{1,\ldots,k\}) = \frac{1/\alpha(x)}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}} \prod_{i=1}^k \mathbb{P}(G^\circ = b_i^\circ)\mathbb{P}(G^\bullet = b_i^\bullet)$$

where $G^\bullet \sim \mathcal{G}(\alpha_\bullet)$ and $G^\circ \sim \mathcal{G}(\alpha_\circ)$, and $\alpha(1) = \alpha_\bullet$ and $\alpha(0) = \alpha_\circ$.

The law μ of Z is characterized by the properties given in this Theorem and the invariance by translation, since this characterizes the finite dimensional distributions of μ. We have no combinatorial explanation to the properties of the blocks, but these distributions may be guessed from the result of Bousquet-Mélou cited at the beginning of Section 3.3.

Comments 3.4 • *The proof that we provide below shows that the distribution described in Theorem 3.3 is a solution of equation (15) not only for $p \in (0, p_{crit})$ but also for any $p \in (0,1)$. For $p \in [p_{crit}, 1)$ we didn't find an argument to show that there is a unique solution to (15). In the case $p = 1$, the solution is not unique since an alternating sequence of blocks, the occupied block having size k and the empty one, $k+1$ is conserved by the gas evolution for any $k \geq 1$. For $p = 0$, the process Z null a.s. is the unique solution.*

• *One may also view Z as built as juxtaposing alternatively occupied and empty blocks with the prescribed distributions, starting from $-\infty$. Viewed like this, the density can be computed using a simple renewal argument or by the law of large number.*

• *Assume that $Z_1 = (Z_1(x))_{x\in\mathbb{Z}}$ and $Z_0 = (Z_0(x))_{x\in\mathbb{Z}}$ have the property announced in the Theorem, but that $\alpha_\bullet$ and $\alpha_\circ$ are unknown. To compute $\alpha_\bullet$ and $\alpha_\circ$ we derive and solve the two following equations*

$$(1-\alpha_\circ)p = 1-\alpha_\bullet \text{ and } \alpha_\circ = \alpha_\bullet(1-\alpha_\circ)p. \tag{17}$$

Indeed, a block of 1 in Z_0 is placed "under" a block of 0 in Z_1. Once a block of 1 is begun in Z_0, there is an additional 1 at the right of this block with probability $(1-\alpha_\circ)p$ (an additional 0 is needed in Z_1 and the right Bernoulli tossing is needed). This gives the first equation, since in Z_1 an additional 1 occurs with probability $1-\alpha_\bullet$. On the other hand, since $\{x, Z_0(x) = 1\} = \big\{x, C_x := a, Z_1(x) = 0, Z_1(x+1) = 0\big\}$, the probability $\frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}} = \mathbb{P}_p(Z_0(x) = 1) = p\mathbb{P}(Z_1(x) = 0, Z_1(x+1) = 0)$ equals to $p\mathbb{P}(Z_1(x+1) = 0|Z_1(x) = 0)\mathbb{P}(Z_1(x) = 0) = p\frac{\alpha_\circ^{-1}}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}}(1-\alpha_0)$. This gives the second equation (that can also be obtained by a renewal type argument without using the density).

3.2 Consequences of Theorem 3.3

It is worth mentioning that Theorem 3.3 is in fact equivalent to the following representation, allowing also to compute the gas density.

Proposition 3.5 *Under $\mathbb{P}_p$, the process $(Z(x))_{x\in\mathbb{Z}}$ is a Markov chain with transition matrix*

$$M = \begin{pmatrix} \mathbb{P}(Z(1)=1|Z(0)=1) & \mathbb{P}(Z(1)=0|Z(0)=1) \\ \mathbb{P}(Z(1)=1|Z(0)=0) & \mathbb{P}(Z(1)=0|Z(0)=0) \end{pmatrix} = \begin{pmatrix} 1-\alpha_\bullet & \alpha_\bullet \\ \alpha_\circ & 1-\alpha_\circ \end{pmatrix} \quad (18)$$

under the stationary distribution.

One may compute from this Proposition and Theorem 2.16 some results about the GF of DA with general sources on the square lattice.

Proposition 3.6 *Let $S=\{s_1,\dots,s_k\}$ where $s_i=(x_i,-x_i)$ be some points on the principal line L_0, such that $d_i := x_{i+1}-x_i$ for $i\in\{2,\dots,k\}$ are positive integers. The GF of DA on the square lattice with source S is given by*

$$\mathbf{G}_S^{\mathsf{Sq}}(-p) = (-1)^{|S|}\frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}}\prod_{i=1}^{k-1}\frac{\alpha_\bullet(1-\alpha_\bullet-\alpha_\circ)^{d_i}+\alpha_\circ}{\alpha_\bullet+\alpha_\circ}.$$

Proof. This is a simple consequence of the Markovianity of Z and of Theorem 2.16. Using a diagonalization of M (given in (18)), one gets $\mathbb{P}(Z_k=1|Z_0=1) = \left(M^k\right)_{1,1} = \frac{\alpha_\bullet(1-\alpha_\bullet-\alpha_\circ)^k+\alpha_\circ}{\alpha_\bullet+\alpha_\circ}$. $\square$

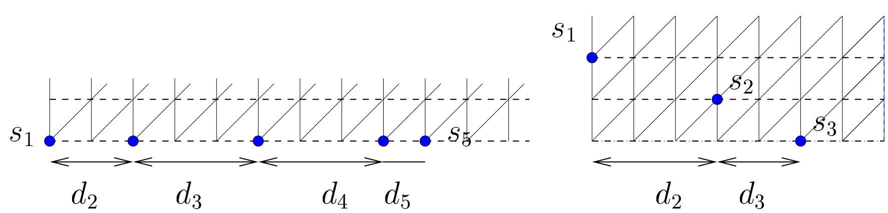

Figure 9: On the first picture, the sources considered in Proposition 3.6. The second picture illustrates the sources considered in Proposition 3.7.

For example, if $S_n := \{(i,-i), i=1,\dots,n\}$ then $\mathbf{G}_{S_n}^{\mathsf{Sq}}(-p) = \frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}}(1-\alpha_\bullet)^{n-1}(-1)^n$. Then, the GF of DA on the square lattice with compact sources satisfies

$$\sum_{n\geq 1}\mathbf{G}_{S_n}^{\mathsf{Sq}}(-p) = \frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1}+\alpha_\circ^{-1}}\sum_{n\geq 1}(1-\alpha_\bullet)^{n-1}(-1)^n = \frac{-p}{1+3p}.$$

A combinatorial explanation of this formula is given in Gouyou-Beauchamps & Viennot [11]. One may also compute the GF of DA having their sources on different lines. For example:

Proposition 3.7 *Let $S = \{s_1, \ldots, s_k\}$ where $s_i = (x_i, -y_i)$ be some points of* Sq *such that $y_i = x_i + i$ (see Figure 9 (ii)) and $d_i := x_{i+1} - x_i$ for $i \in \{2, \ldots, k\}$ are positive integers. The GF of DA on the square lattice with source S is given by*

$$\mathbf{G}_S^{\mathsf{Sq}}(-p) = (-1)^{|S|} \frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1} + \alpha_\circ^{-1}} \prod_{i=1}^{k-1} \alpha_\circ \frac{(1 - \alpha_\bullet - \alpha_\circ)^{d_i} - 1}{\alpha_\bullet + \alpha_\circ}.$$

See Figure 9 for an illustration of the considered sources.

Proof. We use again the Markovian properties of Z but this time on several lines in the same time. We want to compute $\mathbb{P}(X_s = 1, s \in S)$. Notice that the cells of S are not ancestors from each others. For any $s \in S$, denote by $c_1(s)$ and $c_2(s)$ the two children of s in Sq (c_1 is at the left of c_2); we have

$$\mathbb{P}(X_s = 1, s \in S) = \mathbb{P}(X_s = 1, X_{c_1(s)} = X_{c_2(s)} = 0, s \in S)$$

since if $X_s = 1$ then a.s. $X_{c_1(s)} = X_{c_2(s)} = 0$. We condition on the gas occupation on s_1, $c_1(s_2)$ and $c_2(s_2)$:

$$\begin{aligned}\mathbb{P}(X_s = 1, s \in S) &= \mathbb{P}(X_s = 1, s \in S \setminus \{s_1\} | X_{s_1} = 1, X_{c_1(s_2)} = 0, X_{c_2(s_2)} = 0) \\ &\quad \times \mathbb{P}(X_{s_1} = 1, X_{c_1(s_2)} = 0, X_{c_2(s_2)} = 0).\end{aligned}$$

By Markovianity we have $\mathbb{P}(X_{s_1} = 1, X_{c_1(s_2)} = 0, X_{c_2(s_2)} = 0) = \frac{\alpha_\bullet^{-1}}{\alpha_\bullet^{-1} + \alpha_\circ^{-1}} (M^{d_1})_{1,2} M_{2,2}$ and $\mathbb{P}(X_s = 1, s \in S \setminus \{s_1\} | X_{s_1} = 1, X_{c_1(s_2)} = X_{c_2(s_2)} = 0)$

$$\begin{aligned} &= \mathbb{P}(X_s = 1, s \in S \setminus \{s_1\} | X_{c_1(s_2)} = X_{c_2(s_2)} = 0) \\ &= \frac{\mathbb{P}(X_s = 1, s \in S \setminus \{s_1\})}{\mathbb{P}(X_{c_1(s_2)} = X_{c_2(s_2)} = 0)}.\end{aligned}$$

Thus, $\mathbb{P}(X_s = 1, s \in S) = \mathbb{P}(X_s = 1, s \in S \setminus \{s_1\}) \dfrac{\alpha_\circ}{\alpha_\bullet} (M^{d_1})_{1,2}$. Using that $(M^k)_{1,2} = \frac{\alpha_\bullet (1 - \alpha_\bullet - \alpha_\circ)^k - \alpha_\bullet}{\alpha_\bullet + \alpha_\circ}$ we get the result. $\square$

Remark 3.8 *Let $S = \{s_1, \ldots, s_k\}$ be a free subset of* Sq. *The serie $\mathbf{G}_S$ may be computed (theoretically) using the properties of the gas model, since it suffices to compute $\mathbb{P}(X_s = 1, s \in S)$. This can be done by writing a finite sum involving the occupations of the cells that are above the s_i's and under the line L_j with the largest index such that $L_j \cap S$ is not empty (inside the region surrounded by the dotted lines in Fig. 10), and using the Markovianity of the gas occupation on L_j.*

The Markovianity of Z implies some (weighted) properties of DA taking into account the occupation or not of their sources; here, by weighted we mean that here we deal with probabilities, and then each DA A has weight $p^{|A|}(1-p)^{|\mathcal{P}(A)|}$ which depends on its area and perimeter. Let $z := (z_0, \ldots, z_k)$ be fixed in $\{0, 1\}$. The probability of

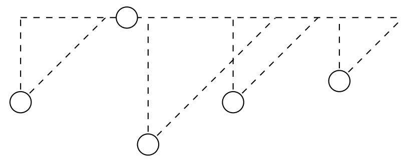

Figure 10: The points stands for the points of S; the dotted lines surround the (finite) set of cells where the summation has to be done.

the event $(Z(0),\dots,Z(k)) = z$ depends only on the four quantities $n_k^{\bullet\bullet}(z) = \#\{i, i \in \{0\dots k-1\}, (z_i, z_{i+1}) = (1,1)\}$, $n_k^{\bullet\circ}(z)$, $n_k^{\circ\bullet}(z)$ and $n_k^{\circ\circ}(z)$, defined accordingly. Hence $\mathbb{P}((Z(0),\dots,Z(5)) = z) = \mathbb{P}((Z(0),\dots,Z(5)) = z')$ for $z = (0,0,1,1,0,0)$ and $z' = (0,1,1,0,0,0)$. In terms of DA with sources $S = \{(i,-i)\}_{i=0\dots5} = \{v_i\}_{i=0\dots5}$ it leads to

$$\sum_{A,\chi_{v_i}(A)=z_i, i\in\{0,\dots,5\}} p^{|A|}(1-p)^{|\mathcal{P}(A)|} = \sum_{A,\chi_{v_i}(A)=z'_i, i\in\{0,\dots,5\}} p^{|A|}(1-p)^{|\mathcal{P}(A)|}.$$

A bijection between DA with the considered occupation of their sources and preserving moreover the area and the perimeter would be a direct explanation of this identity, but such a map does not exist since the enumeration of these two sets of DA according to the area and the perimeter are distinct for DA of area 5:

$$\sum_{A,|A|\le 5,\chi_{v_i}(A)=z_i, i\in\{0,\dots,5\}} t^{|A|}u^{|\mathcal{P}(A)|} = u^7t^2 + (2u^8+10u^9)t^4 + (2u^8+7u^9)t^5$$

and

$$\sum_{A,|A|\le 5,\chi_{v_i}(A)=z'_i, i\in\{0,\dots,5\}} t^{|A|}u^{|\mathcal{P}(A)|} = u^7t^2 + (2u^8+10u^9)t^4 + (1u^8+8u^9)t^5.$$

3.3 Proof of Theorem 3.3

One may propose several proofs of Theorem 3.3: following the proof of Bousquet-Mélou [4], one may first work on the cylinder as done in Section 5.3. There the process Z is in some sense Markovian on the "horizontal lines", and conditioned to come back at its starting point. Theorem 3.3 may be obtained by a passage to the limit on the length of the cylinder (the limit in distribution here is just a convergence of the finite dimensional distribution).

More directly, another method consists in proving first Proposition 3.5: three steps are needed. At the beginning, suppose that Z_1 is a Markov chain with two states 0 and 1. Then using that is must be preserved by the gas evolution, there is at most one Markov chain satisfying these constraints (easy). Let M be its transition matrix. It remains to show that this Markov chain is indeed preserved by the gas evolution. This is done via some linear algebra on M, showing that if $(Z_1(i),\dots,Z_1(i+k))$ is a M-Markov chain, so do $(Z_0(i), \ldots, Z_0(i+k-1))$ (this is done via some quite laborious computation). Proposition 3.1 allows to conclude.

The proof of Theorem 3.3 that we propose below may be viewed as a translation of what is said above. But we think that it is in fact different and is more likely to be generalized to other models. Another pleasant reason is the natural appearance of the gas density in the beginning of the considerations (see Comment 3.4): using the other approaches, this is not the case.

Proof of Theorem 3.3. Consider $(b_i^\circ)_{i\in\{1,\ldots,k\}}$ and $(b_i^\bullet)_{i\in\{1,\ldots,k\}}$ be two fixed vectors in $\{1,2,\ldots\}^k$. For j in $\{0,1\}$ we denote the block sizes of Z_j by $B_{i,j}^\circ$ and $B_{i,j}^\bullet$. We assume that $Z_1(0)$ and the block sizes of Z_1 have the distribution described in the Theorem, and we prove then that $Z_0(0)$ and the blocks of Z_0 have the same distribution. By Proposition 3.1 this is sufficient to prove the Theorem.

To compute $\mathbb{P}(Z_0(0) = 1, B_{i,0}^\circ = b_i^\circ, B_{i,0}^\bullet = b_i^\bullet, i \in \{1,\ldots,k\})$, we will sum on all possible block configurations of Z_1 on the positions $\{0,\ldots,1+\sum_{i=0}^k b_i^\circ + b_i^\bullet\}$. We recall that by the gas model of type 1, the process Z_0 is a solution of (15) (see Fig. 11). Thus, knowing k consecutive occupied cells in Z_0 allows to guess k Bernoulli random variables and a block of $k+1$ empty positions in Z_1 (that maybe included in a larger empty block). In particular, if $Z_0(0) = 1$ then $Z_1(0) = 0$.

The empty blocks of Z_0 are a little bit more tricky to handle: for each empty block in Z_0 with size b_i°, there are $b_i^\circ - 1$ cells in Z_1 that can be occupied or empty (see Fig. 11). We call these cells *uncertain cells.*

Notice also that $\{Z_0(0) = 1, B_{i,0}^\circ = b_i^\circ, B_{i,0}^\bullet = b_i^\bullet, i \in \{1,\ldots,k\}\} = \{Z_0(0) = 1, B_{i,0}^\circ = b_i^\circ, B_{i,0}^\bullet = b_i^\bullet, i \in \{1,\ldots,k\}, Z_0(\sum_{i=1}^k b_i^\bullet + b_i^\circ) = 1\}$ since knowing the size of the $2k$ first blocks and the values of $Z_0(0)$ implies the occupation of the cell that follows.

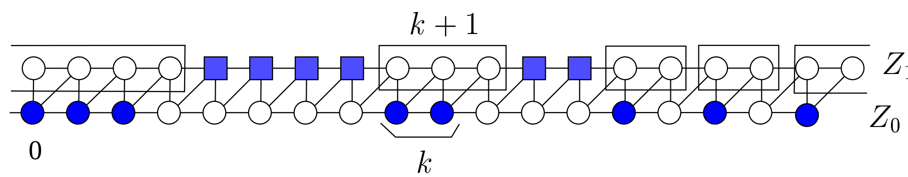

Figure 11: A block of k occupied positions in Z_0 determines $k+1$ empty position on Z_1 (surrounded by rectangles). The little squares figure out the uncertain cells, that can not be determined just in view of the gas occupation in Z_0.

Hence, we have

$$\begin{aligned}\mathbb{P}(Z_0(0) = 1, B_{i,0}^\circ = b_i^\circ, B_{i,0}^\bullet = b_i^\bullet, i \in \{1,\ldots,k\}) &= \frac{\alpha_\circ^{-1}}{\alpha_\circ^{-1} + \alpha_\bullet^{-1}}[(1-\alpha_\circ)p]^{\sum_{i=1}^k b_i^\bullet} \\ &\quad \times \left(\prod_{i=1}^k S_{b_i^\circ}\right)(1-\alpha_0)p.\end{aligned}$$

The first factor $\frac{\alpha_\circ^{-1}}{\alpha_\circ^{-1}+\alpha_\bullet^{-1}}$ on the right hand side comes from $Z_1(0)=0$. The other term are computed successively, as transition probabilities:
– the term $(1-\alpha_\circ)p$ corresponds to the creation of an occupied cell x in Z_0 knowing that x is empty in Z_1. Using that the events $\{Z_0(x)=1, Z_1(x)=0\}$ and $\{Z_0(x)=1, Z_1(x)=0, Z_1(x+1)=0\}$, we have:

$$\mathbb{P}(Z_0(x)=1|Z_1(x)=0) \quad = \quad \frac{\mathbb{P}(Z_0(x)=1, Z_1(x)=Z_1(x+1)=0)}{\mathbb{P}(Z_1(x)=0))}.$$

This is also $\mathbb{P}(Z_0(x)=1|Z_1(x)=Z_1(x+1)=0)\mathbb{P}(Z_1(x+1)=0|Z_1(x+1)=0)=p(1-\alpha_\circ)$. In other words knowing that x in empty in Z_1 it is occupied in Z_0 if there is a transition empty-empty in Z_1 between x and $x+1$ and a favorable coloring $C_x=a$ of x (proba. p). There are $\sum_{i=1}^k b_i^\bullet$ such terms.
– The last term $(1-\alpha_0)p$ comes from the creation of an occupied cell after these blocks in Z_0.
– The factor $S_{b_i^\circ}$ corresponds to the contributions of the uncertain cells in Z_1 above the b_i° empty cells of ith empty considered block of Z_0: more precisely

$$S_l=\mathbb{P}(Z_0(1)=0,\ldots,Z_0(l)=0, Z_1(l+1)=0|Z_1(1)=0). \tag{19}$$

Above the empty cells $1,\ldots,l$ in Z_0 there are $l-1$ uncertain cells in Z_1, at position $2,\ldots,l$.

For $l=1$ there is no uncertain cell above and we find $S_1=(1-\alpha_\circ)(1-p)$. For $k\geq 2$, we write $S_k=S_k^\bullet+S_k^\circ$, where $S_k^\bullet$ (resp. S_k°) corresponds to formula (19), where in the RHS is added $Z_1(l)=1$ (resp. $Z_1(l)=0$), in other words where the occupation of the last uncertain cell is specified. For $k=1$, we take the convention $S_1^\circ=S_1$ and $S_1^\bullet=0$. For $k\geq 2$, there is a simple decomposition of $S_k^\bullet$ and S_k° obtained by removing the last cells of this block:

$$\left\{\begin{array}{lcl} S_l^\circ & = & S_{l-1}^\circ(1-\alpha_\circ)(1-p)+S_{l-1}^\bullet(1-\alpha_\circ)(1-p) \\ S_l^\bullet & = & S_{l-1}^\circ\dfrac{\alpha_\circ\alpha_\bullet}{(1-\alpha_\circ)(1-p)}+S_{l-1}^\bullet(1-\alpha_\bullet) \end{array}\right.$$

Using (17), this finally gives $S_l^\circ=(1-p)(1-\alpha_\circ)S_{l-1}$ and $S_l^\bullet=(1-\alpha_\circ)pS_{l-1}$, then $S_l=(1-\alpha_\circ)^{l-1}S_1=(1-\alpha_\circ)^l(1-p)$. Thus

$$\begin{array}{l} \mathbb{P}(Z_0(0)=1, B_{i,0}^\circ=b_i^\circ, B_{i,0}^\bullet=b_i^\bullet, i\in\{1,\ldots,k\}) \\ \qquad\qquad = \quad \dfrac{(1-\alpha_\circ)p}{\dfrac{\alpha_\circ}{\alpha_\circ^{-1}+\alpha_\bullet^{-1}}}[(1-\alpha_\circ)p]^{\sum_{i=1}^k b_i^\bullet}(1-\alpha_\circ)^{\sum_{i=1}^k b_i^\circ}(1-p)^k \end{array}$$

using that $(1-\alpha_\circ)(1-\alpha_\bullet)(1-p)=\alpha_\circ\alpha_\bullet$, and $(1-\alpha_\circ)p/\alpha_\circ=\alpha_\bullet^{-1}$, we get

$$\mathbb{P}(Z_0(0)=1, B_{i,0}^\circ=b_i^\circ, B_{i,0}^\bullet=b_i^\bullet, i\in\{1,\ldots,k\}) \quad = \quad \frac{\alpha_\bullet^{-1}}{\alpha_\circ^{-1}+\alpha_\bullet^{-1}}\prod_{i=1}^k\mathbb{P}(G^\bullet=b_i^\bullet)\mathbb{P}(G^\circ=b_i^\circ)$$

which is the expected result.
The computation of $\mathbb{P}(Z_0(0) = 0, B^\circ_{i,0} = b^\circ_i, B^\bullet_{i,0} = b^\bullet_i, i \in \{1,\dots,k\})$ can be performed along the same lines, except that here the two cases $Z_1(0) = 1$ and $Z_1(1) = 1$ have to be considered. $\square$

4 Simultaneous construction of DA and gas model of type 2

We construct in this section a probability space on G in such a way that the density of the gas model equals up to some change of variables the area and perimeter GF for DA. This approach is very similar to that of Section 2. Here a probability measure having two parameters is needed. The idea is again to transfer a quantity revealing the perimeter size via a DA of calculus. The main difference with Section 2 is that here, the revealing quantity is random when it was deterministic in Section 2 (it was $\chi(A)$). We endow the perimeter sites with i.i.d Bernoulli(p) random variables, and we will transfer to the source the minimum of those random variables. Assume that N is random and that you know the law of $M := \min\{B_1,\dots,B_N\}$ for $B_1,\dots,B_N$ i.i.d. Bernoulli(p) random variables, independent of N. It is straightforward that the distribution of M characterizes that of N. This is morally what we use here.

Consider G an agreeable graph, and the probability space $\Omega^G_\star = \{a,b,c\}^G$, endowed with the probability measure

$$\mathbb{P}_{p_a,p_b,p_c} = (p_a\delta_a + p_b\delta_b + p_c\delta_c)^{\otimes G},$$

where p_a, p_b, p_c are three non negative parameters summing to 1. In other words, we have again a random coloring of G, where the nodes colors are i.d.d. and follows the following distribution (we write $C^\star_x$ the color of x). Under $\mathbb{P}_{p_a,p_b,p_c}$, for any x, $C^\star_x$ is a with probability p_a, $C^\star_x$ is b with probability p_b and $C^\star_x$ is c with probability p_c.

We define now $\mathbf{A}^S_\star(\omega)$ in the same way as $\mathbf{A}^S(\omega)$ is defined in the beginning of Section 2.1: $\mathbf{A}^S_\star(\omega)$ has again as set of cells the maximal DA with over-source S whose cells are all a-colored; now the perimeter sites of $\mathbf{A}^S_\star$ may have the color b or c. In order to take into account these colors, we introduce

$$\mathcal{P}_b(A) = \{x, x \in \mathcal{P}(A), C^\star_x = b\}, \quad \text{and} \quad \mathcal{P}_c(A) = \{x, x \in \mathcal{P}(A), C^\star_x = c\},$$

the perimeter sites of A having color b and c. The set of DA on G with bi-colored perimeter sites and source S is denoted by $\mathcal{A}^\star(S,G)$, and we add in the same way a star to the variables already defined in the previous section to make visible that we are working with colored objects.

The following proposition is the analogous, or more precisely a refinement, of Proposition 2.1:

Proposition 4.1 *Let $G = (V, E)$ be an agreeable graph.*
(i) Let $B \in \mathcal{A}^\star(S, G)$ be a finite DA with source (exactly) the free set S. We have

$$\mathbb{P}_{p_a,p_b,p_c}(\mathbf{A}_\star^S = B) = p_a^{|B|} p_b^{|\mathcal{P}_b(B)|} p_c^{|\mathcal{P}_c(B)|}.$$

(ii) Let $B \in \overline{\mathcal{A}^\star(S, G)}$ be a finite DA with over-source the free set S. We have

$$\mathbb{P}_{p_a,p_b,p_c}(\mathbf{A}_\star^S = B) = p_a^{|B|} p_b^{|\overline{\mathcal{P}_{S,b}}(B)|} p_c^{|\overline{\mathcal{P}_{S,c}}(B)|},$$

where for $d \in \{b, c\}$, $\overline{\mathcal{P}_{S,d}}(B) = \mathcal{P}_d(A) \cup \{x, x \in S, C_x^\star = d\}$.

Our gas model of type 2 on G is built as follows. For any $x \in V$ and $\omega \in \Omega$, set

$$X_x^\star(\omega) := \begin{cases} 0 & \text{if } C_x(\omega) = b \\ 1 & \text{if } C_x(\omega) = c \\ \min\limits_{d\text{: children of } x} X_d^\star(\omega) & \text{if } C_x(\omega) = a \end{cases} \tag{20}$$

$$= \mathbb{1}_{C_x^\star = c} + \mathbb{1}_{C_x^\star = a} \min_{d\text{: children of } x} X_d(\omega). \tag{21}$$

Once again this recursive definition allows to eventually compute $X_x^\star$ for any x, if $p_a < p_{crit}^{\{x\}}$. By a simple recursion one easily notices that the values $X_x^\star$ all belong to $\{0, 1\}$.

Remark 4.2 *Since the values $X_x^\star$ all belong to $\{0, 1\}$, the* min *operator coincides with the product, and also with the "and" operator, interpreting the gas occupation as Boolean variables. Bousquet-Mélou [4] when she considers the case (p_1, p_2, p_2, p_2) in the square lattice case works under an equivalent model even if she uses a different vocabulary (we refer to Section 5.3 for some hints).*

Theorem 4.3 *Let $G = (V, E)$ be an agreeable graph.*
(i) For any $x \in V$ and $p_a < p_{crit}^{\{x\}}$ we have

$$\mathbb{E}_{p_a,p_b,p_c}(X_x^\star) = p_c + \mathbb{P}_{p_a,p_b,p_c}(|\mathcal{P}_b(\mathbf{A}^\star)| = 0) = p_c + \mathbf{G}^x(p_a, p_c),$$

where $\mathbf{G}^x(u, v) = \sum_{A,|A|>0,\mathcal{S}(A)=x} u^{|A|} v^{|\mathcal{P}(A)|}$ is the GF of the set of DA with source $\{x\}$ counted according to their area and perimeter.
(ii) Let S be a free finite subset of G, and $p_a < p_{crit}^S$. Under $\mathbb{P}_{p_a,p_b,p_c}$, we have

$$\mathbb{E}_{p_a,p_b,p_c}\left(\prod_{x \in S} X_x^\star\right) = \mathbb{P}_{p_a,p_b,p_c}(X_x^\star = 1, x \in S) = \overline{\mathbf{G}}^S(p_a, p_c). \tag{22}$$

where $\overline{\mathbf{G}}^S(u, v) = \sum_{A,\mathcal{S}'(A) \subset S} u^{|A|} v^{|\overline{\mathcal{P}}(A)|}$ is the GF of the set of DA with over source S counted according to their area and perimeter.

Proof (i) The first equality follows the fact that $X_x^\star = 1$ if and only if $C_x^\star = c$ or $\mathbf{A}_\star^{\{x\}}$ contains x and has only c-colored perimeter sites. For the second equality, write

$$\mathbb{P}_{p_a,p_b,p_c}(|\mathcal{P}_b(\mathbf{A}_\star^{\{x\}})| = 0) = \sum_{A,|A|>0,\mathcal{S}(A)=x} p_a^{|A|} p_c^{|\mathcal{P}(A)|} = \mathbf{G}^x(p_a, p_c).$$

The proof of (ii) follows the same lines: if all the values $X_x^\star$ for $x \in S$ equals 1, then the DA A^S has all its perimeter sites c-colored, and the points of $S \setminus S_\bullet$ must be c-colored. □

4.1 Comments

The gas model of type 2 on the square lattice (or on another lattice), under $\mathbb{P}_p$ for $p < p_{crit}$ is well defined. We can establish a formula similar to (15) and a Proposition similar to Proposition 3.1 for the gas model of type 2, using the same arguments. To compute the density of the gas model of type 2 (this would provide a way to compute p_{crit} which is unknown) a strong property of the gas must be found: roughly speaking it seems that a perfect knowledge of the distribution of the gas on the whole lattice is necessary. In the case of the gas model of type 1 the process X_x for x moving along a choosing line is Markovian. The gas model of type 2 even if very similar is not Markovian on the lines except for very special values of p_a, p_b and p_c (and no more Markovian of order 2, and we think no Markovian of order k for any k).

We want here to point out that some gas models generalizing the gas model of type 2 may have the wanted property to be Markovian (or have some suitable structural properties). An idea would be to enrich the gas in building a model of gas taking its values in a set larger than $\{0,1\}$ (the gas model of type 2 will appear as a kind of projection of the generalized model), and again use the min operator inside the DA of calculus. The minimum of some random variables with a prescribed law ν would replace the minimum of Bernoulli random variables. By chance, maybe there exists a parameter ν (or a family of distribution ν) for which the gas has a simple description.

5 Appendix

5.1 Proof of Proposition 2.2

Consider a random DA $\mathbf{A}^S$. Denote by $L_0 = S \cap \mathbf{A}^S$. Then for any $i \geq 1$, set

$$L_i = \{v, v \in \mathbf{A}^S, v \text{ has a father in } L_{i-1}, v \notin \cup_{j \leq i-1} L_j\}.$$

The random sequence (L_i) gives a decomposition of $\mathbf{A}^S$ into layers: the cells in L_i are the children of the nodes in L_{i-1} that are not children of any cells belonging to L_j, $j \leq i-2$. We want to prove that if $p < 1/K$ then $\mathbb{P}_p$ a.s., there exists k such that $|L_k| = 0$.

The number N_c of children in A of a given cell c of A has law Binomial$(o(c), p)$, where $o(c)$ is the out degree of c in G, that is

$$\mathbb{P}_p(N_c = j) = \binom{o(c)}{j} p^j (1-p)^{o(c)-j} \text{ for } j \in \{0, \dots, o(c)\}.$$

Therefore, given $L_i = l_i$ for $i \leq k$, the random variable $|L_{k+1}|$ has law Binomial$(o(C), p)$ where $o(C)$ is the number of children of the nodes of l_k in G that are not children of any nodes of the l_i,$i < k$. Hence, given $L_i = l_i$ for $i \leq k$, the random variable $|L_{k+1}|$ is smaller for the stochastic order than Binomial$(K\,|l_k|, p)$: we say that X is smaller than Y for the stochastic order if $\mathbb{P}(X \geq x) \leq \mathbb{P}(Y \geq x)$ for any x (we write $X \leq_S Y$). This is simple property of the binomial distribution clear via the representation of binomial random variables as sum of i.i.d. Bernoulli random variables. A simple iteration, shows that $|L_k|$ is smaller for the stochastic order than Z_k where $Z_0 = |L_0|$, and where given $Z_{k-1} = z_{k-1}$, Z_k is Binomial(Kz_{k-1}, p) distributed. There exists a probability space, on which one may construct two sequences (l'_k) and (Z'_k) such that

$$(l'_k) \overset{(d)}{=} (|L_k|) \text{ and } (Z'_k) \overset{(d)}{=} (Z_k)$$

and such that a.s. $l'_k \leq Z'_k$. On this space, $\inf\{k, l'_k = 0\} \leq T := \inf\{k, Z'_k = 0\}$.

But, Z'_k is simply a Galton-Watson process with offspring distribution Binomial(K, p) (see Athreya & Ney [1]), and then it is classical that such a Galton-Watson process eventually dies out ($T < +\infty$ a.s.) if the mean of its offspring distribution is smaller than 1 (if $|L_0|$ is finite, which is the case here); here it is the case if $p \leq 1/K$. $\square$

Comments 5.1 *If G is a tree in which at depth k, all the nodes have $k+1$ children, then $p_{crit} = 0$. Indeed, under $\mathbb{P}_p$, the mean number of children of an individual having depth larger than $1/p$ is larger than 1. In the following levels, the number of individuals are larger for the stochastic order than a super-critical Galton-Watson process, for which $T = +\infty$ with positive probability.*

5.2 Proof of Theorem 2.16

The first equality is clear, and to prove the second one a generalization of the proof of Theorem 2.12 is needed; we skip the details and give only the main lines of this generalization.

First, following the lines of Proposition 2.9, we have

$$-\mathbf{G}_S^G(-p) \;\; = \;\; \mathbb{E}_p\left(D_S(\mathcal{A}^S)\right) \tag{23}$$

where for any finite DA A,

$$D_S(A) = \mathbb{1}_{\mathcal{S}(A)=S} \sum_{A':\; A' \prec A, \mathcal{S}(A')=S} (-1)^{|A'|+1}. \tag{24}$$

Let $\mathbf{f} = (\mathbf{t}_1, \dots, \mathbf{t}_k) \in \mathcal{T}^k$ be a forest with k trees. We say that $\mathbf{f}$ is embeddable in G at S if for any $i \in \{1, \dots, k\}$, $\mathbf{t}_i$ is embeddable in G at s_i. We set

$$\pi_S(\mathbf{f}) = \bigcup_{i \in \{1,k\}} \pi_{s_i}(\mathbf{t}_i)$$

the union of the embeddings of the $\mathbf{t}_i$'s on the $s'_i s$. For any forest $\mathbf{f}$ embeddable in G at S, $\pi_S(\mathbf{f})$ is a DA on G with source S. For any DA A with source S in G, we set

$$Y_S(A) := \sum_{\mathbf{f}=(\mathbf{t}_1,\dots,\mathbf{t}_k):\; \mathbf{f} \in \mathcal{T}^k, \pi_S(\mathbf{f}) \prec A} (-1)^{|\mathbf{t}_1|+\cdots+|\mathbf{t}_k|+k},$$

where the sum is taken on the set of forests embeddable in G, and

$$\Delta_S(A) := \mathbb{1}_{\mathcal{S}(A)=S} \sum_{A':\ A' \prec A, \mathcal{S}(A')=S} Y_S(A'). \tag{25}$$

A simple analysis yields $Y_S(A) := \prod_{i=1}^{k} \Delta_{s_i}(A_i)$ where A_i is the maximal sub-DA of A with source s_i. For $A = S$, we have $Y_S(A) = 1$. Adapting the proof of Lemma 2.15 to forests, we get

$$Y_S(A) = (-1)^{|A|-|S|}.$$

By (24) for any A with source S,

$$D_S(A) = (-1)^{1+|S|} \Delta_S(A). \tag{26}$$

It remains to relate $\Delta_S(A)$ with $\prod_{s \in S} X_s$. By definition of the gas model of type 1, we have

$$\begin{aligned} \{\omega, X_s(\omega) = 1, s \in S\} &= \{\omega, C_s(\omega) = a, s \in S, X_y(\omega) = 0, y \in S_1\} \\ &= \left\{\omega, \mathcal{S}(A^S(\omega)) = S, \prod_{y \in S_1} \left(1 - \chi_y(A^{(y)}(\omega))\right) = 1\right\} \end{aligned}$$

where S_1 is those vertices in G, children of elements of S. Write

$$\{\omega, \Delta_S(A^S(\omega)) = 1\} = \{\omega, C_s(\omega) = a, s \in S, \prod_{s_i \in S} \Delta_{s_i}(A_i^{(s_i)}(\omega)) = 1\}$$

Now, consider $S_1(i)$ the children of s_i in G. By the proof of Theorem 2.12,

$$\Delta_{s_i}(A_i^{(s_i)}(\omega)) = \mathbb{1}_{\{\omega, C_{s_i}(\omega)=a\}} \prod_{y \in S_1(i)} (1 - \Delta_y(A^y(\omega))) \tag{27}$$

and then

$$\prod_{s_i \in S} \Delta_{s_i}(A_i^{(s_i)}(\omega)) = \prod_{s_i \in S} \left(\mathbb{1}_{\{\omega, C_{s_i}(\omega)=a\}} \prod_{y \in S_1(i)} (1 - \Delta_y(A^y(\omega))) \right).$$

Now, notice that a node y in S_1 may be in several $S_1(i)$'s, but since $1 - \Delta_y(A^y(\omega)) \in \{0, 1\}$ the repetition of this factor in the product (27) may be simplified; we then obtain

$$\prod_{s_i \in S} \Delta_{s_i}(A_i^{(s_i)}(\omega)) = \left(\mathbb{1}_{\{\omega, C_s(\omega)=a, \forall s \in S\}} \prod_{y \in S_1} (1 - \Delta_y(A^y(\omega))) \right).$$

Using (10), this is equal to $\mathbb{1}_{\{\omega, C_s(\omega)=a, \forall s \in S\}} \prod_{y \in S_1} (1 - \chi_y(A^{(y)}(\omega))$. Finally we get

$$\{\omega, X_s(\omega) = 1, s \in S\} = \{\omega, \Delta_S(A^S(\omega)) = 1\}$$

and this, together with (26) and (23) allow to conclude. $\square$

5.3 Square lattice DA counted according to the area: The historical approach

The content of this section 5.3 is a reformulation of a part of Bousquet-Mélou [4] concerning the enumeration of DA on the square lattice using a gas model of type 1 (this terminology is not introduced in [4]). She avoids the statistical mechanics arguments used by Dhar [9];

First, instead of working on the square lattice, a cyclic directed square lattice having n cells in each row is introduced: the cylinder $\mathsf{cy}(n) = (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z}$, for $n \geq 2$ which is the lattice Sq quotiented by a congruence relation: (x, y) has children $(x \mod n, y+1)$ and $(x+1 \mod n, y+1)$.

Consider a DA A with source C in $\mathsf{cy}(n)$. Removing the first row of A, the following formula is easily checked,

$$\mathbf{G}_C^{\mathsf{cy}}(x) = x^{|C|} \sum_{D \subset \mathcal{N}(C)} \mathbf{G}_D^{\mathsf{cy}}(x), \tag{28}$$

where $\mathcal{N}(C) = \{i, i+1 \mod n, i \in C\}$ is the perimeter sites of C.

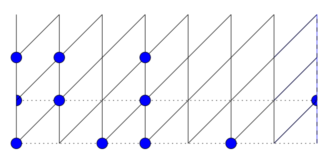

Figure 12: A DA on the cylinder. The maximum source has 4 cells, the area is 10. The right most and left most vertices are identified.

Then, Bousquet-Mélou introduced a gas model equivalent to the gas model of type 1. The rows of the cylinder are labeled by integers, the $i+1$ row being above the ith (see Fig. 12). The gas occupation on a row is random vector $X = (X_1, \ldots, X_n)$ taking its values in $\{0,1\}^n$; its distribution is described by $F_C = \mathbb{P}(X_i = 1, i \in C, X_i = 0, i \in {}^cC)$ or by $f_C = \mathbb{P}(X_i = 1, i \in C)$ using the inclusion-exclusion principle. For any j, the vector X^j giving the distribution of the gas on the jth row is given from the X^{j+1}th by the following stochastic evolution (gas model of type 1):

$$X_i^j = B_i^j(p)(1 - X_i^{j+1})(1 - X_{(i+1 \mod n)}^{j+1}) \quad \text{for any } i \in \mathbb{Z}/n\mathbb{Z}, j \in \mathbb{Z}$$

where the $B_i^j(p)$ are i.i.d. Bernoulli(p) random variables (in other words, a cell i in the jth row is occupied with probability p if and only if i and $i+1$ are empty in the $j+1$th cell, and if a Bernoulli(p) tossing is a success, the Bernoulli tossing being independent).

One gets $f_C^j(p) = p^{|C|}\mathbb{P}(X_i^{j+1} = 0, i \in \mathcal{N}(C))$ and by the inclusion-exclusion principle,

$$f_C^i(p) = p^{|C|} \sum_{D \subset \mathcal{N}(C)} (-1)^{|D|} f_D^{i+1}(p), \tag{29}$$

this in terms of $F^{(i)}$ reads

$$F_C^i(p) = \left(\frac{p}{1-p}\right)^{|C|} \sum_{D\in\{1,\ldots,n\}\setminus\mathcal{N}(C)} (1-p)^{n-|\mathcal{N}(D)|} F_D^{i+1}(p).$$

The sequence (X^j) forms a Markov chain which is aperiodic, irreducible with a finite number of states. Hence, there exists a unique invariant distribution F; the corresponding function f satisfies

$$f_C(p) = p^{|C|} \sum_{D\subset\mathcal{N}(C)} (-1)^{|D|} f_D(p). \tag{30}$$

Now, the system (30) is solved explicitly by checking that

$$F_D(p) = \frac{1}{Z_n}\left(\frac{p}{1-p}\right)^{|D|}(1-p)^{|\mathcal{N}(D)|} \text{ where } Z_n = \sum_{D\subset\{1,\ldots,n\}}\left(\frac{p}{1-p}\right)^{|D|}(1-p)^{|\mathcal{N}(D)|} \tag{31}$$

is a solution. Notice that for each C, F_C is a rational function of p since it is a solution of a linear system with polynomial coefficient in p. Observe (28) and (30). If f is a solution of (30) then

$$\mathbf{G}_C^{\mathsf{cy}}(-p) = (-1)^{|C|} f_C(p)$$

is a solution of (28) (in fact there is only one solution up to a multiplicative constant, which is fixed taking $-\sum_C \mathbf{G}_C^{\mathsf{cy}}(-p) = 1$). Denote by $\mathbf{G}_c^{\mathsf{cy}}$ the series of DA with source c, a unique cell. We have

$$-\mathbf{G}_c^{\mathsf{cy}}(-p) = f_c(p) = \mathbb{P}(X_1 = 1),$$

the so-called *density* of the gas model. The series $\mathbf{G}_c^{\mathsf{cy}}$, we should write $\mathbf{G}_c^{\mathsf{cy}(n)}$, is the series of DA in the cylinder. It is clear that its first n coefficients coincide with that of the series $\mathbf{G}_c^{\mathsf{sq}}$ of DA on the square lattice with source O. Hence in the space of formal series

$$\mathbf{G}_c^{\mathsf{cy}(n)} \underset{n\to\infty}{\longrightarrow} \mathbf{G}_c^{\mathsf{Sq}}.$$

The explicit solution of the gas model on $\mathsf{cy}(n)$ is derived as follows: using that $|\mathcal{N}(D)| = |D| + |\mathcal{N}_r(D)|$ where $\mathcal{N}_r(D) = \{i, i\in D, i+1\notin D\}$, Z_n reads $\sum_{D\subset\{1,\ldots,n\}} p^{|D|}(1-p)^{|\mathcal{N}_r(D)|}$ and is guessed (or viewed) to have the following form

$$Z_n = \sum_{(y_1,\ldots,y_n)\in\{0,1\}^n} \prod_{i=1}^{n} V(y_i, y_{i+1})$$

where $V(0,0) = V(0,1) = 1$, $V(1,0) = p(1-p)$, $V(1,1) = p$ and $y_{n+1} := y_1$. This corresponds in some sense to a "cyclical Markovian model" where a transition $0\to 1$ or $0\to 0$ is "counted" 1, a transition $1\to 0$ is counted $p(1-p)$, a transition $1\to 1$, counted p. In other words, since $y_{n+1} = y_1$, Z_n is $tr(Y^n)$ where $Y = \begin{pmatrix} 1 & 1 \\ p(1-p) & p \end{pmatrix}$;

this is equal to $\lambda_1^n + \lambda_2^n$ where $\lambda_1 > \lambda_2$ are the two eigenvalues of Y, that are: $\lambda_{1,2} = \frac{1+p\pm\sqrt{1+2p-3p^2}}{2}$. It remains to express the density. In Bousquet-Mélou [4] this is done by differentiation of (31). One may also compute this density by computing W_n/Z_n where $W_n = (Y^n)_{2,2}$ which corresponds to the contribution of the configurations where the first cell is occupied (in other words, $y_1 = y_{n+1} = 1$). With very simple linear algebra, we get $W_n = (\lambda_1^n(\lambda_1 - 1) + \lambda_2^n(\lambda_2 - 1))/(\lambda_2 - \lambda_1)$. Since $\lambda_1 > \lambda_2$,

$$W_n/Z_n \to (1-\lambda_1)/(\lambda_2 - \lambda_1) \tag{32}$$

which is equal to $\alpha_\bullet^{-1}/(\alpha_\bullet^{-1} + \alpha_\circ^{-1})$. The convergence in (32) is considered in the formal sense.

Comments 5.2 *In fact, the solution of (31) is more or less guessed by Dhar [9] who works on the whole lattice. He "notices" that the distribution of particles on two consecutive lines coincides with the hard particle distribution of activity $p/(1-p)$ on two consecutive lines (seemingly using Verhagen [15] work). On the cylinder, this can be interpreted saying that the probability to see j occupied cells on $G_i \cup G_{i+1}$ is $\lambda(p/(1-p))^j$ where λ is a constant. Given this, the distribution on a line is then just a marginal of this distribution, easy to compute.*

In [4], Bousquet-Mélou generalized the study of gas model allowing some new evolutions between lines. With these tools, she is able to give some formal link between density of a gas and GF of DA on the square lattice (and also on other lattices) counting DA according to several parameters, the area, the perimeter, the right perimeter and the "loops". These links are once again formal. For example, she derived the following formula: the area and perimeter GF for one-source DA on the cyclic square lattice is

$$\sum_A t^{|A|} x^{|\mathcal{P}(A)|} = 1 - x - \rho(p_1, p_2, p_2, p_2) \tag{33}$$

where $p_1 = 1 - x - t$ and $p_2 = 1 - x$. The quantity $\rho(p_1, p_2, p_3, p_4)$ is the density of a model of gas obtained by the rules evolution given in the following table.

Value of (X_x, X_{x+1}) in Z_1	0 and 0	1 and 0	0 and 1	1 and 1
Value of X_x in Z_0	Bernoulli(p_1)	Bernoulli(p_2)	Bernoulli(p_3)	Bernoulli(p_4).

References

[1] K.B. Athreya and P.E. Ney. *Branching processes.* Die Grundlehren der mathematischen Wissenschaften. Band 196. Berlin-Heidelberg-New York: Springer-Verlag. XI, 1972.

[2] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. Directed animals, forests and permutations. *Discrete Math.*, 204(1-3):41–71, 1999.

[3] M. Bousquet-Mélou and A. R. Conway. Enumeration of directed animals on an infinite family of lattices. *J. Phys. A*, 29(13):3357–3365, 1996.

[4] M. Bousquet-Mélou. New enumerative results on two-dimensional directed animals. *Discrete Math.*, 180(1-3):73–106, 1998.

[5] J. Bétréma and J.G. Penaud. Animaux et arbres guingois. (Animals and guingois trees). *Theor. Comput. Sci.*, 117(1-2):67–89, 1993.

[6] J. Bétréma and J.G. Penaud. Modèles avec particlues dures, animaux dirigés, et séries en variables partiellement commutatives. *Arxiv: math.CO/0106210*, pages 1–49, 1993.

[7] S. Corteel, A. Denise, and D. Gouyou-Beauchamps. Bijections for directed animals on infinite families of lattices. *Ann. Comb.*, 4(3-4):269–284, 2000. Conference on Combinatorics and Physics (Los Alamos, NM, 1998).

[8] D. Dhar. Equivalence of the two-dimensional directed-site animal problem to Baxter's hard-square lattice-gas model. *Phys. Rev. Lett.*, 49(14):959–962, 1982.

[9] D. Dhar. Exact solution of a directed-site animals-enumeration problem in three dimensions. *Phys. Rev. Lett.*, 51(10):853–856, 1983.

[10] D. Dhar, M.K. Phani, and M. Barma. Enumeration of directed site animals on two-dimensional lattices. *J. Phys. A*, 15(6):L279–L284, 1982.

[11] D. Gouyou-Beauchamps and G. Viennot. Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem. *Adv. in Appl. Math.*, 9(3):334–357, 1988.

[12] V. Hakim and J. P. Nadal. Exact results for 2D directed animals on a strip of finite width. *J. Phys. A*, 16(7):L213–L218, 1983.

[13] Y. Le Borgne. Variations combinatoires sur des classes d'objets comptées par la suite de catalan. *PhD thesis, Bordeaux*, 2004.

[14] J. P. Nadal, B. Derrida, and J. Vannimenus. Directed lattice animals in 2 dimensions: numerical and exact results. *J. Physique*, 43(11):1561–1574, 1982.

[15] A.M.W. Verhagen An exactly soluble case of the triangular Ising model in a magnetic field. *J. Statist. Phys.*, 15(3), 219–231, 1976.

[16] G. Viennot. Problèmes combinatoires posés par la physique statistique. In *Sémin. Bourbaki, 36e année, Vol. 1983/84, Exp. 626, Astérisque 121- 122, 225-246* . 1985.

[17] G. Viennot. Heaps of pieces. I. Basic definitions and combinatorial lemmas. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 321–350. Springer, Berlin, 1986.