Unbounded regions of Infinitely Logconcave Sequences

David Uminsky and Karen Yeats*

Department of Mathematics and Statistics
Boston University, Boston, USA

Submitted: Mar 20, 2007; Accepted: Oct 26, 2007; Published: Nov 5, 2007
Mathematics Subject Classifications: Primary 05A10; Secondary 39B12

Abstract

We study the properties of a logconcavity operator on a symmetric, unimodal subset of finite sequences. In doing so we are able to prove that there is a large unbounded region in this subset that is ∞-logconcave. This problem was motivated by the conjecture of Boros and Moll in [1] that the binomial coefficients are ∞-logconcave.

1 Introduction

In this paper we study the asymptotic behavior of the logconcavity operator on finite sequences. Before we can state the problem we will need a few definitions.

We say that a sequence $\{c_0, c_1, ...c_n\}$ is **1-logconcave** (or logconcave) if $c_i \geq 0$ for $0 \leq i \leq n$ and $c_i^2 - c_{i+1}c_{i-1} \geq 0$ for $1 \leq i \leq n-1$.

We can extend this idea of logconcave as follows: Since $\{c_i\}$ is a finite sequence of length n we define $c_i = 0$ for $i < 0$ and $i > n+1$, then define the operator

$$\mathcal{L}\{c_i\} = \{c_i^2 - c_{i-1}c_{i+1}\}. \tag{1}$$

If $\{c_i\}$ is logconcave then $\mathcal{L}\{c_i\}$ is a new non-negative sequence. We now define a sequence $\{c_i\}$ to be **∞-logconcave** if $\mathcal{L}^k\{c_i\}$ is a non-negative sequence for all $k \geq 1$.

While studying a new class of integrals related to Ramanujan's Master Theorem, Boros and Moll proposed that a particular family of finite sequences of coefficients $\{d_l(m)\}$ were ∞-logconcave. Boros and Moll then point out that showing that the binomial coefficients are ∞-logconcave (project 7.9.3 in [1]) would go a long way in showing the sequence $\{d_l(m)\}$ is ∞-logconcave. Kauers and Paule in [3] show that the $\{d_l(m)\}$ are 1-logconcave. These conjectures motivated us to investigate the operator $\mathcal{L}$ on the space of finite sequences.

*The first author is partially supported by NSF grant DMS-0405724. Thanks to Cameron Morland for making better figures and to the referee for a very close reading.

Numerical experiments suggest that the binomial coefficients are ∞-logconcave. Moreover, many sequences "near" the binomial sequence also appear to be ∞-logconcave. These numerics led us to take an alternative approach. We begin to study the properties of $\mathcal{L}$ on the subset of finite sequences of the forms

$$\{\ldots, 0, 0, 1, x_0, x_1, \ldots, x_n, \ldots, x_1, x_0, 1, 0, 0, \ldots\}$$

$$\{\ldots, 0, 0, 1, x_0, x_1, \ldots, x_n, x_n, \ldots, x_1, x_0, 1, 0, 0, \ldots\}$$

We will refer to the first sequence above as the odd case and to the second sequence as the even case because of the repetition of the middle term in the sequence. Notice that all the binomial coefficients belong to one of the above cases.

Our approach to the problem is the following: for a given sequence of the form above of length $2n+3$ or $2n+4$ we analyze the dynamics of $\mathcal{L}$ on the subset of Euclidean space $\mathbb{R}^n$ with all non-negative coordinates. This differs from the approach of Moll in [4].

Our main result in this paper is to show that there is a large unbounded region $\mathcal{R}$ in this orthant that contains only ∞-logconcave sequences. Moreover $\mathcal{R}$ can act like a trapping region for ∞-logconcave sequences, i.e., sequences not starting in $\mathcal{R}$ can land in $\mathcal{R}$ after a number of iterates of $\mathcal{L}$.

The paper is organized as follows: In section 2 we present some of the simple cases along with some numerical evidence. The general arguments are presented in detail for the even case in section 3 and the odd case is briefly covered in similar fashion in section 4.

2 Low Dimensional Cases

2.1 The one dimensional cases, $\{1, x, 1\}$ and $\{1, x, x, 1\}$

For these two cases the underlying dynamics of $\mathcal{L}$ is rather easy to compute explicitly. We first consider the sequence $\{1, x, 1\}$.

Theorem 2.1. $\mathcal{L}\{1, x, 1\} = \{1, x^2 - 1, 1\}$. *Thus the positive fixed point for this sequence is* $x = \frac{1+\sqrt{5}}{2}$. *Moreover, if* $x \geq \frac{1+\sqrt{5}}{2}$ *then our sequence is* ∞*-logconcave and not otherwise.*

Proof. A simple calculation shows that $x = \frac{1+\sqrt{5}}{2}$ is a fixed point of $\mathcal{L}$ for the sequence $\{1, x, 1\}$. Moreover it is easy to see that if if $x > \frac{1+\sqrt{5}}{2}$ then x grows under the iterates of $\mathcal{L}$ and hence is always positive. It is also easy to see that the interval $[1, \frac{1+\sqrt{5}}{2}]$ is mapped over the interval $[0, \frac{1+\sqrt{5}}{2}]$ monotonically so that any values of $x \in [1, \frac{1+\sqrt{5}}{2})$ are eventually mapped below $x = 1$. Therefore our sequence is no longer unimodal and, hence, not logconcave. □

Notice that the binomial sequence $\{1, 2, 1\}$ lies securely in this region thus we have shown that $\{1, 2, 1\}$ is ∞-logconcave. The sequence $\{1, x, x, 1\}$ is handled in a similar fashion.

Theorem 2.2. $\mathcal{L}\{1, x, x, 1\} = \{1, x^2 - x, x^2 - x, 1\}$*, thus the fixed point for this sequence is* $x = 2$*. If* $x \geq 2$ *then our sequence is* ∞*-logconcave and not otherwise.*

Proof. Nearly identical to the one above. $\square$

Notice that the key to the easy theorems above is finding the fixed points of $\mathcal{L}$ in our underlying Euclidean space and then monotonicity leads us to the rest. Fixed "points" will no longer remain the key in the general argument but as we will see in the 2-D cases below we will have hypersurfaces that bound open regions of ∞-logconcavity.

2.2 The 2-D cases, $\{1, x, y, x, 1\}$ and $\{1, x, y, y, x, 1\}$

The 2-D cases are more complicated than the 1-D cases but they give better insight into how we might hope to find "regions" of ∞-logconcavity. To better see the general techniques to finding such regions, we first focus on the even case.

If one is to investigate this question numerically one can compute the following picture.

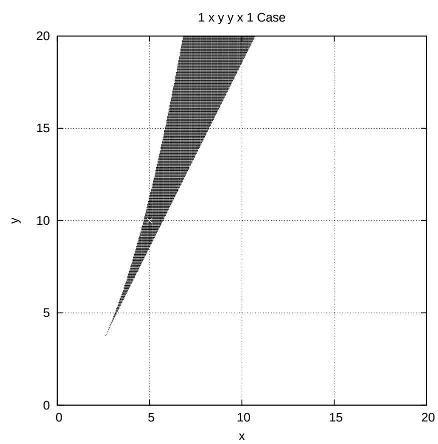

Figure 1: The filled region is the numerical region of ∞-logconcavity for the 2-D even case. The X indicates the position of the binomial coefficient.

The first thing to notice is that the binomial coefficient $x = 5, y = 10$ is in the numerical region of ∞-logconcavity. This picture also suggests that there is an ∞-logconcave region bounded away from the origin, below by some line and above by some curve. This picture is remarkably stable. The boundary points in the 1-D cases have now been replaced by curves.

For the case $\{1, x, y, x, 1\}$ the results are similar. The numerical picture looks as follows.

Again, we notice that the binomial coefficient $x = 4, y = 6$ is in the numerical region of ∞-logconcavity, which is quite encouraging. This picture also suggests that there is a region of ∞-logconcavity bounded away from the origin. It is important to point out that the regions in both cases are different with the odd case containing a wider region.

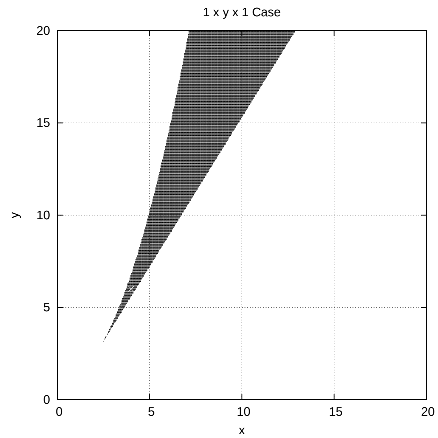

Figure 2: The filled region is the numerical region of ∞-logconcavity for the 2-D odd case. The X indicates the position of the binomial coefficient.

2.3 Note on a 3-D case

If one is to investigate the even 3-D case $\{1, x, y, z, z, y, x, 1\}$, we would arrive at the following boundary hypersurfaces.

In this case the binomial coefficient sequence $x = 7, y = 21, z = 35$ is not in the region. However its first iterate $x = 28, y = 196, z = 490$ is in the region of interest.

It is the observation of these hypersurfaces as boundaries of the ∞-logconcave region that is important. As it turns out, we can construct these boundaries for arbitrarily long finite sequences as will be shown in section 3 for the even cases and section 4 for the odd cases. While the 1-D and 2-D cases seem to have an "exclusive" region of ∞-logconcavity, this is not true in general as we saw in the 3-D case.

3 The General Case for Even Length

In this section we will prove the existence of an unbounded region of infinite logconcavity for even length symmetric sequences. Before beginning in earnest, let us mention the main steps. By looking at the leading order behavior we find hypersurfaces bounding the region of infinite logconcavity. We proceed to show the region is nonempty and unbounded by an explicit example, which we can control by matching all but one of the coordinates of the example with each hypersurface in turn. Then we show that sequences within the region are indeed ∞-logconcave by again matching with the hypersurfaces in turn and iterating. The technical key to controlling the iteration is understanding the effect of increasing one coordinate while decreasing all the others.

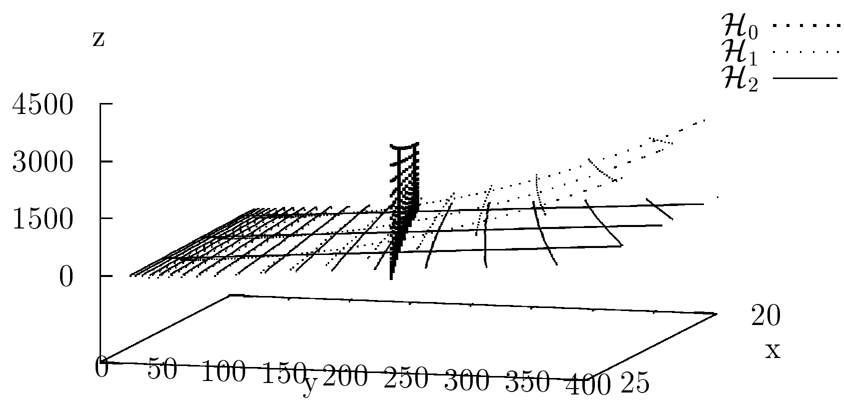

Figure 3: The region of ∞-logconcavity for the 3-D even case. From the picture above the region of interest is bounded on the left by the "vertical" plane ($\mathcal{H}_0$), below by the "horizontal" plane ($\mathcal{H}_2$), and above by the curved surface ($\mathcal{H}_1$).

3.1 Leading order behavior

Consider the sequence of length $2n+4$:

$$s = \{1, a_0 x, a_1 x^{1+d_1}, a_2 x^{1+d_1+d_2}, \ldots, a_n x^{1+d_1+\cdots+d_n}, a_n x^{1+d_1+\cdots+d_n}, \ldots, a_0 x, 1\}.$$

For the moment we are interested in the leading terms of elements of $\mathcal{L}(s)$ viewed as polynomials in x. We will restrict ourselves to values of the d_i for which $a_i^2 x^{2(1+d_1+\cdots+d_i)}$ contributes to the leading term of the corresponding element in the first iteration

$$\begin{aligned}\{1, x(a_0^2 x - a_1 x^{d_1}), x^{2+d_1}(a_1^2 x^{d_1} - a_2 a_0 x^{d_2}), x^{2+2d_1+d_2}(a_2^2 x^{d_2} - a_1 a_3 x^{d_3}), \ldots,\\ x^{2+2d_1+\cdots+2d_{n-1}+d_n}(a_n^2 x^{d_n} - a_n a_{n-1}), x^{2+2d_1+\cdots+2d_{n-1}+d_n}(a_n^2 x^{d_n} - a_n a_{n-1}), \ldots, 1\}\end{aligned} \tag{2}$$

and restrict ourselves to values of $a_i > 0$ which give that the leading terms of $\mathcal{L}(s)$ have the same form as s itself for some new x.

Using (2), the leading term condition is equivalent to

$$0 \le d_n \le d_{n-1} \le \cdots \le d_1 \le 1$$

which we can view as defining a simplex. The values of a_i may then be determined for each face by solving the systems of equations arising from matching the coefficients of the leading terms in $\mathcal{L}(s)$ with the coefficients of s.

Of greatest interest are the $(n-1)$-faces of the simplex since they define the boundaries of what will be our open region of convergence. The $(n-1)$-faces are defined by $d_1 = 1$,

$d_j = d_{j+1}$ for $0 < j < n$, and $d_n = 0$; in all cases the unspecified d_i are distinct and strictly between 0 and 1.

For $d_1 = 1$ the leading terms of $\mathcal{L}(s)$ are

$$\{1, x^2(a_0^2 - a_1), x^4 a_1^2, x^{4+2d_2} a_2^2, \ldots, x^{4+2d_2+\cdots+2d_n} a_n^2, x^{4+2d_2+\cdots+2d_n} a_n^2, \ldots, 1\}$$

so we are led to the system

$$\begin{aligned} a_0^2 - a_1 &= a_0 \\ a_1^2 &= a_1 \\ &\vdots \\ a_n^2 &= a_n \end{aligned}$$

We are interested in positive solutions so $a_i = 1$ for $0 < i \leq n$ and $a_0 = (1 + \sqrt{5})/2$.

For $d_j = d_{j+1}$ the leading terms of $\mathcal{L}(s)$ are

$$\{1, x^2 a_0^2, x^{2+2d_1} a_1^2, \ldots, x^{2+2d_1+\cdots+2d_j}(a_j^2 - a_{j-1}a_{j+1}), x^{2+2d_1+\cdots+4d_j} a_{j+1}^2, \ldots, \\ x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} a_n^2, x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} a_n^2, \cdots, 1\}$$

so we are led to the system

$$\begin{aligned} a_0^2 &= a_0 \\ &\vdots \\ a_{j-1}^2 &= a_{j-1} \\ a_j^2 - a_{j-1}a_{j+1} &= a_j \\ a_{j+1}^2 &= a_{j+1} \\ &\vdots \\ a_n^2 &= a_n \end{aligned}$$

which has unique positive solution $a_i = 1$ for $i \neq j$ and $a_j = (1 + \sqrt{5})/2$.

Finally for $d_n = 0$ the leading terms of $\mathcal{L}(s)$ are

$$\{1, x^2 a_0^2, x^{2+2d_1} a_1^2, \ldots, x^{2+2d_1+\cdots+2d_{n-1}} a_{n-1}^2, \\ x^{2+2d_1+\cdots+2d_{n-1}}(a_n^2 - a_n a_{n-1}), x^{2+2d_1+\cdots+2d_{n-1}}(a_n^2 - a_n a_{n-1}), \ldots, 1\}$$

so we are led to the system

$$\begin{aligned} a_0^2 &= a_0 \\ &\vdots \\ a_{n-1}^2 &= a_{n-1} \\ a_n^2 - a_n a_{n-1} &= a_n \end{aligned}$$

which has unique positive solution $a_i = 1$ for $0 \leq i < n$ and $a_n = 2$.

3.2 Interior

In the region of $\mathbb{R}^{n+1}$ where the coordinates are all positive and increasing, consider the following parametrically defined hypersurfaces:

$$\mathcal{H}_0 = \left\{ \left(\frac{1+\sqrt{5}}{2} x, x^2, x^{2+d_2}, \ldots, x^{2+d_2+\cdots+d_n} \right) : 1 \le x, 1 > d_2 > \cdots > d_n > 0 \right\}$$

$$\mathcal{H}_j = \left\{ \left(x, x^{1+d_1}, \ldots, \frac{1+\sqrt{5}}{2} x^{1+d_1+\cdots+d_j}, x^{1+d_1+\cdots+2d_j}, \ldots, x^{1+d_1+\cdots+2d_j+\cdots+d_n} \right) \right.$$
$$\left. : 1 \le x, 1 > d_1 > \cdots > d_j > d_{j+2} > \cdots > d_n > 0 \right\}$$

$$\mathcal{H}_n = \left\{ \left(x, x^{1+d_1}, \ldots, x^{1+d_1+\cdots+d_{n-1}}, 2x^{1+d_1+\cdots+d_{n-1}} \right) : 1 \le x, 1 > d_1 > \cdots > d_{n-1} > 0 \right\}$$

for $0 < j < n$. These are precisely the results of the leading order analysis of the previous subsection.

Let $\mathcal{R}$ be the region with positive increasing coordinates defined as greater in the ith coordinate than $\mathcal{H}_i$. For example in the 3-D case handled in section 2.3, figure 3, the region in question is above $\mathcal{H}_2$, below $\mathcal{H}_1$ and to the right of $\mathcal{H}_0$.

We say a sequence $\{1, x_0, \ldots, x_n, x_n, \ldots, x_0, 1\}$ is in $\mathcal{R}$ if $(x_0, \ldots, x_n) \in \mathcal{R}$.

Before we discuss $\mathcal{R}$ further we must first recall that the n^{th} **triangular number**, $\tilde{T}(n)$, is defined as

$$\tilde{T}(n) = \tilde{T}(n-1) + n,$$

with $\tilde{T}(0) = 0$. The first few elements of the sequence are $0, 1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots$. We will need the following lemma about triangular numbers.

Lemma 3.1. *Define $T(n) \equiv 2\tilde{T}(n)$ for $n \ge 0$. Then $T(n)$ satisfies the following:*

1. $T(0) - \frac{T(1)}{2} = -1$
2. $T(n+1) = 2T(n) - T(n-1) + 2$

Proof. (1) is trivial. (2) follows since $T(n) = n(n+1)$. □

We also need another straightforward result.

Lemma 3.2. *Suppose $x > 0$ and*

$$s = \{1, x, x^{1+d_1}, x^{1+d_1+d_2}, \cdots, x^{1+d_1+\cdots+d_n}, x^{1+d_1+\cdots+d_n}, \cdots, x, 1\}.$$

Then s is 1-logconcave iff $1 \ge d_1 \ge \cdots \ge d_n \ge 0$ with strict inequalities in the logconcavity condition iff $1 > d_1 > \cdots > d_n > 0$.

Proof. Compute $x^2 \geq x^{1+d_1} \Leftrightarrow 1 \geq d_1$ and $x^2 > x^{1+d_1} \Leftrightarrow 1 > d_1$. If $0 < j < n$ then $x^{2+2d_1+\cdots+2d_j} \geq x^{2+2d_1+\cdots+2d_{j-1}+d_j+d_{j+1}} \Leftrightarrow x^{d_j} \geq x^{d_{j+1}} \Leftrightarrow d_j \geq d_{j+1}$ and likewise with strict inequalities. Finally $x^{2+2d_1+\cdots+2d_n} \geq x^{2+2d_1+\cdots+2d_{n-1}+d_n} \Leftrightarrow x^{d_n} \geq 1 \Leftrightarrow d_n \geq 0$ and likewise with strict inequalities. □

We are now ready to prove some important properties of $\mathcal{R}$.

Lemma 3.3. *$\mathcal{R}$ is nonempty and unbounded.*

Proof. Let $\{1, x_0, \ldots, x_n, x_n, \ldots, x_0, 1\}$ be any 1-logconcave sequence with $x_0 > 0$, for instance the binomial sequence of appropriate length. Also, choose C such that $0 < C < \frac{2}{1+\sqrt{5}}$ and consider the following sequence:

$$s = \left\{1, C^{T(0)}ax_0, C^{T(1)}a^2x_1, C^{T(2)}a^3x_2, \ldots, C^{T(n)}a^{n+1}x_n, C^{T(n)}a^{n+1}x_n, \ldots, 1\right\}$$

for $a > 2C^{T(n-1)-T(n)}$.

Notice that a is dependent on n which is not a problem since n is fixed.

It is clear that s is 1-logconcave and, moreover, the inequalities are strict since

$$C^{2T(0)}a^2x_0^2 = a^2x_0^2 \geq a^2x_1 > C^{T(1)}a^2x_1,$$

for $0 < j < n$

$$C^{2T(j)}a^{2j+2}x_j^2 \geq C^{2T(j)}a^{2j+2}x_{j-1}x_{j+1} > C^{T(j-1)}a^jx_{j-1}C^{T(j+1)}a^{j+2}x_{j+1}$$

by (2) of Lemma 3.1 and the fact that $C < 2/(1+\sqrt{5}) < 1$, and

$$C^{T(n)}a^{n+1}x_n \geq C^{T(n)}a^{n+1}x_{n-1} > C^{T(n-1)}a^nx_{n-1}.$$

Define $\tilde{x} = ax_0 > 0$, define $\tilde{d}_1$ so that $\tilde{x}^{1+\tilde{d}_1} = C^{T(1)}a^2x_1$, and continue recursively so that, for $0 < j \leq n$, $\tilde{d}_j$ is defined so that $\tilde{x}^{1+\tilde{d}_1+\cdots+\tilde{d}_j} = C^{T(j)}a^{j+1}x_j$. By Lemma 3.2, $1 > \tilde{d}_1 > \cdots > \tilde{d}_n > 0$.

Let us next consider each $\mathcal{H}_j$ in turn. For $0 < j < n$ choose $x = \tilde{x}$, $d_i = \tilde{d}_i$ for $i < j$, $d_j = (\tilde{d}_j + \tilde{d}_{j+1})/2$, and $d_i = \tilde{d}_i$ for $i > j+1$. Consequently $1 > d_1 > \cdots > d_j > d_{j+2} > \cdots > d_n > 0$ and these choices match all the coordinates of s with the corresponding coordinates of $\mathcal{H}_i$ except possibly for the jth. But $x^{1+d_1+\cdots+2d_j}/x^{1+d_1+\cdots+d_{j-1}} = x^{2d_j}$, so

$$\begin{aligned} Cx^{1+d_1+\cdots+d_j} &= Cx^{1+d_1+\cdots+d_{j-1}}\sqrt{\frac{x^{1+d_1+\cdots+2d_j}}{x^{1+d_1+\cdots+d_{j-1}}}} \\ &= C\sqrt{x^{1+d_1+\cdots+d_{j-1}}x^{1+d_1+\cdots+2d_j}}. \end{aligned}$$

Comparing with s we have that

$$\begin{aligned} C^{T(j)}a^{j+1}x_j &\geq C^{T(j)}a^{j+1}\sqrt{x_{j-1}x_{j+1}} \\ &= \sqrt{C^{2T(j)-T(j+1)-T(j-1)}C^{T(j-1)}a^jx_{j-1}C^{T(j+1)}a^{j+2}x_{j+1}} \\ &= C^{-1}\sqrt{x^{1+d_1+\cdots+d_{j-1}}x^{1+d_1+\cdots+2d_j}} \\ &> \frac{1+\sqrt{5}}{2}x^{1+d_1+\cdots+d_j} \end{aligned}$$

where the fourth line follows from (2) of Lemma 3.1, thus s is on the correct side of $\mathcal{H}_j$.

For $\mathcal{H}_0$ choose $x = \sqrt{C^{T(1)}a^2x_1} = \tilde{x}^{(1+\tilde{d}_1)/2} > 0$ and, for $2 \leq j \leq n$, $d_j = 2\tilde{d}_j/(1+\tilde{d}_1)$. Consequently $1 > d_2 > \cdots > d_n > 0$ and

$$C^{T(j)}a^{j+1}x_j = \tilde{x}^{1+\tilde{d}_1+\cdots+\tilde{d}_j} = x^{2+d_2+\cdots+d_j}$$

hence matching all the coordinates of s other than the 0th with $\mathcal{H}_0$. So we check,

$$C^{T(0)}ax_0 \geq C^{T(0)}\sqrt{a^2x_1} = C^{T(0)}C^{-\frac{T(1)}{2}}x = C^{-1}x > \frac{1+\sqrt{5}}{2}x$$

Thus s is on the correct side of $\mathcal{H}_0$.

For $\mathcal{H}_n$ simply choose $x = \tilde{x}$ and $d_i = \tilde{d}_i$ for $i < n$, which gives $1 > d_1 > \cdots > d_{n-1} > 0$ and matches all the coordinates of s other than the nth. Then $x_n^2 \geq x_{n-1}x_n$ giving

$$\begin{aligned} C^{T(n)}a^{n+1}x_n &\geq C^{T(n)}a^{n+1}x_{n-1} \\ &= aC^{T(n)-T(n-1)}x^{1+d_1+\cdots+d_{n-1}} \\ &> 2x^{1+d_1+\cdots+d_{n-1}}. \end{aligned}$$

So s is also on the correct side of $\mathcal{H}_n$. Consequently s is in $\mathcal{R}$. So we see that $\mathcal{R}$ is nonempty, and, by the freedom to increase a, is unbounded. □

Definition 3.1. Let $\mathcal{H}$ be a hypersurface in $\mathbb{R}^{n+1}$. We say we view $\mathcal{H}$ as a function $f : \mathbb{R}^n \to \mathbb{R}$ with the jth variable as the dependent variable if for $(x_0, \ldots, x_n)$ a point on $\mathcal{H}$ we have $x_j = f(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

Definition 3.2. Let $\mathcal{H}$ be a hypersurface in $\mathbb{R}^{n+1}$. Call it **j-monotone** if when $\mathcal{H}$ is viewed as a function $f : \mathbb{R}^n \to \mathbb{R}$ with the jth variable as the dependent variable then $f(y_1, \ldots, y_n) \geq f(z_1, \ldots, z_n)$ if $y_i \geq z_i$ for all i.

Lemma 3.4. *Let $\mathcal{H}$ be a j-monotone hypersurface in $\mathbb{R}^{n+1}$. Let $(x_0, x_1, \ldots, x_n)$ be a point on $\mathcal{H}$. Then for $\epsilon_i > 0$, $\eta > 0$,*

$$(x_0 - \epsilon_0, \ldots, x_{j-1} - \epsilon_{j-1}, x_j + \epsilon_j, x_{j+1} - \epsilon_{j+1}, \ldots, x_n - \epsilon_n)$$

and

$$(x_0, \ldots, x_{j-1}, x_j + \eta, x_{j-1}, \ldots, x_n)$$

lie on the same side of $\mathcal{H}$.

Proof. View $\mathcal{H}$ as $f : \mathbb{R}^n \to \mathbb{R}$ with x_j as the dependent variable. Then

$$\begin{aligned} &f(x_0 - \epsilon_0, \ldots, x_{j-1} - \epsilon_{j-1}, x_{j+1} - \epsilon_{j+1}, \ldots, x_n - \epsilon_n) \\ &\leq f(x_0, \ldots, x_{j-1}, x_{j-1}, \ldots, x_n) = x_j < x_j + \epsilon_j \end{aligned}$$

So both points lie on the side of $\mathcal{H}$ which is greater in the jth coordinate. □

Lemma 3.5. *Each of the $\mathcal{H}_j$ is j-monotone.*

Proof. For $\mathcal{H}_0$, x_0 is determined by x_1 and increases when x_1 increases, so $\mathcal{H}_0$ is 0-monotone. For $\mathcal{H}_n$, x_n is determined by x_{n-1} and increases when x_{n-1} increases, so $\mathcal{H}_n$ is n-monotone. For $\mathcal{H}_j$, $0 < j < n$, x_j is $(1+\sqrt{5})\sqrt{x_{j-1}x_{j+1}}/2$ which increases when either x_{j-1} or x_{j+1} increase so $\mathcal{H}_j$ is j-monotone. $\square$

Theorem 3.3. *Any sequence in $\mathcal{R}$ is ∞-logconcave.*

Proof. Suppose $s = \{1, y_0, \ldots, y_n, y_n, \ldots, y_0, 1\}$ is in $\mathcal{R}$. Then for any $0 < j < n$, by the definition of $\mathcal{R}$, we can choose x, $\epsilon > 0$, and the d_i, $i \neq j+1$, such that

$$s = \left\{1, x, \ldots, x^{1+d_1+\cdots+d_{j-1}}, \frac{1+\sqrt{5}}{2}x^{1+d_1+\cdots+d_j} + \epsilon, x^{1+d_1+\cdots+2d_j}, \ldots, \right.$$
$$\left. x^{1+d_1+\cdots+2d_j+\cdots+d_n}, x^{1+d_1+\cdots+2d_j+\cdots+d_n}, \ldots, 1\right\}.$$

Iterate to get

$$\begin{aligned} \mathcal{L}(s) = & \left\{1, x^2 - x^{1+d_1}, \ldots, x^{2+2d_1+\cdots+2d_{j-1}}\right. \\ & -\frac{1+\sqrt{5}}{2}x^{2+2d_1+\cdots+2d_{j-2}+d_{j-1}+d_j} - \epsilon x^{1+d_1+\cdots+d_{j-2}}, \\ & \left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right)x^{2+2d_1+\cdots+2d_j} + (1+\sqrt{5})x^{1+d_1+\cdots+d_j}\epsilon + \epsilon^2, \\ & x^{2+2d_1+\cdots+4d_j} - \frac{1+\sqrt{5}}{2}x^{2+2d_1+\cdots+3d_j+d_{j+1}} - \epsilon x^{1+d_1+\cdots+2d_j+d_{j+1}}, \ldots, \\ & x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} - x^{2+2d_1+\cdots+4d_j+\cdots+d_n}, \\ & \left. x^{2+2d_1+\cdots+4d_j+\cdots+2d_n} - x^{2+2d_1+\cdots+4d_j+\cdots+d_n}, \ldots, 1\right\} \end{aligned}$$

Since $(1+\sqrt{5})/2)^2 - 1 = (1+\sqrt{5})/2$ by using x^2 in place of x in the definition of $\mathcal{H}_j$ and applying Lemma 3.4, which is valid in view of Lemma 3.5, we can conclude that $\mathcal{L}(s)$ is on the side of $\mathcal{H}_j$ which is larger in the jth coordinate. This is the same side which s is on.

Similarly for $\mathcal{H}_0$ we can choose x, $\epsilon > 0$, and the d_i, such that

$$s = \left\{1, \frac{1+\sqrt{5}}{2}x + \epsilon, x^2, \ldots, x^{2+d_2+\cdots+d_n}, x^{2+d_2+\cdots+d_n}, \ldots, 1\right\}.$$

Iterate to get

$$\mathcal{L}(s) = \left\{1, \left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right) x^2 + \epsilon(1+\sqrt{5})x + \epsilon^2, x^4 - \frac{1+\sqrt{5}}{2}x^{3+d_2} - \epsilon x^{2+d_2}, \ldots,\right.$$
$$\left. x^{4+2d_2+\cdots+2d_n} - x^{4+2d_2+\cdots+d_n}, x^{4+2d_2+\cdots+2d_n} - x^{4+2d_2+\cdots+d_n}, \ldots, 1\right\}$$

which, by Lemmas 3.4 and 3.5 shows that $\mathcal{L}(s)$ is on the same side of $\mathcal{H}_0$ as s is, as above.

Finally for $\mathcal{H}_n$ choose x, $\epsilon > 0$, and the d_i, such that

$$s = \left\{1, x, \ldots, x^{1+d_1+\cdots+d_{n-1}}, 2x^{1+d_1+\cdots+d_{n-1}} + \epsilon, 2x^{1+d_1+\cdots+d_{n-1}} + \epsilon, \ldots, 1\right\}.$$

Iterate to get

$$\mathcal{L}(s) = \left\{1, x^2 - x^{1+d_1}, \ldots, x^{2+2d_1+\cdots+2d_{n-1}} - 2x^{2+2d_1+\cdots+d_{n-1}},\right.$$
$$(4-2)x^{2+2d_1+\cdots+2d_{n-1}} + 4\epsilon x^{1+d_1+\cdots+d_{n-1}} + \epsilon^2,$$
$$\left.(4-2)x^{2+2d_1+\cdots+2d_{n-1}} + 4\epsilon x^{1+d_1+\cdots+d_{n-1}} + \epsilon^2, \ldots, 1\right\}$$

which again by Lemmas 3.4 and 3.5 shows that $\mathcal{L}(s)$ is on the same side of $\mathcal{H}_n$ as s is.

Consequently $\mathcal{L}(s)$ is in $\mathcal{R}$. Since $\mathcal{R}$ is a subregion of the region of $\mathbb{R}^{n+1}$ with positive coordinates, this implies that any sequence in $\mathcal{R}$ is ∞-logconcave. □

4 The General Case for Odd Length

4.1 Leading order behavior

Consider the sequence of length $2n+3$

$$s = \{1, a_0x, a_1x^{1+d_1}, a_2x^{1+d_1+d_2}, \ldots, a_nx^{1+d_1+\cdots+d_n}, \ldots, a_0x, 1\}$$

Again we are interested in the leading terms of elements of $\mathcal{L}(s)$ viewed as polynomials in x. We will restrict ourselves to values of the d_i for which $a_i^2x^{2(1+d_1+\cdots+d_i)}$ contributes to the leading term of the corresponding element in the first iteration:

$$\begin{aligned}\{1, x(a_0^2x - a_1x^{d_1}), x^{2+d_1}(a_1^2x^{d_1} - a_2a_0x^{d_2}), x^{2+2d_1+d_2}(a_2^2x^{d_2} - a_1a_3x^{d_3}), \ldots,\\ x^{2+2d_1+\cdots+2d_{n-1}}(a_n^2x^{2d_n} - a_{n-1}^2), \ldots, 1\}\end{aligned} \tag{3}$$

and to values of $a_i > 0$ which give that the leading terms of $\mathcal{L}(s)$ have the same form as s itself for some new x.

Using (3), the leading term condition is equivalent to

$$0 \le d_n \le d_{n-1} \le \cdots \le d_1 \le 1$$

which we can again view as defining a simplex. The $(n-1)$-faces are defined by $d_1 = 1$, $d_j = d_{j+1}$ for $0 \le j < n$, and $d_n = 0$; in all cases with the unspecified d_i distinct and strictly between 0 and 1.

For $d_n = 0$ the leading terms of $\mathcal{L}(s)$ are

$$\{1, x^2 a_0^2, x^{2+2d_1} a_1^2, \ldots, x^{2+2d_1+\cdots+2d_{n-1}} a_{n-1}^2, x^{2+2d_1+\cdots+2d_{n-1}}(a_n^2 - a_{n-1}^2), \ldots, 1\}$$

so we are led to the system

$$\begin{aligned} a_0^2 &= a_0 \\ &\vdots \\ a_{n-1}^2 &= a_{n-1} \\ a_n^2 - a_{n-1}^2 &= a_n \end{aligned}$$

which has unique positive solution $a_i = 1$ for $0 \le i < n$ and $a_n = (1+\sqrt{5})/2$.

For $d_1 = 1$ and $d_j = d_{j+1}$ the systems are identical to the even case.

4.2 Interior

The hypersurfaces $\mathcal{H}_j$, $0 \le j < n$ are the same.

$$\mathcal{H}_n = \left\{ \left(x, x^{1+d_1}, \ldots, x^{1+d_1+\cdots+d_{n-1}}, \frac{1+\sqrt{5}}{2} x^{1+d_1+\cdots+d_{n-1}} \right) \right. \\ \left. : 1 \le x, 1 > d_1 > \cdots > d_{n-1} > 0 \right\}$$

Let $\mathcal{R}$ be the region with positive increasing coordinates defined as greater in the ith coordinate than $\mathcal{H}_i$.

As in the even case

Lemma 4.1. *Suppose $x > 0$ and*

$$s = \{1, x, x^{1+d_1}, x^{1+d_1+d_2}, \cdots, x^{1+d_1+\cdots+d_n}, x^{1+d_1+\cdots+d_{n-1}}, \cdots, x, 1\}.$$

Then s is 1-logconcave iff $1 \ge d_1 \ge \cdots \ge d_n \ge 0$ with strict inequalities in the logconcavity condition iff $1 > d_1 > \cdots > d_n > 0$.

Proof. The only case which differs from the proof of Lemma 3.2 is the nth. $x^{2+2d_1+\cdots+2d_n} \ge x^{2+2d_1+\cdots+2d_{n-1}} \Leftrightarrow x^{2d_n} \ge 1 \Leftrightarrow d_n \ge 0$ and likewise with strict inequalities. $\square$

Lemma 4.2. *$\mathcal{R}$ is nonempty and unbounded.*

Proof. The proof begins as before, but with $a > (1+\sqrt{5})C^{T(n-1)-T(n)}/2$. We only need consider $\mathcal{H}_n$. For $\mathcal{H}_n$ choose x and d_i as before to match all the coordinates of s other than the nth. Then $x_n^2 \geq x_{n-1}^2$ giving

$$C^{T(n)}a^{n+1}x_n \geq C^{T(n)}a^{n+1}x_{n-1} = aC^{T(n)-T(n-1)}x^{1+d_1+\cdots+d_{n-1}} > \frac{1+\sqrt{5}}{2}x^{1+d_1+\cdots+d_{n-1}}$$

So s is also on the correct side of $\mathcal{H}_n$. Consequently s is in $\mathcal{R}$. So we see that $\mathcal{R}$ is nonempty, and, by the freedom to increase a, is unbounded. □

Lemma 4.3. *Each of the $\mathcal{H}_j$ is j-monotone.*

Proof. We only need to consider $\mathcal{H}_n$, in which x_n is determined by x_{n-1} and increases when x_{n-1} increases, so $\mathcal{H}_n$ is n-monotone. □

Theorem 4.1. *Any sequence in $\mathcal{R}$ is ∞-logconcave.*

Proof. Again, in view of the even case, we only need to check $\mathcal{H}_n$. Using notation from the even case choose x, $\epsilon > 0$, and the d_i, such that

$$s = \left\{1, x, \ldots, x^{1+d_1+\cdots+d_{n-1}}, \frac{1+\sqrt{5}}{2}x^{1+d_1+\cdots+d_{n-1}} + \epsilon, \ldots, 1\right\}.$$

Iterate to get

$$\mathcal{L}(s) = \left\{1, x^2 - x^{1+d_1}, \ldots, x^{2+2d_1+\cdots+2d_{n-1}} - \left(\frac{1+\sqrt{5}}{2}\right)x^{2+2d_1+\cdots+d_{n-1}} - x^{1+d_1+\cdots+d_{n-2}}\epsilon,\right.$$
$$\left.\left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right)x^{2+2d_1+\cdots+2d_{n-1}} + (1+\sqrt{5})\epsilon x^{1+d_1+\cdots+d_{n-1}} + \epsilon^2, \ldots, 1\right\}$$

which by Lemmas 3.4 and 4.3 shows that $\mathcal{L}(s)$ is on the same side of $\mathcal{H}_n$ as s is.

Consequently $\mathcal{L}(s)$ is in $\mathcal{R}$. Since $\mathcal{R}$ is a subregion of the region of $\mathbb{R}^{n+1}$ with positive coordinates, this implies that any sequence in $\mathcal{R}$ is ∞-logconcave. □

References

[1] Boros, G. and Moll, V.: *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals.* Cambridge University Press, 2003.

[2] Brenti, Francesco: *Log-concave and Unimodal sequences in Algebra, Combinatorics, and Geometry: an update.* Contemporary Math., 178 (1994), 71-89.

[3] Kauers, Manuel and Paule, Peter: *A Computer Proof of Moll's Log-Concavity Conjecture.* SFB F13. Technical report no. 2006-15, Altenbergerstrasse 69, 2006.

[4] Moll, Victor H.: *Combinatorial sequences arising from a rational integral.* Online Journal of Analytic Combinatorics, **2** (2007), #4.