Color Neighborhood Union Conditions for Long Heterochromatic Paths in Edge-Colored Graphs *

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Abstract

Let G be an edge-colored graph. A heterochromatic (rainbow, or multicolored) path of G is such a path in which no two edges have the same color. Let $CN(v)$ denote the color neighborhood of a vertex v of G. In a previous paper, we showed that if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G, then G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$. In the present paper, we prove that G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$, and give examples to show that the lower bound is best possible in some sense.

Keywords: edge-colored graph, color neighborhood, heterochromatic (rainbow, or multicolored) path.

1. Introduction

We use Bondy and Murty [3] for terminology and notations not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we will mean a function $C : E \to \mathbb{N}$, the set of natural numbers. If G is assigned such a coloring, then we say that G is an *edge-colored graph*. Denote the edge-colored graph by (G, C), and call $C(e)$ the *color* of the edge $e \in E$. We say that $C(uv) = \emptyset$ if $uv \notin E(G)$ for $u, v \in V(G)$. For a subgraph H of G, we denote $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$ and the *color degree* is $d^c(v) = |CN(v)|$. A path is called *heterochromatic (rainbow, or*

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multicolored) if any two edges of it have different colors. If u and v are two vertices on a path P, uPv denotes the segment of P from u to v, whereas $vP^{-1}u$ denotes the same segment but from v to u.

There are many existing literature dealing with the existence of paths and cycles with special properties in edge-colored graphs. In [6], the authors showed that for a 2-edge-colored graph G and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [10], Rödl and Winkler (see [9]), Frieze and Reed [9], and Albert, Frieze and Reed [1]. For more references, see [2, 7, 8, 11, 12]. Many results in these papers were proved by using probabilistic methods.

Suppose $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices u and v of G. In [4], the authors showed that G has a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$. In [5], we proved that G has a heterochromatic path of length at least $\lfloor \frac{2s+4}{5} \rfloor$. In the present paper, we prove that G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$, and give examples to show that the lower bound is best possible in some sense.

2. Long heterochromatic paths for $s \leq 7$

First, we consider the case when $1 \leq s \leq 7$, which will serve as the induction initial for our main result Theorem 3.6 in next section.

Lemma 2.1 *Let G be an edge-colored graph and $1 \leq s \leq 7$ an integer. Suppose that $|CN(u) \cup CN(v)| \geq s$ for every pair of vertices u and v of G. Then G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$.*

Proof. (1) $s = 1$.

Then any edge in G is a heterochromatic path of length $1 = \lceil \frac{s+1}{2} \rceil$.

(2) $s = 2$.

Let $e = uv$ be an arbitrary edge in G.

Since $|CN(u) \cup CN(v)| \geq s = 2$, there exists a $v' \in V(G) - \{u, v\}$ such that $v'u \in E(G)$ and $C(v'u) \neq C(uv)$, or $v'v \in E(G)$ and $C(v'v) \neq C(uv)$.

If $v'u \in E(G)$ and $C(v'u) \neq C(uv)$, then $v'uv$ is a heterochromatic path of length $2 = \lceil \frac{s+1}{2} \rceil$.

If $v'v \in E(G)$ and $C(v'v) \neq C(uv)$, then $v'vu$ is a heterochromatic path of length $2 = \lceil \frac{s+1}{2} \rceil$.

(3) $s = 3$.

Since $|CN(u) \cup CN(v)| \geq s = 3 > 2$ for every pair of vertices u and v of G, there is a heterochromatic path of length $2 = \lceil \frac{s+1}{2} \rceil$ in G.

(4) $s = 4$.

Since $|CN(u) \cup CN(v)| \geq s = 4 > 2$ for every pair of vertices u and v of G, there is a heterochromatic path of length 2, let $u_0u_1u_2$ be such a path.

Since $|CN(u_0) \cup CN(u_2)| \geq 4$, there exists a $v \in V(G) - \{u_0, u_1, u_2\}$ such that $C(vu_0) \notin \{C(u_0u_1), C(u_1u_2)\}$ or $C(vu_2) \notin \{C(u_0u_1), C(u_1u_2)\}$.

If $C(vu_0) \notin \{C(u_0u_1), C(u_1u_2)\}$, then $vu_0u_1u_2$ is a heterochromatic path of length $3 = \lceil \frac{s+1}{2} \rceil$.

If $C(vu_2) \notin \{C(u_0u_1), C(u_1u_2)\}$, then $u_0u_1u_2v$ is a heterochromatic path of length $3 = \lceil \frac{s+1}{2} \rceil$.

(5) $s = 5$.

Since $|CN(u) \cup CN(v)| \geq s = 5 > 4$ for every pair of vertices u and v of G, there is a heterochromatic path of length $3 = \lceil \frac{s+1}{2} \rceil$ in G.

(6) $s = 6$.

Since $|CN(u) \cup CN(v)| \geq s = 6 > 4$ for every pair of vertices u and v of G, there is a heterochromatic path of length 3, let $P = u_0u_1u_2u_3$ be such a path.

If there exists a $v \in V(G) - \{u_0, u_1, u_2, u_3\}$ such that $C(vu_0) \notin C(P)$ or $C(vu_3) \notin C(P)$, then $vu_0u_1u_2u_3$ or $u_0u_1u_2u_3v$ is a heterochromatic path of length $4 = \lceil \frac{s+1}{2} \rceil$.

Otherwise, $|C(u_0u_2, u_0u_3, u_1u_3) - C(P)| = 3$, since $|CN(u_0) \cup CN(u_3) - C(P)| \geq |CN(u_0) \cup CN(u_3)| - |C(P)| \geq 6 - 3 = 3$. On the other hand, since $|CN(u_0) \cup CN(u_3)| \geq 6$, there exists a $v \in V(G) - \{u_0, u_1, u_2, u_3\}$ such that $C(vu_0) = C(u_1u_2)$ or $C(vu_3) = C(u_1u_2)$, then $vu_0u_1u_3u_2$ or $vu_3u_2u_0u_1$ is a heterochromatic path of length $4 = \lceil \frac{s+1}{2} \rceil$.

(7) $s = 7$.

Since $|CN(u) \cup CN(v)| \geq s = 7 > 6$ for every pair of vertices u and v of G, there is a heterochromatic path of length $4 = \lceil \frac{s+1}{2} \rceil$ in G. ∎

3. Long heterochromatic paths for all $s \geq 1$

In this section we will give a best possible lower bound for the length of the longest heterochromatic path in G when $s \geq 7$. First, we will do some preparations.

Lemma 3.1 *Suppose $P = u_0u_1u_2 \ldots u_l$ is a heterochromatic path of length $l \geq 4$, $u_0u_l \in E(G)$ and $C(u_0u_l) \notin C(P)$. If there exists a $v \in N(u_0) - V(P)$ such that $C(u_0v) = C(u_{i-1}u_i)$ for some $1 \leq i \leq l$ that satisfies $|\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_l)| \geq l - 1$, then there is a heterochromatic path of length $l + 1$ in G.*

Proof. Let $C_0 = C(P) \cup C(u_0u_l)$.

We distinguish the following 5 cases:

Case 1. $i = 1$

Then $vu_0u_lP^{-1}u_1$ is a heterochromatic path of length $l + 1$.

Case 2. $i = 2$

Let

$$\begin{aligned} X &= \{3 \leq j \leq l-1 : C(u_1u_j) \notin C_0\}, \\ Y &= \{3 \leq j \leq l-1 : C(u_{j-1}u_l) \notin C_0 \cup \{C(u_1u_j : j \in X)\}\}. \end{aligned}$$

Then we have

$$\{C(u_1w) : w \in V(P)\} - C_0 = \cup_{i=3}^{l} C(u_1u_i) - C_0 = \{C(u_1u_j) : j \in X\} \cup (C(u_1u_l) - C_0),$$

$$\begin{aligned}\{C(u_lw) : w \in V(P)\} - C_0 - \{C(u_1u_j) : j \in X\}& \\ &= \cup_{j=1}^{l-1} C(u_lu_{j-1}) - C_0 - \{C(u_1u_j) : j \in X\} \\ &\subseteq \{C(u_lu_{j-1}) : j \in Y\} \cup (C(u_1u_l) - C_0).\end{aligned}$$

So

$$\begin{aligned}\{C(u_1w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0& \\ &\subseteq \{C(u_1u_j) : j \in X\} \cup \{C(u_lu_{j-1}) : j \in Y\} \cup (C(u_1u_l) - C_0).\end{aligned}$$

If $C(u_1u_l) \notin C_0$, then $vu_0u_1u_lP^{-1}u_2$ is a heterochromatic path of length $l+1$.

Otherwise, we have $C(u_1u_l) \in C_0$, then

$$\begin{aligned} l-1 \leq\ & |\{C(u_1w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0| \\ \leq\ & |\{C(u_1u_j) : j \in X\}| + |\{C(u_lu_{j-1}) : j \in Y\}| \\ \leq\ & |X| + |Y|.\end{aligned}$$

On the other hand, $X, Y \subseteq \{3, \ldots, l-1\}$, and $|\{3, \ldots, l-1\}| = l-3$, so $|X| + |Y| \geq |\{3, \ldots, l-1\}| + 1$. Then we can conclude that there exists a $j \in X \cap Y$. In this case, $vu_0u_1u_jPu_lu_{j-1}P^{-1}u_2$ is a heterochromatic path of length $l+1$.

So there exists a heterochromatic path of length $l+1$ if $i = 2$.

Case 3. $i = l$

Let

$$\begin{aligned} X &= \{1 \leq j \leq l-2 : C(u_{j-1}u_{l-1}) \notin C_0\}, \\ Y &= \{1 \leq j \leq l-2 : C(u_ju_l) \notin C_0 \cup \{C(u_{j-1}u_{l-1}) : j \in X\}\}.\end{aligned}$$

Then

$$\begin{aligned}\{C(u_{l-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0& \\ &\subseteq \{C(u_{l-1}u_{j-1}) : j \in X\} \cup \{C(u_lu_j) : j \in Y\}.\end{aligned}$$

So

$$\begin{aligned} l-1 \leq\ & |\{C(u_{l-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0| \\ \leq\ & |\{C(u_{l-1}u_{j-1}) : j \in X\} \cup \{C(u_lu_j) : j \in Y\}| \\ \leq\ & |X| + |Y|.\end{aligned}$$

Since $X, Y \subseteq \{1, 2, \ldots, l-2\}$ and $|\{1, 2, \ldots, l-2\}| = l-2$, there exists a $j \in X \cap Y$. In this case, $vu_0Pu_{j-1}u_{l-1}P^{-1}u_ju_l$ is a heterochromatic path of length $l+1$.

So there exists a heterochromatic path of length $l+1$ if $i = l$.

Case 4. $i = l-1$

Let

$$\begin{aligned} X &= \{1 \leq j \leq l-3 : C(u_{j-1}u_{l-2}) \notin C_0\}, \\ Y &= \{1 \leq j \leq l-3 : C(u_ju_l) \notin C_0 \cup \{C(u_{l-2}u_{j-1}) : j \in X\}\}.\end{aligned}$$

Then we have

$$\begin{aligned}\{C(u_{l-2}w) : w \in V(P)\} - C_0 &= \cup_{j=1}^{l-3} C(u_{j-1}u_{l-2}) \cup C(u_{l-2}u_l) - C_0 \\ &= \{C(u_{j-1}u_{l-2}) : j \in X\} \cup (C(u_{l-2}u_l) - C_0),\end{aligned}$$

$$\begin{aligned}&\{C(u_l w) : w \in V(P)\} - C_0 - \{C(u_{j-1}u_{l-2}) : j \in X\} \\ &\quad = \cup_{j=0}^{l-2} C(u_l u_j) - C_0 - \{C(u_{j-1}u_{l-2}) : j \in X\} \\ &\quad \subseteq \{C(u_l u_j) : j \in Y\} \cup (C(u_{l-2}u_l) - C_0).\end{aligned}$$

So

$$\begin{aligned}&\{C(u_{l-2}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0 \\ &\quad \subseteq \{C(u_{j-1}u_{l-2}) : j \in X\} \cup \{C(u_l u_j) : j \in Y\} \cup (C(u_{l-2}u_l) - C_0).\end{aligned}$$

If $C(u_{l-2}u_l) \notin C_0$, $vu_0Pu_{l-2}u_lu_{l-1}$ is a heterochromatic path of length $l+1$.

Otherwise, we have $C(u_{l-2}u_l) \in C_0$, then

$$\begin{aligned}l-1 &\leq |\{C(u_{l-2}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0| \\ &\leq |\{C(u_{j-1}u_{l-2}) : j \in X\} \cup \{C(u_l u_j) : j \in Y\}| \\ &\leq |X| + |Y|.\end{aligned}$$

Now we can conclude that there exists a $j \in X \cap Y$, since $|X| + |Y| \geq l - 1 > |\{1, \ldots, l-3\}| + 1$ and $X, Y \subseteq \{1, 2, \ldots, l-3\}$. In this case, $vu_0Pu_{j-1}u_{l-2}P^{-1}u_ju_lu_{l-1}$ is a heterochromatic path of length $l+1$.

So there exists a heterochromatic path of length $l+1$ if $i = l-1$.

Case 5. $3 \leq i \leq l-2$

Then we have $l \geq 5$.

Let

$$\begin{aligned}X_1 &= \{1 \leq j \leq i-2 : C(u_{i-1}u_{j-1}) \notin C_0\}, \\ X_2 &= \{i+1 \leq j \leq l-1 : C(u_{i-1}u_j) \notin C_0\}, \\ C_1 &= \{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\}\} \\ Y_1 &= \{1 \leq j \leq i-2 : C(u_l u_j) \notin C_0 \cup C_1\}, \\ Y_2 &= \{i+1 \leq j \leq l-1 : C(u_l u_{j-1}) \notin C_0 \cup C_1\}.\end{aligned}$$

Then

$$\begin{aligned}&\{C(u_{i-1}w) : w \in V(P)\} - C_0 \\ &\quad = (\cup_{j=1}^{i-2} C(u_{i-1}u_{j-1})) \cup (\cup_{j=i+1}^{l} C(u_{i-1}u_j)) - C_0 \\ &\quad \subseteq \{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \cup (C(u_{i-1}u_l) - C_0) \\ &\quad = C_1 \cup (C(u_{i-1}u_l) - C_0),\end{aligned}$$

$$\begin{aligned}&\{C(u_l w) : w \in V(P)\} - C_0 - C_1 \\ &\quad = (\cup_{j=0}^{i-2} C(u_j u_l)) \cup C(u_{i-1}u_l) \cup (\cup_{j=i+1}^{l-1} C(u_{j-1}u_l)) - C_0 - C_1 \\ &\quad \subseteq \{C(u_l u_j) : j \in Y_1\} \cup \{C(u_l u_{j-1}) : j \in Y_2\} \cup (C(u_{i-1}u_l) - C_0).\end{aligned}$$

So

$$\begin{aligned}
&\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0 \\
&\qquad \subseteq C_1 \cup \{C(u_l u_j) : j \in Y_1\} \cup \{C(u_l u_{j-1}) : j \in Y_2\} \cup (C(u_{i-1}u_l) - C_0) \\
&\qquad = \{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \\
&\qquad\quad \cup \{C(u_l u_j) : j \in Y_1\} \cup \{C(u_l u_{j-1}) : j \in Y_2\} \cup (C(u_{i-1}u_l) - C_0).
\end{aligned}$$

If $C(u_{i-1}u_l) \notin C_0$, then $vu_0Pu_{i-1}u_lP^{-1}u_i$ is a heterochromatic path of length $l+1$.

Otherwise, we have $C(u_{i-1}u_l) \in C_0$, then

$$\begin{aligned}
l-1 \leq\ & |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0| \\
\leq\ & |\{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \\
& \cup \{C(u_l u_j) : j \in Y_1\} \cup \{C(u_l u_{j-1}) : j \in Y_2\}| \\
\leq\ & |X_1| + |X_2| + |Y_1| + |Y_2|.
\end{aligned}$$

Since $X_1, Y_1 \subseteq \{1, \ldots, i-2\}$, $X_2, Y_2 \subseteq \{i+1, \ldots, l-1\}$, and $l-1 > |\{1, \ldots, i-2\} \cup \{i+1, \ldots, l-1\}| + 1$, we can conclude that there exists a $j \in (X_1 \cap Y_1) \cup (X_2 \cap Y_2)$. If $j \in X_1 \cap Y_1$, then $vu_0Pu_{j-1}u_{i-1}P^{-1}u_ju_lP^{-1}u_i$ is a heterochromatic path of length $l+1$. If $j \in X_2 \cap Y_2$, then $vu_0Pu_{i-1}u_jPu_lu_{j-1}P^{-1}u_i$ is a heterochromatic path of length $l+1$.

So there exists a heterochromatic path of length $l+1$ if $3 \leq i \leq l-2$.

From all the cases above, we can conclude that if all the conditions in the lemma are satisfied, there exists a heterochromatic path of length $l+1$ in G. ∎

Lemma 3.2 *Suppose $P = u_0u_1 \ldots u_l$ is a heterochromatic path of length l ($l \geq 4$), $C(u_0u_l) \in C(P)$, $2 \leq i_0 \leq l-1$ and $|\{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\} - C(P)| = 2$. If there exists a $v \in N(u_0) - V(P)$ such that $C(u_0v) = C(u_{i-1}u_i)$ for some $1 \leq i \leq i_0 - 1$ and $|\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)| \geq l-2$, then there is a heterochromatic path of length $l+1$ in G.*

Proof. Let $C_0 = C(P) \cup C(u_0u_{i_0}) \cup C(u_{i_0-1}u_l)$.

We distinguish the following three cases:

Case 1. i=1

Then $vu_0u_{i_0}Pu_lu_{i_0-1}P^{-1}u_1$ is a heterochromatic path of length $l+1$.

Case 2. i=2

Let

$$\begin{aligned}
X &= \{j : 3 \leq j \leq l-1, j \neq i_0, C(u_1u_j) \notin C_0\}, \\
Y &= \{j : 3 \leq j \leq l-1, j \neq i_0, C(u_{j-1}u_l) \notin C_0 \cup \{C(u_1u_j) : j \in X\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
\{C(u_1w) : w \in V(P)\} - C_0 &= \cup_{j=3}^{l} C(u_1u_j) - C_0 \\
&= \{C(u_1u_j) : j \in X\} \cup (C(u_1u_{i_0}) - C_0) \cup (C(u_1u_l) - C_0),
\end{aligned}$$

$$\begin{aligned}
&\{C(u_l w) : w \in V(P)\} - C_0 - \{C(u_1u_j) : j \in X\}\\
&\quad = \cup_{j=1}^{l-1} C(u_{j-1}u_l) - C_0 - \{C(u_1u_j) : j \in X\}\\
&\quad = \{C(u_{j-1}u_l) : j \in Y\} \cup (C(u_0u_l) - C_0) \cup (C(u_1u_l) - C_0)\\
&\quad = \{C(u_{j-1}u_l) : j \in Y\} \cup (C(u_1u_l) - C_0).
\end{aligned}$$

So

$$\begin{aligned}
&\{C(u_1w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0\\
&\quad = \{C(u_1u_j) : j \in X\} \cup \{C(u_{j-1}u_l) : j \in Y\} \cup (C(u_1u_{i_0}) - C_0) \cup (C(u_1u_l) - C_0).
\end{aligned}$$

If $C(u_1u_{i_0}) \notin C_0$, then $vu_0u_1u_{i_0}Pu_lu_{i_0-1}P^{-1}u_2$ is a heterochromatic path of length $l+1$.

If $C(u_1u_l) \notin C_0$, then $vu_0u_1u_lP^{-1}u_2$ is a heterochromatic path of length $l+1$.

Otherwise, we consider the case when $\{C(u_1u_{i_0}), C(u_1u_l)\} \subseteq C_0$, then

$$\begin{aligned}
|X| + |Y| &\geq |\{C(u_1u_j) : j \in X\} \cup \{C(u_{j-1}u_l) : j \in Y\}|\\
&\geq |\{C(u_1w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0|\\
&\geq l-2 > l-3 = |\{3, \dots, i_0-1, i_0+1, \dots, l-1\}| + 1.
\end{aligned}$$

Since $X, Y \subseteq \{3, \dots, i_0-1, i_0+1, \dots, l-1\}$, there exists a $j \in X \cap Y$, then $vu_0u_1u_jPu_lu_{j-1}$ $P^{-1}u_2$ is a heterochromatic path of length $l+1$.

Case 3. $3 \leq i \leq i_0 - 1$

Let

$$\begin{aligned}
X_1 &= \{j : 1 \leq j \leq i-2, C(u_{i-1}u_{j-1}) \notin C_0\},\\
X_2 &= \{j : i+1 \leq j \leq l-1, j \neq i_0, C(u_{i-1}u_j) \notin C_0\},\\
C_1 &= \{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\},\\
Y_1 &= \{j : 1 \leq j \leq i-2, C(u_ju_l) \notin C_0 \cup C_1\},\\
Y_2 &= \{j : i+1 \leq j \leq l-1, j \neq i_0, C(u_{j-1}u_l) \notin C_0 \cup C_1\}.
\end{aligned}$$

Then

$$\begin{aligned}
\{C(u_{i-1}w) : w \in V(P)\} - C_0 &= (\cup_{j=1}^{i-2} C(u_{j-1}u_{i-1})) \cup (\cup_{j=i+1}^{l} C(u_{i-1}u_j)) - C_0\\
&= C_1 \cup (C(u_{i-1}u_{i_0}) - C_0) \cup (C(u_{i-1}u_l) - C_0),
\end{aligned}$$

$$\begin{aligned}
&\{C(u_lw) : w \in V(P)\} - C_0 - C_1\\
&\quad = (\cup_{j=0}^{i-1} C(u_ju_l)) \cup (\cup_{j=i+1}^{l-1} C(u_{j-1}u_l)) - C_0 - C_1\\
&\quad \subseteq \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\}\\
&\qquad \cup (\{C(u_0u_l), C(u_{i-1}u_l), C(u_{i_0-1}u_l)\} - C_0)\\
&\quad = \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\} \cup (C(u_{i-1}u_l) - C_0).
\end{aligned}$$

So

$$\begin{aligned}
&\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0\\
&\quad \subseteq \{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \cup \{C(u_ju_l) : j \in Y_1\}\\
&\qquad \cup \{C(u_{j-1}u_l) : j \in Y_2\} \cup (C(u_{i-1}u_{i_0}) - C_0) \cup (C(u_{i-1}u_l) - C_0).
\end{aligned}$$

If $C(u_{i-1}u_{i_0}) \notin C_0$, then $vu_0Pu_{i-1}u_{i_0}Pu_lu_{i_0-1}P^{-1}u_i$ is a heterochromatic path of length $l+1$. If $C(u_{i-1}u_l) \notin C_0$, then $vu_0Pu_{i-1}u_lP^{-1}u_i$ is a heterochromatic path of length $l+1$.

Otherwise, we have $\{C(u_{i-1}u_{i_0}), C(u_{i-1}u_l)\} \subseteq C_0$, then

$$\begin{aligned}
&|X_1| + |X_2| + |Y_1| + |Y_2| \\
&\qquad \geq |\{C(u_{i-1}u_{j-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \\
&\qquad\quad \cup \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\}| \\
&\qquad \geq |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0| \\
&\qquad \geq l - 2 > l - 3 = |\{1, \ldots, i-2\} \cup \{i+1, \ldots, i_0-1, i_0+1, \ldots, l-1\}| + 1.
\end{aligned}$$

Since $X_1, Y_1 \subseteq \{1, 2, \ldots, i-2\}$, $X_2, Y_2 \subseteq \{i+1, \ldots, i_0-1, i_0+1, \ldots, l-1\}$, we can conclude that there exists a $j \in (X_1 \cap Y_1) \cup (X_2 \cap Y_2)$. If $j \in X_1 \cap Y_1$, then $vu_0Pu_{j-1}u_{i-1}P^{-1}u_ju_lP^{-1}u_i$ is a heterochromatic path of length $l+1$, otherwise $j \in X_2 \cap Y_2$, and in that case $vu_0Pu_{i-1}u_jPu_lu_{j-1}P^{-1}u_i$ is a heterochromatic path of length $l+1$.

From all the cases above, we can conclude that if all the conditions in this lemma are satisfied, there is a heterochromatic path of length $l+1$ in G. ∎

Lemma 3.3 *Suppose $P = u_0u_1 \ldots u_l$ is a heterochromatic path of length l ($l \geq 4$), $C(u_0u_l) \in C(P)$, $2 \leq i_0 \leq l-1$ and $|\{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\} - C(P)| = 2$. If there exists a $v \in N(u_0) - V(P)$ such that $C(u_0v) = C(u_{i-1}u_i)$ for some $i_0+1 \leq i \leq l$, and $|\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)| \geq l-2$, then there is a heterochromatic path of length $l+1$ in G.*

Proof. Let $C_0 = C(P) \cup C(u_0u_{i_0}) \cup C(u_{i_0-1}u_l)$.

We distinguish the following three cases:

Case 1. $i = l$

Let

$$\begin{aligned}
X &= \{j : 1 \leq j \leq l-2, j \neq i_0 - 1, C(u_{j-1}u_{l-1}) \notin C_0\}, \\
Y &= \{j : 1 \leq j \leq l-2, j \neq i_0 - 1, C(u_ju_l) \notin C_0 \cup \{C(u_{j-1}u_{l-1}) : j \in X\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\{C(u_{l-1}w) : w \in V(P)\} - C_0 \\
&\qquad = \cup_{j=1}^{l-2} C(u_{j-1}u_{l-1}) - C_0 \\
&\qquad = \{C(u_{j-1}u_{l-1}) : j \in X\} \cup (C(u_{i_0-2}u_{l-1}) - C_0),
\end{aligned}$$

$$\begin{aligned}
&\{C(u_lw) : w \in V(P)\} - C_0 - \{C(u_{j-1}u_{l-1}) : j \in X\} \\
&\qquad = \cup_{j=0}^{l-2} C(u_ju_l) - C_0 - \{C(u_{j-1}u_{l-1}) : j \in X\} \\
&\qquad = \{C(u_ju_l) : j \in Y\} \cup (\{C(u_0u_l), C(u_{i_0-1}u_l)\} - C_0 - \{C(u_{j-1}u_{l-1}) : j \in X\}) \\
&\qquad = \{C(u_ju_l) : j \in Y\}.
\end{aligned}$$

So

$$\begin{aligned}&\{C(u_{l-1}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0\\ &\qquad = \{C(u_{j-1}u_{l-1}) : j \in X\} \cup \{C(u_j u_l) : j \in Y\} \cup (C(u_{i_0-2}u_{l-1}) - C_0).\end{aligned}$$

If $C(u_{i_0-2}u_{l-1}) \notin C_0$, then $vu_0Pu_{i_0-2}u_{l-1}P^{-1}u_{i_0-1}u_l$ is a heterochromatic path of length $l+1$.

Otherwise, we have $C(u_{i_0-2}u_{l-1}) \in C_0$, then

$$\begin{aligned}|X| + |Y| &\geq |\{C(u_{j-1}u_{l-1}) : j \in X\} \cup \{C(u_j u_l) : j \in Y\}|\\ &\geq |\{C(u_{l-1}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0|\\ &\geq l-2 = |\{1, \ldots, i_0-2, i_0, \ldots, l-2\}| + 1.\end{aligned}$$

Since $X, Y \subseteq \{1, \ldots, i_0-2, i_0, \ldots, l-2\}$, $X \cap Y \neq \emptyset$, i.e., there exists a $j \in X \cap Y$, then $vu_0Pu_{j-1}u_{l-1}P^{-1}u_ju_l$ is a heterochromatic path of length $l+1$.

Case 2. $i = l-1$

Let

$$\begin{aligned}X &= \{j : 1 \leq j \leq l-3, j \neq i_0-1, C(u_{j-1}u_{l-2}) \notin C_0\},\\ Y &= \{j : 1 \leq j \leq l-3, j \neq i_0-1, C(u_j u_l) \notin C_0 \cup \{C(u_{j-1}u_{l-2}) : j \in X\}\}.\end{aligned}$$

Then

$$\begin{aligned}&\{C(u_{l-2}w) : w \in V(P)\} - C_0\\ &\qquad = (\cup_{j=1}^{l-3} C(u_{j-1}u_{l-2}) \cup C(u_{l-2}u_l) - C_0)\\ &\qquad = \{C(u_{j-1}u_{l-2}) : j \in X\} \cup (C(u_{i_0-2}u_{l-2}) - C_0) \cup (C(u_{l-2}u_l) - C_0),\end{aligned}$$

$$\begin{aligned}&\{C(u_l w) : w \in V(P)\} - C_0 - \{C(u_{j-1}u_{l-2}) : j \in X\}\\ &\qquad = \cup_{j=0}^{l-2} C(u_j u_l) - C_0 - \{C(u_{j-1}u_{l-2}) : j \in X\}\\ &\qquad \subseteq \{C(u_j u_l) : j \in Y\} \cup (C(u_0u_l) - C_0) \cup (C(u_{l-2}u_l) - C_0) \cup (C(u_{i_0-1}u_l) - C_0)\\ &\qquad = \{C(u_j u_l) : j \in Y\} \cup (C(u_{l-2}u_l) - C_0).\end{aligned}$$

So

$$\begin{aligned}&\{C(u_{l-2}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0\\ &\qquad = \{C(u_{j-1}u_{l-2}) : j \in X\} \cup \{C(u_j u_l) : j \in Y\} \cup (\{C(u_{i_0-2}u_{l-2}), C(u_{l-2}u_l)\} - C_0).\end{aligned}$$

If $C(u_{l-2}u_l) \notin C_0$, $vu_0Pu_{l-2}u_lu_{l-1}$ is a heterochromatic path of length $l+1$. If $C(u_{i_0-2}u_{l-2}) \notin C_0$, then $vu_0Pu_{i_0-2}u_{l-2}P^{-1}u_{i_0-1}u_lu_{l-1}$ is a heterochromatic path of length $l+1$.

Otherwise, we have $\{C(u_{l-2}u_l), C(u_{i_0-2}u_{l-2})\} \subseteq C_0$, then

$$\begin{aligned}|X| + |Y| &\geq |\{C(u_{j-1}u_{l-2}) : j \in X\} \cup \{C(u_j u_l) : j \in Y\}|\\ &\geq |\{C(u_{l-2}w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C_0|\\ &\geq l-2 > l-3 = |\{1, \ldots, i_0-2, i_0, \ldots, l-3\}| + 1.\end{aligned}$$

Since $X, Y \subseteq \{1, \ldots, i_0 - 2, i_0, \ldots, l-3\}$, $X \cap Y \neq \emptyset$, i.e., there exists a $j \in X \cap Y$, then $vu_0Pu_{j-1}u_{l-2}P^{-1}u_ju_lu_{l-1}$ is a heterochromatic path of length $l+1$.

Case 3. $i_0 + 1 \leq i \leq l-2$

Let

$$\begin{aligned} X_1 &= \{j : 1 \leq j \leq i-2, j \neq i_0 - 1, C(u_{j-1}u_{i-1}) \notin C_0\}, \\ X_2 &= \{j : i+1 \leq j \leq l-1, C(u_ju_{i-1}) \notin C_0\}, \\ C_1 &= \{C(u_{j-1}u_{i-1}) : j \in X_1\} \cup \{C(u_ju_{i-1}) : j \in X_2\}, \\ Y_1 &= \{j : 1 \leq j \leq i-2, j \neq i_0 - 1, C(u_ju_l) \notin C_0 \cup C_1\}, \\ Y_2 &= \{j : i+1 \leq j \leq l-1, C(u_{j-1}u_l) \notin C_0 \cup C_1\}. \end{aligned}$$

Then

$$\begin{aligned} &\{C(u_{i-1}w) : w \in V(P)\} - C_0 \\ &\quad = (\cup_{j=1}^{i-2} C(u_{i-1}u_{j-1})) \cup (\cup_{j=i+1}^{l} C(u_{i-1}u_j)) - C_0 \\ &\quad = C_1 \cup (C(u_{i-1}u_{i_0-2}) - C_0) \cup (C(u_{i-1}u_l) - C_0), \end{aligned}$$

$$\begin{aligned} &\{C(u_lw) : w \in V(P)\} - C_0 - C_1 \\ &\quad = (\cup_{j=0}^{i-2} C(u_lu_j)) \cup (C(u_lu_{i-1})) \cup (\cup_{j=i+1}^{l-1} C(u_{j-1}u_l)) - C_0 - C_1 \\ &\quad \subseteq \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\} \cup (\{C(u_{i_0-1}u_l) \cup C(u_{i-1}u_l)\} - C_0) \\ &\quad = \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\} \cup (C(u_{i-1}u_l) - C_0). \end{aligned}$$

So

$$\begin{aligned} &\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0 \\ &\quad \subseteq \{C(u_{j-1}u_{i-1}) : j \in X_1\} \cup \{C(u_ju_{i-1}) : j \in X_2\} \cup \{C(u_ju_l) : j \in Y_1\} \\ &\qquad \cup \{C(u_{j-1}u_l) : j \in Y_2\} \cup (C(u_{i-1}u_{i_0-2}) - C_0) \cup (C(u_{i-1}u_l) - C_0). \end{aligned}$$

If $C(u_{i-1}u_{i_0-2}) \notin C_0$, then $vu_0Pu_{i_0-2}u_{i-1}P^{-1}u_{i_0-1}u_lP^{-1}u_i$ is a heterochromatic path of length $l+1$. If $C(u_{i-1}u_l) \notin C_0$, then $vu_0Pu_{i-1}u_lP^{-1}u_i$ is a heterochromatic path of length $l+1$.

Otherwise, we have $\{C(u_{i-1}u_{i_0-2}), C(u_lu_{i-1})\} \subseteq C_0$, then

$$\begin{aligned} &|X_1| + |X_2| + |Y_1| + |Y_2| \\ &\quad \geq |\{C(u_{j-1}u_{i-1}) : j \in X_1\} \cup \{C(u_{i-1}u_j) : j \in X_2\} \\ &\qquad \cup \{C(u_ju_l) : j \in Y_1\} \cup \{C(u_{j-1}u_l) : j \in Y_2\}| \\ &\quad \geq |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C_0| \\ &\quad \geq l-2 > l-3 = |\{1, \ldots, i_0-2, i_0, \ldots, i-2\} \cup \{i+1, \ldots, l-1\}| + 1. \end{aligned}$$

Since $X_1, Y_1 \subseteq \{1, \ldots, i_0-2, i_0, \ldots, i-2\}$, $X_2, Y_2 \subseteq \{i+1, \ldots, l-1\}$, $(X_1 \cap Y_1) \cup (X_2 \cap Y_2) \neq \emptyset$, i.e., there exists a $j \in (X_1 \cap Y_1) \cup (X_2 \cap Y_2)$. If $j \in X_1 \cap Y_1$, then $vu_0Pu_{j-1}u_{i-1}P^{-1}u_ju_lP^{-1}u_i$ is a heterochromatic path of length $l+1$. If $j \in X_2 \cap Y_2$, then $vu_0Pu_{i-1}u_jPu_lu_{j-1}P^{-1}u_i$ is a heterochromatic path of length $l+1$.

From all the cases above, we can conclude that if all the conditions in the lemma are satisfied, there exists a heterochromatic path of length $l+1$ in G. ∎

Theorem 3.4 *Let G be an edge-colored graph and $|CN(u) \cup CN(v)| \geq s \geq 1$ for any two vertices u and v in G. Then there exists a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$ in G.*

Proof. We will prove the theorem by induction.

If $1 \leq s \leq 7$, our Theorem 2.1 shows that G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$.

Now we shall only consider the case when $s \geq 8$. Assume that if $|CN(u) \cup CN(v)| \geq s-1$ for any $u, v \in V(G)$, G has a heterochromatic path of length at least $\lceil \frac{(s-1)+1}{2} \rceil \geq \lceil \frac{7+1}{2} \rceil = 4$. Then we need only to show that if $|CN(u) \cup CN(v)| \geq s$ for any $u, v \in V(G)$, G has a heterochromatic path of length $\lceil \frac{s+1}{2} \rceil$. Since if s is odd then $\lceil \frac{s}{2} \rceil = \lceil \frac{s+1}{2} \rceil$, we need only to show that if s is even, G has a heterochromatic path of length at least $\lceil \frac{s+1}{2} \rceil$.

By the assumption we know that G has a heterochromatic path of length at least $\lceil \frac{(s-1)+1}{2} \rceil = \lceil \frac{s}{2} \rceil$. Assume that the longest heterochromatic path in G is of length $l = \lceil \frac{s}{2} \rceil$ and $P = u_0 u_1 \ldots u_l$ is such a path.

Now we will show that $N(u_0) \subseteq V(P)$ by contradiction. Assume $N(u_0) - V(P) \neq \emptyset$ and $v \in N(u_0) - V(P)$. Then $C(u_0 v) \notin C(P)$ or $C(u_0 v) \in C(P)$.

If $C(u_0 v) \notin C(P)$, $v u_0 P u_l$ is a heterochromatic path of length $l+1$, a contradiction to the assumption that the longest heterochromatic path in G is of length l.

Now we shall only consider the case when $C(u_0 v) = C(u_{i-1} u_i)$ for some $1 \leq i \leq l$. We distinguish the following cases:

Case 1. $C(u_0 u_l) \notin C(P)$

If there exists a $w \in N(u_{i-1}) - V(P)$ such that $C(u_{i-1} w) \notin C(P) \cup C(u_0 u_l)$, then $w u_{i-1} P^{-1} u_0 u_l P^{-1} u_i$ is a heterochromatic path of length $l+1$, a contradiction. So we have that $CN(u_{i-1}) - C(P) - C(u_0 u_l) \subseteq \{C(u_{i-1} w) : w \in V(P)\} - C(P) - C(u_0 u_l)$.

On the other hand, if there exists a $w \in N(u_l) - V(P)$ such that $C(u_l w) \notin C(P)$, $u_0 P u_l w$ is a heterochromatic path of length $l+1$, a contradiction. So we also have that $CN(u_l) - C(P) \subseteq \{C(u_l w) : w \in V(P)\} - C(P)$, then $CN(u_l) - C(P) - C(u_0 u_l) \subseteq \{C(u_l w) : w \in V(P)\} - C(P) - C(u_0 u_l)$.

So $CN(u_{i-1}) \cup CN(u_l) \subseteq \{C(u_{i-1} w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} \cup C(P) \cup C(u_0 u_l)$. Now we can get that

$$\begin{aligned}
s \leq\ & |CN(u_{i-1}) \cup CN(u_l)| \\
\leq\ & |\{C(u_{i-1} w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} \cup C(P) \cup C(u_0 u_l)| \\
\leq\ & |\{C(u_{i-1} w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C(P) - C(u_0 u_l)| \\
& + |C(P)| + |C(u_0 u_l)| \\
=\ & |\{C(u_{i-1} w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C(P) - C(u_0 u_l)| + l + 1.
\end{aligned}$$

So $|\{C(u_{i-1} w) : w \in V(P)\} \cup \{C(u_l w) : w \in V(P)\} - C(P) - C(u_0 u_l)| \geq s - l - 1 = 2l - l - 1 = l - 1$. Then by Lemma 3.1, there is a heterochromatic path of length $l+1$ in G, a contradiction.

Case 2. $C(u_0 u_l) \in C(P)$

Since P is one of the longest heterochromatic path in G, there does not exist any $w \in N(u_0) \cup N(u_l) - V(P)$ such that $C(u_0 w) \notin C(P)$ or $C(u_l w) \notin C(P)$, otherwise

wu_0Pu_l or u_0Pu_lw is a heterochromatic path of length $l+1$, a contradiction.

Let

$$\begin{aligned} X &= \{2 \leq i \leq l-1 : C(u_0u_i) \notin C(P)\}, \\ Y &= \{2 \leq i \leq l-1 : C(u_{i-1}u_l) \notin C(P) \cup \{C(u_0u_i) : i \in X\}\}. \end{aligned}$$

Then

$$CN(u_0) \cup CN(u_l) - C(P) \subseteq \{C(u_0u_i) : i \in X\} \cup \{C(u_{i-1}u_l) : i \in Y\},$$

so

$$\begin{aligned} |X| + |Y| &\geq |\{C(u_0u_i) : i \in X\} \cup \{C(u_{i-1}u_l) : i \in Y\}| \\ &\geq |CN(u_0) \cup CN(u_l) - C(P)| \\ &\geq |CN(u_0) \cup CN(u_l)| - |C(P)| \\ &\geq s - l = l > |\{2, 3, \ldots, l-1\}| + 1. \end{aligned}$$

Now we can conclude that there exists an i_0 $(2 \leq i_0 \leq l-1)$ such that $i_0 \in X \cap Y$, i.e., $|\{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\} - C(P)| = 2$, since $X, Y \subseteq \{2, \ldots, l-1\}$.

Then we distinguish the following 3 subcases:

Subcase 1. $1 \leq i \leq i_0 - 1$

If there exists a $w \in N(u_{i-1}) - V(P)$ such that $C(u_{i-1}w) \notin C(P) \cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\}$, $wu_{i-1}P^{-1}u_0u_{i_0}Pu_lu_{i_0-1}P^{-1}u_i$ is a heterochromatic path of length $l+1$, a contradiction. So we have that $CN(u_{i-1}) - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l) \subseteq \{C(u_{i-1}w) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)$.

On the other hand, if there exists a $w \in N(u_l) - V(P)$ such that $C(u_lw) \notin C(P)$, u_0Pu_lw is a heterochromatic path of length $l+1$, a contradiction. So we also have that $CN(u_l) - C(P) \subseteq \{C(u_lw) : w \in V(P)\} - C(P)$, then $CN(u_l) - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l) \subseteq \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)$.

So

$$\begin{aligned} CN(u_{i-1}) \cup CN(u_l) \subseteq\ &\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} \cup C(P) \\ &\cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\}. \end{aligned}$$

Now we can get that

$$\begin{aligned} s &\leq |CN(u_{i-1}) \cup CN(u_l)| \\ &\leq |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} \cup C(P) \cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\}| \\ &\leq |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) \\ &\quad -C(u_{i_0-1}u_l)| + |C(P)| + |C(u_0u_{i_0})| + |C(u_{i_0-1}u_l)| \\ &= |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)| \\ &\quad +l + 2. \end{aligned}$$

So $|\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)| \geq s - l - 2 = 2l - l - 2 = l - 2$. Then by Lemma 3.2, there is a heterochromatic path of length $l+1$ in G, a contradiction.

Subcase 2. $i = i_0$

Then $vu_0u_{i_0}Pu_lu_{i_0-1}P^{-1}u_1$ is a heterochromatic path of length $l+1$, a contradiction.

Subcase 3. $i_0+1 \le i \le l$

If there exists a $w \in N(u_{i-1}) - V(P)$ such that $C(u_{i-1}w) \notin C(P) \cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\}$, $wu_{i-1}P^{-1}u_{i_0}u_0Pu_{i_0-1}u_lP^{-1}u_i$ is a heterochromatic path of length $l+1$, a contradiction. So we have that $CN(u_{i-1}) - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l) \subseteq \{C(u_{i-1}w) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)$.

On the other hand, if there exists a $w \in N(u_l) - V(P)$ such that $C(u_lw) \notin C(P)$, u_0Pu_lw is a heterochromatic path of length $l+1$, a contradiction. So we also have that $CN(u_l) - C(P) \subseteq \{C(u_lw) : w \in V(P)\} - C(P)$, then $CN(u_l) - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l) \subseteq \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)$.

So

$$\begin{aligned} CN(u_{i-1}) \cup CN(u_l) \subseteq\ & \{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} \cup C(P) \\ & \cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\} \end{aligned}$$

Now we can get that

$$\begin{aligned} s \le\ & |CN(u_{i-1}) \cup CN(u_l)| \\ \le\ & |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} \cup C(P) \cup \{C(u_0u_{i_0}), C(u_{i_0-1}u_l)\}| \\ \le\ & |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) \\ & - C(u_{i_0-1}u_l)| + |C(P)| + |C(u_0u_{i_0})| + |C(u_{i_0-1}u_l)| \\ =\ & |\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l))| \\ & + l + 2. \end{aligned}$$

So $|\{C(u_{i-1}w) : w \in V(P)\} \cup \{C(u_lw) : w \in V(P)\} - C(P) - C(u_0u_{i_0}) - C(u_{i_0-1}u_l)| \ge s - l - 2 = 2l - l - 2 = l - 2$. Then by Lemma 3.3, there is a heterochromatic path of length $l+1$ in G, a contradiction.

From all the cases above, we can conclude that if $C(u_0v) = C(u_{i-1}u_i)$ for some $1 \le i \le l$, we will get a contradiction.

So we can conclude that $N(u_0) \subseteq V(P)$.

In the same way, we can also get that $N(u_l) \subseteq V(P)$.

Now we have

$$CN(u_0) \cup CN(u_l) = \{C(u_0u_i) : 1 \le i \le l-1\} \cup \{C(u_iu_l) : 1 \le i \le l-1\} \cup C(u_0u_l).$$

Then

$$\begin{aligned} |CN(u_0) \cup CN(u_l)| &= |\{C(u_0u_i) : 1 \le i \le l-1\} \cup \{C(u_iu_l) : 1 \le i \le l-1\} \cup C(u_0u_l)| \\ &\le 2|\{1, 2, \ldots, l-1\}| + 1 = 2l - 1 = s - 1 < s, \end{aligned}$$

a contradiction to the assumption that $|CN(u) \cup CN(v)| \ge s$ for any $u, v \in V(G)$.

So the longest heterochromatic path is of length greater than l, then there must exist a heterochromatic path of length $l + 1 = \lceil \frac{s+1}{2} \rceil$ in G.

The proof is now complete. ∎

Finally, we give examples to show that our lower bound is best possible. Let s be a positive integer. If s is even, let G_s be the graph obtained from the complete graph

$K_{\frac{s+4}{2}}$ by deleting an edge; if s is odd, let G_s be the complete graph $K_{\frac{s+3}{2}}$. Then, color the edges of G_s by different colors for any two different edges. So, for any $s \geq 1$ we have that $|CN(u) \cup CN(v)| \geq s$ for any pair of vertices u and v in G, and any longest heterochromatic path in G is of length $\lceil \frac{s+1}{2} \rceil$.

References

[1] M. Albert, A. Frieze and B. Reed, Multicolored Hamilton cycles, Electronic J. Combin. **2** (1995), ♯R10.

[2] M. Axenovich, T. Jiang and Zs. Tuza, Local anti-Ramsey numbers of graphs, Combin. Probab. Comput. 12(2003), 495-511.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan London and Elsvier, New York (1976).

[4] H.J. Broersma, X.L. Li, G. Woeginger and S.G. Zhang, Paths and cycles in colored graphs, Australasian J. Combin. 31(2005), 297-309.

[5] H. Chen and X.L. Li, Color degree and color neighborhood union conditions for long heterochromatic paths in edge-colored graphs, arXiv:math.CO/0512144 v1 7 Dec 2005.

[6] W.S. Chou, Y. Manoussakis, O. Megalaki, M. Spyratos and Zs. Tuza, Paths through fixed vertices in edge-colored graphs, Math. Inf. Sci. Hun. 32(1994), 49-58.

[7] P. Erdős and Zs. Tuza, Rainbow Hamiltonian paths and canonically colored subgraphs in infinite complete graphs, Mathematica Pannonica 1(1990), 5-13.

[8] P. Erdős and Zs. Tuza, Rainbow subgraphs in edge-colorings of complete graphs, Ann. Discrete Math. 55(1993), 81-88.

[9] A.M. Frieze and B.A. Reed, Polychromatic Hamilton cycles, Discrete Math. **118**(1993), 69-74.

[10] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and edge-coloring conjecture, Discrete Math. **62(1)**(1986), 29-33.

[11] Y. Manoussakis, M. Spyratos and Zs. Tuza, Cycles of given color patterns, J. Graph Theory 21(1996), 153-162.

[12] Y. Manoussakis, M. Spyratos, Zs. Tuza and M. Voigt, Minimal colorings for properly colored subgraphs, Graphs and Combin. 12(1996), 345-360.