Matroid automorphisms of the F_4 root system *

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Dedicated to Thomas Brylawski.

Abstract

Let F_4 be the root system associated with the 24-cell, and let $M(F_4)$ be the simple linear dependence matroid corresponding to this root system. We determine the automorphism group of this matroid and compare it to the Coxeter group W for the root system. We find non-geometric automorphisms that preserve the matroid but not the root system.

1 Introduction

Given a finite set of vectors in Euclidean space, we consider the linear dependence matroid of the set, where dependences are taken over the reals. When the original set of vectors has 'nice' symmetry, it makes sense to compare the geometric symmetry (the *Coxeter* or *Weyl* group) with the group that preserves sets of linearly independent vectors. This latter group is precisely the automorphism group of the linear matroid determined by the given vectors.

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For the root systems A_n and B_n, matroid automorphisms do not give any additional symmetry [4]. One can interpret these results in the following way: combinatorial symmetry (preserving dependence) and geometric symmetry (via isometries of the ambient Euclidean space) are essentially the same. In contrast to these root systems, the root system associated with the icosahedron (usually denoted H_3) does possess matroid automorphisms that do not correspond to any geometric symmetries [3]. We can interpret this as follows: it is possible to map the vectors of H_3 to themselves so that all linear dependences are preserved, but angles between vectors are not.

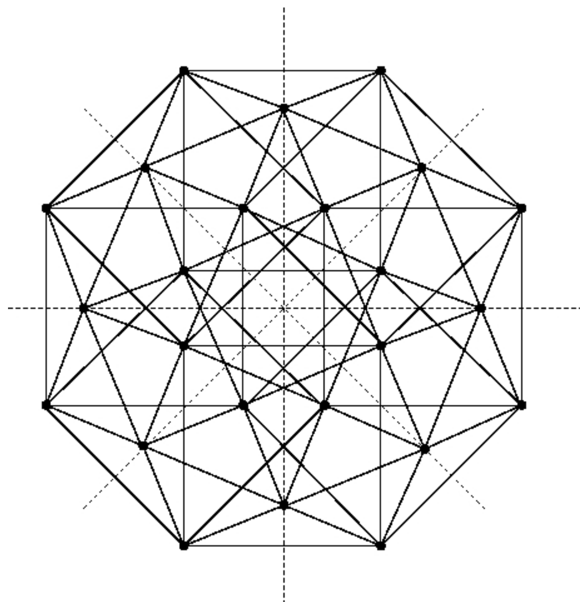

Figure 1: The 24-cell.

In this paper, we study the root system F_4. This root system can be defined in at least two equivalent ways. F_4 is the collection of 48 vectors in $\mathbb{R}^4$ normal to the 24 mirror hyperplanes of the 24-cell, a 4-dimensional regular solid whose 24 facets are 3-dimensional octahedra (see Figure 1). Alternatively, we can view the roots as the vectors of the lattice D_4 of squared lengths 1 and 2. This latter viewpoint will be more useful for us. The Coxeter group W for this root system (which is also the symmetry group of the 24-cell) has order 1152 and factors as a semidirect product: $W \cong (\mathbb{Z}_2^3 \rtimes S_4) \rtimes S_3$. Much more information can be found in [5] and [6].

In one respect, computing the automorphisms of the matroid $M(F_4)$ (or any given matroid) is 'automatic.' The matroid $M(F_4)$ is the column dependence matroid of a 4×24 matrix with entries $0, \pm 1$ (the 24 vectors are described below). Thus, feeding the matrix to a computer program should determine the possible permutations of the columns that preserve all linearly dependent sets. This subset (of the symmetric group S_{24}) is necessarily isomorphic to our matroid automorphism group. Such an approach will not explain the structure of the group, however, or how it relates to the geometry of the

Coxeter group W. It is also unclear how much computation such a computer program would require; it is obviously not feasible to search through all 24! possible permutations.

As is the case in any comparison of the combinatorial and geometric groups, it is clear that any geometric symmetry of the 24-cell will preserve the dependence or independence of any subset of the 24 normal vectors that comprise the matroid $M(F_4)$. Further, since central inversion in W corresponds to the identity operation in the matroid automorphism group, which we denote $\mathrm{Aut}(M(F_4))$, we have a 2-to-1 map from W to $\mathrm{Aut}(M(F_4))$.

Our main theorem, Theorem 3.8, computes the structure of the automorphism group $\mathrm{Aut}(M(F_4))$:

Theorem 3.8. $\mathrm{Aut}(M(F_4)) \cong ((\mathbb{Z}_2^3 \rtimes S_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

As with the root system H_3, there are matroid automorphisms that cannot be realized 'geometrially' in the sense that no element of the Coxeter group W corresponds to these automorphisms. These non-geometric automorphisms correspond to swapping the short and long vectors in the root system.

The paper is organized as follows: Section 2 defines the matroid $M(F_4)$ and introduces a graphic way to represent the rank 2 flats of the matroid. Section 3 computes the matroid automorphism group (after a series of lemmas), and a schematic interpretation of the connection between the two groups is given in section 4.

We thank Derek Smith for very useful discussions on the relationship between F_4 and the D_4 lattice.

2 The matroid $M(F_4)$.

We refer the reader to the first chapter of [7] for an introduction to matroids. We define the matroid $M(F_4)$ through the root system F_4, which we briefly recall now.

Let e_i be the i^{th} basis vector in $\Re^4$. The root system F_4 consists of 48 vectors in $\Re^4$ which can partitioned into 24 *short* vectors and 24 *long* vectors:

- *Short vectors:* $\pm e_i, (\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2$
- *Long vectors:* $\pm e_i \pm e_j (i < j)$.

The short vectors have squared length 1, while the long vectors have squared length 2. Note that if v is a root, then so is $-v$. From the matroid viewpoint, the vectors v and $-v$ correspond to a multiple point. To define a (linear) matroid from this root system, we remove one vector from each of the 24 $\{v, -v\}$ pairs. (The 2^{24} ways to do this all give the same matroid.) Equivalently, we could define $M(F_4)$ to be the simplification of the matroid formed by all 48 of the roots of F_4. Additionally, we can rescale the vectors without changing the matroid.

For the remainder of this paper, we form $M(F_4)$ as a linear dependence matroid over $\mathbb{Q}$ by choosing 24 roots, with labels, as follows:

- For $1 \leq i < j \leq 4$, let a_{ij}^+ and a_{ij}^- denote the vectors $e_i + e_j$ and $e_i - e_j$, resp.

- The vectors e_i are selected.
- Let $f_i = \sum_{k=1}^4 e_k - 2e_i$.
- Let $g_1 = \sum_{k=1}^4 e_k$.
- Let $g_2 = e_1 + e_2 - e_3 - e_4$.
- Let $g_3 = e_1 - e_2 + e_3 - e_4$.
- Let $g_4 = e_1 - e_2 - e_3 + e_4$.

Equivalently, $M(F_4)$ is the matroid on the central hyperplane arrangement formed by the 24 3-dimensional hyperplanes $\overline{v} \cdot \overline{x} = 0$, where $\overline{v}$ ranges over the 24 vectors described above and $\overline{x} = \langle x_1, x_2, x_3, x_4 \rangle$. (These are precisely the 24 hyperplanes of symmetry for the 24-cell.)

Thus, $M(F_4)$ is a (vector) matroid on 24 points with ground set $\{a_{ij}^+, a_{ij}^-, e_k, f_l, g_m\}$ for appropriate values of the indices. For convenience, it will be useful to partition the 24 points of $M(F_4)$ into two classes A and B:

- $A = \{a_{ij}^{\pm}\}$ for $1 \leq i < j \leq 4$;
- $B = \{e_i, f_j, g_k\}$ for $1 \leq i, j, k \leq 4$.

We further partition A and B each into three blocks as follows:

- $E' = \{a_{12}^+, a_{12}^-, a_{34}^+, a_{34}^-\}$;
- $F' = \{a_{13}^+, a_{13}^-, a_{24}^+, a_{24}^-\}$;
- $G' = \{a_{14}^+, a_{14}^-, a_{23}^+, a_{23}^-\}$;
- $E = \{e_1, \dots, e_4\}$;
- $F = \{f_1, \dots, f_4\}$;
- $G = \{g_1, \dots, g_4\}$.

Then $A = E' \cup F' \cup G'$ and $B = E \cup F \cup G$.

The rank 2 flats of $M(F_4)$

We will analyze the set of all matroid automorphisms of $M(F_4)$ by concentrating on its rank 2 flats.

The proof of the next proposition is straightforward.

Proposition 2.1. *Suppose L is a rank 2 flat in $M(F_4)$ with $|L| > 2$. Then $|L| = 3$ or 4. In particular, we have the following:*

1. *If $|L| = 3$, then either $L \subset A$ or $L \subset B$. Further,*
   - *3-point lines in A: L has the form $\{a_{ij}^r, a_{jk}^s, a_{ik}^t\}$ for some $1 \le i < j < k \le 4$ and for signs r, s and t, where $rst = -1$.*
   - *3-point lines in B: Every 3-point line has the form $\{e_i, f_j, g_k\}$ for some $1 \le i, j, k \le 4$.*
2. *If $|L| = 4$, then $|L \cap A| = |L \cap B| = 2$. Further, both points in $L \cap A$ are in the same block of the partition $A = E' \cup F' \cup G'$, and both points in $L \cap B$ are in the same block of the partition $B = E \cup F \cup G$.*

Each point is in four 3-point lines, giving a total of 32 3-point lines in the matroid. Half of these lines are entirely contained in A and the other half are contained in B. For example, the point e_1 is in the following 3-point lines:

$$\{e_1, f_1, g_1\}, \{e_1, f_2, g_2\}, \{e_1, f_3, g_3\}, \{e_1, f_4, g_4\}.$$

The 3-point lines containing a_{12}^+ are

$$\{a_{12}^+, a_{23}^+, a_{13}^-\}, \{a_{12}^+, a_{23}^-, a_{13}^+\}, \{a_{12}^+, a_{14}^+, a_{24}^-\}, \{a_{12}^+, a_{14}^-, a_{24}^+\}.$$

There are a total of 18 4-point lines in $M(F_4)$, with each point in precisely three such lines. For example, the point e_1 is in

$$\{e_1, e_2, a_{12}^+, a_{12}^-\}, \{e_1, e_3, a_{13}^+, a_{13}^-\}, \{e_1, e_4, a_{14}^+, a_{14}^-\}.$$

Motivated by the treatment of signed graphs and matroid automorphisms in [4], we represent the 4-point line incidence structure in $M(F_4)$ via the labeled graph Γ of Figure 2.

We point out an asymmetry in Γ: Each of the 12 points of B appears exactly once (as a vertex of one of the components), but each point of A appears three times (once as an edge in each of the three components E, F and G). Then in the labeled graph Γ, each 4-point line is represented as a pair of parallel edges, together with its two endpoints. For instance, $\{f_2, f_4, a_{13}^+, a_{24}^-\}$ appears as one of the diagonals in the F component of Γ. (We could avoid the vertex-edge mixing by placing a loop at each vertex with label equal to its vertex label.)

It is also possible to define an 'inside out' version of Γ, which we call Γ'. This graph is isomorphic to Γ, but has vertex labels from A and edge labels from B. Further, the partition $A = E' \cup F' \cup G'$ is the corresponding vertex partition of Γ'. We will refer to Γ' in describing certain automorphisms of the matroid.

It is easy to read off all of the 4-point lines of the matroid (there are 18 such lines) from Γ (or, equally well, from Γ'). In addition, it is easy to see the 16 3-point lines contained in A: these correspond to triangles in one of the components E, F or G in which there are an odd number of $-$ labels selected (these are the *balanced circles* of [8] with signs reversed). Each such 3-point line appears exactly once in each of the three components.

We remark that the matroid $M(F_4)$ has 96 2-point lines – each point is contained in six such lines. Each 2-point line consists of one point from A and one from B. It is

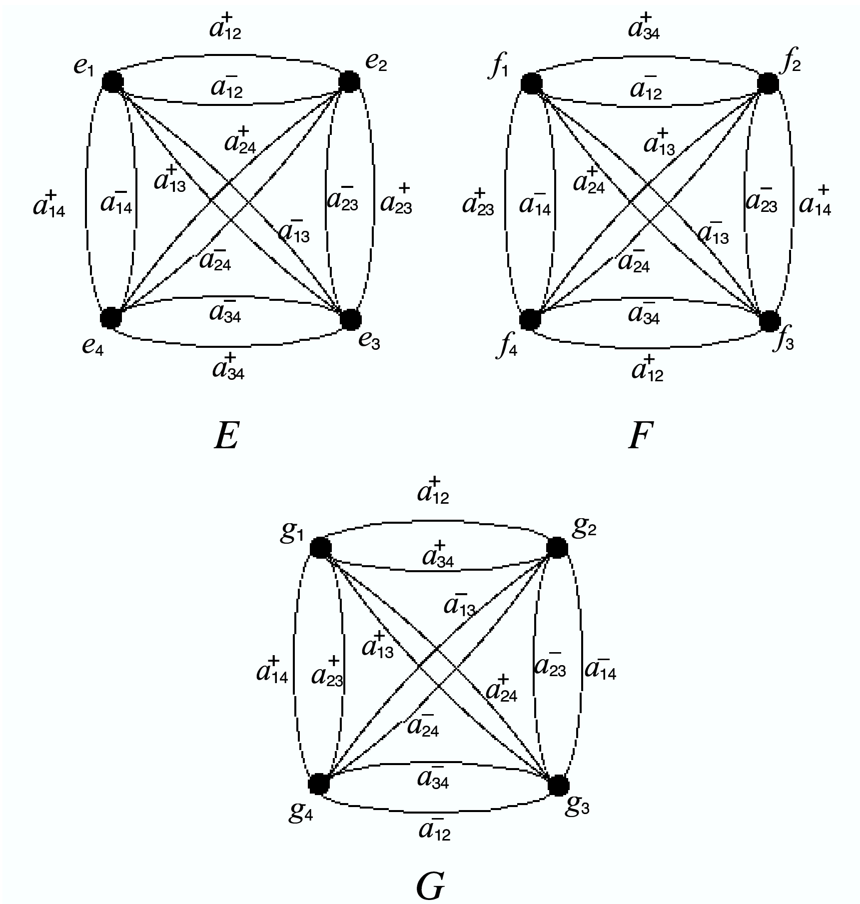

Figure 2: The graph Γ represents the 4-point lines of $M(F_4)$.

possible to see these lines in the graph Γ: they consist of a vertex and an edge from the same component where the vertex is *not* incident to the edge.

There are 24 flats of rank 3, each of which contains 9 points. Half of these flats are visible in Γ and the other half can be seen in Γ'. In Γ, these flats consist of 3 points from A and 6 from B (this is reversed in Γ') and are formed by removing one vertex and all the edges incident to that vertex in one of the three components E, F or G. These flats are all isomorphic to the matroid associated with the 3-dimensional cube from [4]. See Figure 3 for a geometric representation of this flat.

It is clear that any automorphism of $M(F_4)$ induces some permutation of the 18 4-point lines. We will determine the full matroid automorphism group as follows:

1. Determine precisely which permutations of these 18 lines can occur;

2. Show how these permutations correspond to graph automorphisms of Γ;

3. Prove that these permutations completely determine the entire matroid automorphism.

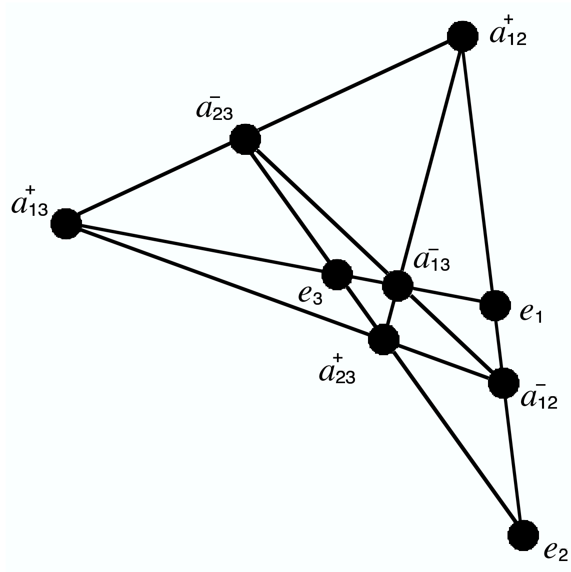

Figure 3: One of the 24 9-point rank 3 flats of $M(F_4)$.

The full automorphism group is computed in the next section.

3 Automorphisms of the matroid $M(F_4)$

In this section we determine the automorphism group $\mathrm{Aut}(M(F_4))$. The proof will follow several lemmas that determine the structure of various subgroups of $\mathrm{Aut}(M(F_4))$. We begin with a lemma that reduces matroid automorphisms to labeled graph automorphisms of Γ. Recall the partition of the ground set of $M(F_4)$ into A and B: $A = \{a_{ij}^{\pm}\}$ and $B = \{e_i, f_j, g_k\}$.

Lemma 3.1. *Let $\sigma \in \mathrm{Aut}(M(F_4))$ be an automorphism of $M(F_4)$, and let $b \in B$. If $\sigma(b) \in B$, then σ is an automorphism of Γ.*

Proof. We must show that σ respects the incidences of Γ, i.e., that σ maps A to A, and also permutes the three blocks E, F, G.

We may suppose $\sigma(e_1) \in B$ (the argument is identical for any element in B). Then, since σ preserves all flats of $M(F_4)$, we have $\{\sigma(e_1), \sigma(f_1), \sigma(g_1)\}$ is a 3-point line of $M(F_4)$. By Proposition 2.1, this forces both $\sigma(f_1) \in B$ and $\sigma(g_1) \in B$. Moreover, we can apply the same argument to the other three 3-point lines containing e_1 to get $\sigma(f_i) \in B$ and $\sigma(g_j) \in B$ for all $1 \leq i, j \leq 4$.

Now choosing $b = f_1$ (say) and repeating the above argument forces $e_i \in B$ for $i = 2, 3, 4$. Thus, the vertices of Γ are mapped to vertices by σ. For σ to be a labeled graph automorphism, we must also show that adjacent vertices of Γ are mapped to adjacent vertices. But, by Proposition 2.1, there are *no* 4-point lines in $M(F_4)$ which contain points corresponding to vertices in different components of Γ, i.e., there are no 4-point lines of the form $\{e_i, f_j, x, y\}$. Thus, if $\sigma(e_1) = f_i$ for some i, then $\sigma(e_2) = f_j$ for some $j \neq i$ since the 4-point line containing e_1 and e_2 is preserved by σ. This shows that σ permutes the three vertex blocks E, F, G.

Since we now have $\sigma(b) \in B$ for all $b \in B$, we know $\sigma(a) \in A$ for all $a \in A$ (since σ is a bijection). To finish the proof, we must show that all of the adjacencies are preserved in the induced edge mapping on Γ. But the vertex-edge incidence of Γ precisely matches the 4-point lines of the matroid, which σ must preserve.

Thus, σ is a labeled graph automorphism of Γ.

□

Since the 2-point lines and the rank 3 flats are also easy to recover from Γ, Lemma 3.1 shows that any permutation of the graph Γ gives a matroid automorphism. Thus, determination of matroid automorphisms essentially will be reduced to finding the automorphisms of the labeled graph Γ. We note that the hypothesis $\sigma(b) \in B$ for some $b \in B$ could be replaced by $\sigma(a) \in A$ for some $a \in A$. In fact, the proof of Lemma 3.1 shows any automorphism of the matroid either fixes the sets A and B or swaps them (although we have not yet shown that there are automorphisms that do swap these sets – see Lemma 3.6).

Lemma 3.2. *Let $H_1 \leq \mathrm{Aut}(M(F_4))$ be those automorphisms that map E to itself and preserve the signs of all $a \in A$. Then $H_1 \cong S_4$.*

Proof. First note that $S_4 \leq H_1$: If σ_i is the map that swaps e_1 and e_i (for $i > 1$), then each σ_i is a product of 7 transpositions:

$$\begin{aligned}
\sigma_2 &= (e_1e_2)(f_1f_2)(g_3g_4)(a_{13}^+a_{23}^+)(a_{13}^-a_{23}^-)(a_{14}^+a_{24}^+)(a_{14}^-a_{24}^-)\\
\sigma_3 &= (e_1e_3)(f_1f_3)(g_2g_4)(a_{12}^+a_{23}^+)(a_{12}^-a_{23}^-)(a_{14}^+a_{34}^+)(a_{14}^-a_{34}^-)\\
\sigma_4 &= (e_1e_4)(f_1f_4)(g_2g_3)(a_{12}^+a_{24}^+)(a_{12}^-a_{24}^-)(a_{13}^+a_{34}^+)(a_{13}^-a_{34}^-)
\end{aligned}$$

Thus each $\sigma_i \in H_1$ and it is clear $\{\sigma_2, \sigma_3, \sigma_4\}$ generate a group isomorphic to S_4.

Conversely, the action of any $\sigma \in H_1$ on E uniquely determines its action on all of A (this follows from the incidence structure of the E component of Γ since σ preserves the signs in A), which, in turn, uniquely determines its action on F and G (by using the incidence structure of the F and G components of Γ, resp.). Thus, each permutation of $\{e_1, \ldots, e_4\}$ determines a unique $\sigma \in H_1$. Thus, $H_1 \cong S_4$.

□

We now examine the subgroup of $\mathrm{Aut}(M(F_4))$ that allows sign changes in A. The proof is similar to the relevant portions of the proofs of Proposition 4.5(3) and Theorem 4.6(3) of [4].

Lemma 3.3. *Let $H_2 \leq \mathrm{Aut}(M(F_4))$ be those automorphisms σ that fix E pointwise, i.e., if $\sigma \in H_2$, then $\sigma(e_i) = e_i$ for $1 \leq i \leq 4$. Then $H_2 \cong \mathbb{Z}_2^3$.*

Proof. Since σ is the identity on E and 4-point lines are preserved, the only allowable permutations on A are those that swap a^+_{ij} and a^-_{ij}. In order to preserve the 3-point lines in the E component of Γ, we must preserve the sign parity of each cycle in the graph. Then Proposition 4.5(3) of [4] gives the following:

> Let S be a collection of edges in the E component of Γ which have been swapped $a^+_{ij} \leftrightarrow a^-_{ij}$ by some $\sigma \in H_2$. Then S forms a cutset in Γ.

Then H_2 is generated by the three elementary vertex cutsets, which we list below:

$$\begin{aligned}
\tau_1 &= (a^+_{12}a^-_{12})(a^+_{13}a^-_{13})(a^+_{14}a^-_{14})(f_1g_1)(f_2g_2)(f_3g_3)(f_4g_4)\\
\tau_2 &= (a^+_{12}a^-_{12})(a^+_{23}a^-_{23})(a^+_{24}a^-_{24})(f_1g_2)(f_2g_1)(f_3g_4)(f_4g_3)\\
\tau_3 &= (a^+_{13}a^-_{13})(a^+_{23}a^-_{23})(a^+_{34}a^-_{34})(f_1g_3)(f_2g_4)(f_3g_1)(f_4g_2)
\end{aligned}$$

New cutsets generated by these are formed by taking the symmetric difference of cutsets. This corresponds precisely to the group operation in the abelian group $\mathbb{Z}_2^3$.

□

Lemma 3.4. *Let H be the automorphisms that map A to A and B to B, and let H_3 be the subgroup of automorphisms that map E to itself. Then*

1. $H_3 \cong \mathbb{Z}_2^3 \rtimes S_4$;
2. $H_3 \lhd H$;
3. $H/H_3 \cong \mathbb{Z}_3$.

Proof.

1. H_3 is the automorphism group of the matroid associated with the 4-dimensional hypercube. It follows that $H_3 \cong \mathbb{Z}_2^3 \rtimes S_4$ by Theorem 4.6(3) of [4].

2. By part (1), we have $H_3 \cong H_2 \rtimes H_1$, the subgroups of Lemmas 3.2 and 3.3. We will show that conjugation of any element of H_3 remains in H_3 by conjugating the generators of H_1 and H_2. Now H_1 is generated by $\sigma_2, \sigma_3, \sigma_4$ from the proof of Lemma 3.2. Suppose π maps E to F (say); then $\pi\sigma_i\pi^{-1}$ is the permutation obtained from σ_i by applying the substitution rules from π. But σ_i fixes F and G (in addition to E), so $\pi\sigma_i\pi^{-1}$ must fix E (and F and G, as well).

   For H_2, which is generated by τ_1, τ_2, τ_3 from the proof of Lemma 3.3, note that $\pi\tau_i\pi^{-1}$ will fix E pointwise (since E is fixed pointwise by each τ_i).

3. Evidently, there are three cosets of H_3 in H: $H_3, \pi_f H_3$ and $\pi_g H_3$, where π_f is any automorphism mapping E to F and π_g is any automorphism mapping E to G.

□

Collecting the results of the preceding lemmas gives the next result.

Lemma 3.5. *Let H be the automorphisms that map A to A and B to B. Then*

$$H \cong \left(\mathbb{Z}_2^3 \rtimes S_4\right) \rtimes \mathbb{Z}_3.$$

There are automorphisms of $M(F_4)$ that swap the vectors of A and B. We give one such automorphism now.

Lemma 3.6. *Let σ be the following permutation of the 24 points of $M(F_4)$:*

$$\begin{aligned}\sigma = (e_1 a_{12}^+)(e_2 a_{12}^-)(e_3 a_{34}^+)(e_4 a_{34}^-)\ &(f_1 a_{23}^-)(f_2 a_{23}^+)(f_3 a_{14}^-)(f_4 a_{14}^+)\\ &(g_1 a_{13}^+)(g_2 a_{13}^-)(g_3 a_{24}^+)(g_4 a_{24}^-).\end{aligned}$$

Then $\sigma \in \mathrm{Aut}(M(F_4))$.

Proving the lemma involves showing that all of the 3- and 4-point lines are preserved by σ. For example, the 3-point line $\{e_3, f_2, g_4\}$ is mapped to the line $\{a_{34}^+, a_{23}^+, a_{24}^-\}$ and the 4-point line $\{e_2, e_3, a_{23}^+, a_{23}^-\}$ is mapped to $\{f_1, f_2, a_{12}^-, a_{34}^+\}$ by σ. The induced map swaps the 32 3-point lines (each of the 16 3-point lines of A is paired with a 3-point line of B), while, for the 4-point lines, σ swaps 14 lines and fixes the remaining four. We leave the remaining details (which are entirely routine) to the reader.

Before we determine the structure of the automorphism group $\mathrm{Aut}(M(F_4))$, we determine its size.

Lemma 3.7. $|\mathrm{Aut}(M(F_4))| = 1152$.

Proof. Combining Lemmas 3.5 and 3.6 allows us to map e_1 to any of the 24 elements of $M(F_4)$, i.e., $\mathrm{Aut}(M(F_4))$ is transitive. Let $\phi \in \mathrm{Aut}(M(F_4))$. Then $\phi(e_1), \phi(e_2), \phi(e_3)$ and $\phi(e_4)$ are in the same block of the partition $E \cup F \cup G \cup E' \cup F' \cup G'$, so we have 24 choices for $\phi(e_1)$, 3 choices for $\phi(e_2)$, 2 choices for $\phi(e_3)$ and one choice for $\phi(e_4)$. This gives a total of 192 automorphisms. (We can view this operation graphically by sending the E component of Γ to one of the six components of $\Gamma \cup \Gamma'$, then permuting $\{\phi(e_1), \phi(e_2), \phi(e_3), \phi(e_4)\}$ in 4! ways.)

Once ϕ has mapped the E block to some block of the partition, by Lemma 3.3, we have 8 possible ways to arrange the 12 elements which comprise the edges of that block (in either Γ or Γ'). These 8 choices do not affect $\phi(e_i)$ for $1 \le i \le 4$, so this gives a total of $192 \times 8 = 1152$ automorphisms.

It remains to show that the choices made now completely determine ϕ. But this follows from Lemmas 3.2 and 3.3.

$\square$

Theorem 3.8. $\mathrm{Aut}(M(F_4)) \cong ((\mathbb{Z}_2^3 \rtimes S_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Proof. Let $\phi \in \mathrm{Aut}(M_4))$. If ϕ maps A to A and B to B, then $\phi \in H$, where H is the subgroup of Lemma 3.5; if ϕ swaps the elements of A and B, then it is clear that $\phi = h\sigma$ for σ from Lemma 3.6 and for some $h \in H$, and this decomposition is unique. Thus H is a subgroup of $\mathrm{Aut}(M(F_4))$ of index 2, so $H \triangleleft G$, and $\mathrm{Aut}(M(F_4)) \cong H \rtimes \{e, \sigma\}$. The result now follows from Lemma 3.5.

$\square$

4 Geometry and matroid automorphisms

This section will develop the connections between the geometry and the matroid maps. In particular, we will interpret the matroid operations geometrically, including the map that swaps A and B in the matroid.

The Weyl or Coxeter groups corresponding to irreducible root systems are very well-studied, primarily because of their deep connections to Lie algebra. For the root system F_4, this group can be decomposed in different ways (which is typical when decomposition involves semidirect products). Following [6] (see p. 45), we have the following.

Theorem 4.1. *Let W be the Weyl group of symmetries of the root system F_4. Then*

$$W \cong \left(\mathbb{Z}_2^3 \rtimes S_4\right) \rtimes S_3.$$

The factor $\mathbb{Z}_2^3 \rtimes S_4$ can be interpreted as the symmetry group of the root system D_4, which consists of the vectors of the form $\pm B$, i.e., the vectors $\{\pm e_i, \pm f_j, \pm g_k\}$. It is possible to interpret this subgroup geometrically: $\mathbb{Z}_2^3 \rtimes S_4$ is the subgroup of the group of symmetries of a 4-dimensional hypercube (the *hyperoctahedral group*) formed by removing the reflecting hyperplanes with normal vectors $\pm e_i$ for $1 \leq i \leq 4$. The remaining factor of S_3 can be interpreted as the symmetries of the Coxeter-Dynkin diagram for F_4, which is a tree with 3 vertices adjacent to a single, central vertex.

Alternatively, we can understand the root system through the D_4 lattice (see [1]). The vectors that comprise this lattice can be written as quaternions, and the lattice is called the 'Hurwitz integral quaternions.' Each vector in the D_4 lattice has the form $a + bi + cj + dk$ where a, b, c, d are either all integers or all half-integers. This lattice is of special interest since it gives the densest lattice sphere-packing in 4 dimensions, where each sphere is centered at a lattice point and has 24 neighbors. To obtain D_4 from the root system F_4, take all integer linear combinations of the vectors in F_4. Much more information on this lattice and these connections can be found in [1].

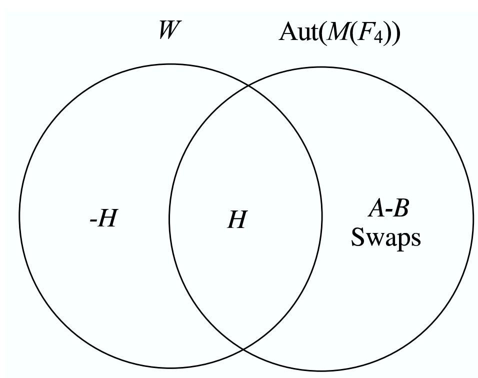

Figure 4: Schematic representation of Coxeter group W and $\mathrm{Aut}(M(F_4))$.

For our purposes, note that both W and $\mathrm{Aut}(M(F_4))$ have 1152 elements. Every isometry of W corresponds to a matroid automorphism; moreover, since $\pm w$ give rise to the same matroid automorphisms, we have a 2-to-1 map from W to $\mathrm{Aut}(M(F_4))$. This relationship is exactly the same as for the root systems B_n (see [4]) and H_3 (see [3]). We view these 576 matroid automorphisms as *geometric* operations.

It should be clear that the matroid automorphisms that correspond to geometric operations (i.e., those elements of $\mathrm{Aut}(M(F_4))$ in the image of the 2-to-1 map from W) are precisely those automorphisms that map A to A and B to B. These are the D_4-lattice preserving maps.

The 576 non-geometric matroid automorphisms correspond to swaps between the short and long vectors of F_4. There are no isometries of W that switch these vectors (since isometries preserve the length of the vectors), but the two sets of vectors have the same dependence structure. In fact, the vectors of A and B each form their own 24-cells, and the matroid maps that swap these two sets of vectors can be viewed as swapping the two 'dual' 24-cells. Alternatively, these maps swap two dual D_4-lattices, one based on $\pm A$ and the other on $\pm B$.

The relation between the two groups is given schematically in Figure 4. A similar relationship exists for the root system H_3. Finally, we remark that it would be interesting to extend this program of analyzing matroid automorphisms to the exceptional root systems E_6, E_7, and E_8.

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