Flexibility of Embeddings of Bouquets of Circles on the Projective Plane and Klein Bottle*

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Abstract

In this paper, we study the flexibility of embeddings of bouquets of circles on the projective plane and the Klein bottle. The numbers (of equivalence classes) of embeddings of bouquets of circles on these two nonorientable surfaces are obtained in explicit expressions. As their applications, the numbers (of isomorphism classes) of rooted one-vertex maps on these two nonorientable surfaces are deduced.

1 Introduction

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a polygon of even edges in the plane whose edges are pairwise identified and directed clockwise or counterclockwise. Such polygonal representations of surfaces can be also written by words. For example, the sphere is written as $O_0 = aa^-$ where a^- is paired with a, but with the opposite direction of a on the boundary of the polygon. The projective plane, the torus and the Klein bottle are represented respectively by aa, aba^-b^- and $aabb$. In general, $O_p = \prod_{i=1}^{p} a_ib_ia_i^-b_i^-$ and $N_q = \prod_{i=1}^{q} a_ia_i$ denote, respectively, a surface of orientable genus p and a surface of nonorientable genus q. Of course, N_1, O_1 and N_2 are, respectively, the projective plane, the torus and the Klein bottle. Every surface is homeomorphic to precisely one of the surface O_p $(p \geq 0)$, or N_q $(q \geq 1)$ [9,14]. Suppose $A = a_1a_2 \cdots a_t, t \geq 1$, is a word, then $A^- = a_t^- \cdots a_2^-a_1^-$ is called the *inverse* of A.

Let $\mathcal{S}$ be the collection of surfaces and let AB be a surface. The following topological transformations and their inverses do not change the orientability and genus of a surface:

TT 1: $Aaa^-B \Leftrightarrow AB$ where $a \notin AB$, TT 2: $AabBab \Leftrightarrow AcBc$ where $c \notin AB$ and

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TT 3: $AB \Leftrightarrow (Aa)(a^{-}B)$ where $AB \neq \emptyset$.

Notice that A and B are both linear orders of letters and permitted to be empty. The parentheses stand for cyclic order when more than one cyclic orders occur, for distinguishing from one to another. In fact, what is determined under these operations is just a topological equivalence $\sim$ on $\mathcal{S}$.

The following relations can be deduced by using TT 1-3, as shown in, e.g.,[9].

Relation 1: $(AxByCx^{-}Dy^{-}) \sim ((ADCB)(xyx^{-}y^{-}))$,
Relation 2: $(AxBx) \sim ((AB^{-})(xx))$,
Relation 3: $(Axxyzy^{-}z^{-}) \sim ((A)(xx)(yy)(zz))$.

In these three relations, A, B, C, and D are all linear orders of letters and permitted to be empty. Parentheses are always omitted when unnecessary to distinguish cyclic or linear order.

An *embedding* of a graph G into a surface S is a homeomorphism $h : G \to S$ of G into S such that every component of $S - h(G)$ is a 2-cell. Two embeddings $h : G \to S$ and $g : G \to S$ of G into a surface S are said to be *equivalent* if there is a homeomorphism $f : S \to S$ such that $f \circ h = g$.

Given a graph G, how many nonequivalent embeddings of G are there into a given surface. This is one of important problems in embedding flexibility, *i.e*, the classification problem [16]. Research on this problem can be tracked back to [3] and some results have been obtained, such as [1-2,4-8,15-18,20-21,23] etc.

In the following, we will introduce the joint tree model of a graph embedding, which was established in [9] by Liu, based on his initial work in [10]. Some works have been done based on the joint tree method [5,20-21,23].

Given a spanning tree T of a graph G, for $1 \leq i \leq \beta$, we split each cotree edge e_i into two semi-edges and label them by the same letter as a_i where β is the betti number of G. The resulting graph is a tree consisting of tree edges in T and 2β semi-edges. We denote this new tree by $\hat{T}$. Then indexing the 2β semi-edges of $\hat{T}$ by $+$(always omitted) or $-$, so that the indices of each pair of semi-edges labelled with same letter can be the same or distinct. A *rotation* at a vertex v, denoted by σ_v, is a cyclic permutation of edges incident with v. Let $\sigma_G = \prod_{v \in V(G)} \sigma_v$ be a rotation system of G.

The tree $\hat{T}$ with an index of each semi-edge and a rotation system of G is called a *joint tree* of G. Denote by $\hat{T}^{\delta}_{\sigma}$ the joint tree, in which $\delta = (\delta_1, \delta_2, \cdots, \delta_\beta)$ be a binary vector, δ_i can be 0 or 1 where $\delta_i = 0$ means that the two indices of a_i are distinct; otherwise, the same. In fact, the edge a_i with $\delta_i = 1$ are the *twisted* edge [7] in the embedding. By reading these lettered semi-edges with indices of a $\hat{T}^{\delta}_{\sigma}$ in a fixed orientation (clockwise or counterclockwise), we can get an algebraic representation for a surface. It is a cyclic order of 2β letters with indices. Such a surface is called an *associated surface* [11] of G. If two associate surfaces of G have the same cyclic order with the same δ in their algebraic representations, then we say that they are the same; otherwise, distinct.

From [11], there is a 1-to-1 correspondence between associate surfaces and embeddings of a graph, hence an embedding of a graph on a surface can be represented by an associate surface of it.

Let G be a graph, $g_p(G), \tilde{g}_q(G), p \geq 0, q \geq 1$ be the numbers (of equivalence classes) of embeddings of G on the surface of orientable genus p and nonorientable genus q, respectively.

Denote the bouquet of n circle by B_n. Gross et al obtain the genus distribution of B_n [4], Kwak and Lee obtained the total embedding polynomial of B_{n+1} from that of B_n inductively and derive the total genus distribution of B_{n+1} from it in [7], but we can't get the numbers of embeddings of B_n on nonorientable surfaces from their results easily and directly. In this paper, by joint tree model, we study the flexibility of embeddings of B_n on the projective plane and the Klein bottle. The numbers of embeddings of B_n on these two nonorientable surface are obtained in explicit expressions. Then the structures of those embeddings are described. As their applications, the numbers (of isomorphism classes) of rooted one-vertex maps on these two nonorientable surfaces are deduced.

2 Some Lemmas

Lemma 2.1[11] *An orientable surface S is a surface of orientable genus 0 if and only if there is no form as $AxByCx^{-1}Dy^{-1}$ in it.*

Lemma 2.2 *Let S be a nonorientable surface, if there is a form as $AxByCx^{-}Dy^{-}$ in S, then the genus of S will be not less than 3; if there is a form as $AxByCx^{-}Dy$ or $AxByCyDx$ in S, then the genus of S will be not less than 2.*

Proof: If the form as $AxByCx^{-}Dy^{-}$ exists in S, by Relation 1, $AxByCx^{-}Dy^{-}$ $\sim ADCBxyx^{-}y^{-}$, and there is at least one pair of semi-edges z, z^{ε} with same indices, because S is nonorientable. By Relation 1-3, we can get $S \sim A'zzxyx^{-}y^{-} \sim A'zzxxyy$. So the genus will be not less than 3.

If the form as $AxByCx^{-}Dy$ exists, by using Relation 2 twice, we get that $AxByCx^{-}Dy \sim AxBD^{-}xC^{-}yy \sim ADB^{-}C^{-}xxyy$, so the genus of S will be not less than 2. In the same way, we can get that the genus of S where the form $AxByCyDx$ exists will be not less than 2. Thus the proof is complete. □

The following lemma can be got from both [4] and [9] easily.

Lemma 2.3[4,9] *The number of embeddings of B_n on the sphere is*

$$g_0(B_n) = (n-1)!2^{n-1}\frac{\binom{2n}{n}}{n+1}.$$

The *Catalan sequence* is the sequence $C_0, C_1, C_2, \ldots, C_n, \ldots$ where $C_n = \frac{\binom{2n}{n}}{n+1} (n = 0, 1, 2, \ldots)$ is the nth *Catalan number*. Let $F(x) = \sum_{n\geq 0} C_n x^n$ be the generating function of $\{C_n\}_{n\geq 0}$, then $F(x) = \frac{1-\sqrt{1-4x}}{2x}$ [19].

Lemma 2.4 *For integers $n \geq k \geq 0$, we have*

$$\sum_{\substack{x_1+\cdots+x_{2k}=n-k \\ x_j\geq 0, 1\leq j\leq 2k}} \prod_{j=1}^{2k} \frac{\binom{2x_j}{x_j}}{(x_j+1)} = \frac{2k(2n-1)!}{(n-k)!(n+k)!}.$$

Proof: For $F(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum\limits_{n=0}^{\infty} C_n x^n$, from [19],

$$F^m(x) = \sum_{n=0}^{\infty} \frac{m(2n+m-1)!}{n!(n+m)!} x^n, m \geq 1, \tag{1}$$

and

$$F^m(x) = (\sum_{n=0}^{\infty} C_n x^n)^m = \sum_{n=0}^{\infty} (\sum_{\substack{x_1+\cdots+x_m=n \\ x_j\geq 0, 1\leq j\leq m}} \prod_{j=1}^{m} C_{x_j}) x^n, \tag{2}$$

from (1) and (2), we can get that

$$\sum_{\substack{x_1+\cdots+x_m=n \\ x_j\geq 0, 1\leq j\leq m}} \prod_{j=1}^{m} C_{x_j} = \frac{m(2n+m-1)!}{n!(n+m)!}.$$

Let $m = 2k, n = n-k$, the lemma follows. □

3 The number of embeddings of bouquets of circles on the projective plane

Let S be a surface. If $x, y \in S$ are in the form as $S = AxByCxDy$, then they are said to be *interlaced*; otherwise, *parallel.*

Theorem 3.1 *All the associate surfaces of B_n on the projective plane have the form as*

$$A_1 a_1 \cdots A_k a_k A_{k+1} a_1 \cdots A_{2k} a_k,\ 1 \leq k \leq n$$

in which A_i is either empty or $A_i \sim O_0, 1 \leq i \leq 2k$.

Proof: Suppose that there are k $(1 \leq k \leq n)$ twisted edges $a_1, \ldots, a_k$ in the embedding of B_n on the projective plane. When $k \geq 2$, each pair of $a_i, a_j, i \neq j$ must be interlaced in the associate surface, otherwise its nonorientable genus will be greater than one, according to Relation 2. So the associate surfaces have the form as $A_1 a_1 \cdots A_k a_k A_{k+1} a_1 \cdots A_{2k} a_k$, it still holds when $k = 1$. For $2(n-k)$ semi-edges corresponding to the $n-k$ untwisted edges in $A_i, 1 \leq i \leq 2k$, we have $\forall x \in A_i, x^- \in A_i$, by Lemma 2.2. And according to Relation 2,

$$A_1 a_1 \cdots A_k a_k A_{k+1} a_1 \cdots A_{2k} a_k \sim A_1 A_{k+1}^- a_k^- A_k^- \cdots a_2^- A_2^- A_{k+2} a_2 \cdots A_{2k} a_k a_1 a_1.$$

$$
\begin{aligned}
& A_1 a_1 \cdots A_k a_k A_{k+1} a_1 \cdots A_{k+k} a_k \sim N_1 \\
\Leftrightarrow \quad & A_1 A_{k+1}^- a_k^- A_k^- \cdots a_2^- A_2^- A_{k+2} a_2 \cdots A_{2k} a_k \sim O_0 \\
\Leftrightarrow \quad & A_i \sim O_0, 1 \le i \le 2k.
\end{aligned}
$$

Thus the theorem is obtained. □

The structure of the embeddings of B_n on the projective plane is shown in Fig.1.

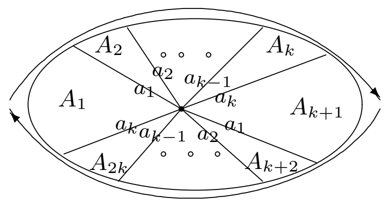

Fig.1 B_n on the projective plane

Theorem 3.2 *The number of the embeddings of B_n on the projective plane is*

$$\tilde{g}_1(B_n) = (n-1)!2^{n-1}\left(2^{2n-1} - \frac{(2n)!}{2(n!)^2}\right).$$

Proof: By the joint tree method, there are $\binom{n}{k}(k-1)!2^{k-1}$ ways to choose and place the k, $1 \le k \le n$ twisted edges, corresponding to $2k$ semi-edges $a_1, a_1, \ldots, a_k, a_k$, in the associate surface. Suppose that there are $2x_j$ semi-edges in A_j, $1 \le j \le 2k$, then each of the $2x_j$ semi-edges can be the first semi-edge in A_j, *i.e.*, each of the $2x_j$ letters can be the first letter in the liner order of letters A_j. According to Theorem 3.1 and Lemmas 2.3, 2.4, the number of ways to put $2(n-k)$ semi-edges, corresponding to $n-k$ untwisted edges, into $A_1, \ldots, A_{2k}$ is

$$
\begin{aligned}
& \sum_{\substack{x_1+\cdots+x_{2k}=n-k \\ x_j \ge 0, 1 \le j \le 2k}} \frac{(n-k)!}{x_1! x_2! \cdots x_{2k}!} \prod_{j=1}^{2k} \left(2x_j g_0(B_{x_j})\right) \\
= \quad & 2^{n-k}(n-k)! \sum_{\substack{x_1+\cdots+x_{2k}=n-k \\ x_j \ge 0, 1 \le j \le 2k}} \prod_{j=1}^{2k} \frac{\binom{2x_j}{x_j}}{(x_j+1)} \\
= \quad & \frac{2^{n-k+1} k(2n-1)!}{(n+k)!}.
\end{aligned}
$$

For $1 \le k \le n$, the number of embeddings of B_n on the projective plane is

$$\sum_{k=1}^{n} \binom{n}{k} (k-1)!2^{k-1} \frac{2^{n-k+1} k(2n-1)!}{(n+k)!} \quad = \quad (n-1)!2^{n-1} \sum_{k=1}^{n} \binom{2n}{n-k},$$

by simplification, the theorem is obtained. □

4 The number of embeddings of bouquets of circles on the Klein bottle

There are at least 4 edges in the polygonal representations of the Klein bottle. And there are only two ways, *i.e.*, $aabb$ and $abab^{-}$. From those words, the associate surfaces of B_n on the Klein bottle can be classified into two cases, according to the places of the twisted edges. Case 1, all the twisted edges are of the form

$$A_1a_1\cdots A_ia_iA_{i+1}a_1\cdots A_{2i}a_iD_1b_1\cdots D_jb_jD_{j+1}b_1\cdots D_{2j}b_j,\ i,j\geq 1, i+j\leq n;$$

Case 2, all the twisted edges are of the form

$$A_1a_1\cdots A_ka_kA_{k+1}a_1\cdots A_{2k}a_k,\ 1\leq k\leq n-1;$$

in which $A_1,\ldots,A_{2i},D_1,\ldots,D_{2j}$ can be empty.

Theorem 4.1 *The associate surfaces in Case* 1 *are Klein bottles if and only if one of the following two conditions holds:*
(1) $A_1D_1\sim O_0,\ A_t\sim D_k\sim O_0,\ t\in[2,2i],\ k\in[2,2j]$;
(2) $A_{i+1}D_{j+1}\sim O_0,\ A_t\sim D_k\sim O_0,\ t\in[1,i]\cup[i+2,2i],\ k\in[1,j]\cup[j+2,2j]$.

Proof: By using Relation 2 twice,

$$\begin{aligned}&A_1a_1\cdots A_ia_iA_{i+1}a_1\cdots A_{2i}a_iD_1b_1\cdots D_jb_jD_{j+1}b_1\cdots D_{2j}b_j\\ \sim\ &A_1a_1\cdots A_{i-1}a_{i-1}A_iA_{2i}^{-1}a_{i-1}^{-1}A_{2i-1}^{-1}\cdots a_1^{-1}A_{i+1}^{-1}\\ &D_1b_1\cdots D_{j-1}b_{j-1}D_jD_{2j}^{-1}b_{j-1}^{-1}D_{2j-1}^{-1}\cdots b_1^{-1}D_{j+1}^{-1}a_ia_ib_jb_j\end{aligned}\qquad(*)$$

$$\begin{aligned}(*)\sim N_2\ \Leftrightarrow\ &A_1a_1\cdots A_{i-1}a_{i-1}A_iA_{2i}^{-1}a_{i-1}^{-1}A_{2i-1}^{-1}\cdots a_1^{-1}A_{i+1}^{-1}\\ &D_1b_1\cdots D_{j-1}b_{j-1}D_jD_{2j}^{-1}b_{j-1}^{-1}D_{2j-1}^{-1}\cdots b_1^{-1}D_{j+1}^{-1}\sim O_0\\ \Leftrightarrow\ &A_iA_{2i}^{-1}\sim\cdots\sim A_2A_{i+2}^{-1}\sim D_jD_{2j}^{-1}\sim\cdots\sim D_2D_{j+2}^{-1}\\ &\sim A_1A_{i+1}^{-1}D_1D_{j+1}^{-1}\sim O_0\end{aligned}$$

Because the semi-edges in $A_t,B_k,1\leq t\leq 2i,1\leq k\leq 2j$, are all corresponding to untwisted edges, according to Lemma 2.1,

$$\forall x\in A_t, x^{-1}\in A_t,\ t\neq 1,i+1,\quad \forall y\in D_k, y^{-1}\in D_k,\ k\neq 1,j+1.$$

$$A_t\sim D_k\sim O_0,\ t\neq 1,i+1,k\neq 1,j+1.$$

For the same reason,

$$\forall x\in A_1, x^{-1}\in A_1 \text{ or } x^{-1}\in D_1;\quad \forall y\in A_{i+1}, y^{-1}\in A_{i+1} \text{ or } y^{-1}\in D_{j+1}.$$

If $\forall x\in A_1, x^{-1}\in A_1$ and $\forall y\in D_1, y^{-1}\in D_1$, then

$$A_1A_{i+1}^{-1}D_1D_{j+1}^{-1}\sim O_0\Leftrightarrow A_1\sim D_1\sim O_0, A_{i+1}D_{j+1}\sim O_0.$$

The condition 2 holds.
If $\exists x \in A_1, x^{-1} \in D_1$, or $\exists y \in D_1, y^{-1} \in A_1$, then $\forall x \in A_{i+1}, x^{-1} \in A_{i+1}$, $\forall y \in D_{j+1}$, $y^{-1} \in D_{j+1}$, according to Lemma 2.1. Hence we have that

$$A_1 A_{i+1}^{-1} D_1 D_{j+1}^{-1} \sim O_0 \Leftrightarrow A_1 D_1 \sim O_0, A_{i+1} \sim D_{j+1} \sim O_0.$$

The condition 1 holds. If $A_1 = \emptyset$, then $D_1 \sim A_{i+1}^{-} D_{j+1}^{-} \sim O_0$, the condition 2 holds. Above all, we get the conclusion. $\square$

Theorem 4.2 *The associate surfaces in Case* 2 *are Klein bottles if and only if one of the following* k *conditions holds:* $A_i A_{k+i}^{-1} \sim N_1, A_j \sim O_0, j \neq i, i+k, 1 \leq i \leq k$.

Proof: By Relation 2,

$$\begin{aligned} & A_1 a_1 \cdots A_k a_k A_{k+1} a_1 \cdots A_{k+k} a_k \sim N_2 \\ \Leftrightarrow \quad & A_1 a_1 \cdots A_{k-1} a_{k-1} A_k A_{2k}^{-1} a_{k-1}^{-1} A_{2k-1}^{-1} \cdots a_1^{-1} A_{k+1}^{-1} \sim N_1, \end{aligned}$$

according to Lemma 2.2, the theorem follows. $\square$

The structures of the embeddings of B_n on the Klein bottle in Cases 1 and 2 are shown in Fig.2 and Fig3, respectively. Fig.2 describes the condition (1) in Theorem 4.1, Fig.3 describes the condition $A_1 A_{k+1}^{-} \sim N_1, A_j \sim O_0, j \neq 1, 1+k$, in Theorem 4.2.

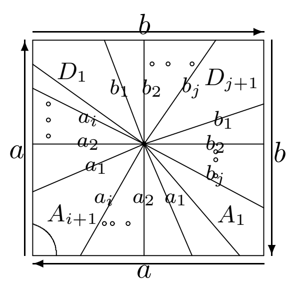

Fig.2 B_n on the Klein bottle in Case 1

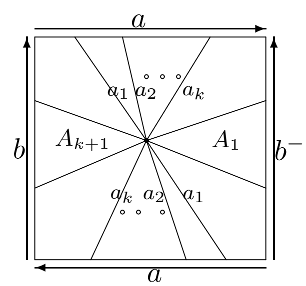

Fig.3 B_n on the Klein bottle in Case 2

For convenience, we write $\sum\limits_{\substack{x_1+\cdots+x_{2k}=n-k \\ x_j \geq 0, 1 \leq j \leq 2k}} \prod\limits_{j=1}^{2k} \frac{\binom{2x_j}{x_j}}{(x_j+1)}$ as $G(n,k)$ in the following.

Theorem 4.3 *The number of embeddings of* B_n *on the Klein bottle in Case* 1 *is*

$$\tilde{g}_2^1(B_n) = (n-1)! 2^{n-1} \sum_{t=1}^{n} t(t-1)^2 \binom{2n}{n-t}.$$

Proof: For the symmetry, we suppose that $i \leq j$. For the convenience of calculation, we subdivide the two conditions in Theorem 4.1 into there conditions as follows:
a) $A_t \sim D_k \sim O_0,\ 1 \leq t \leq 2i, 1 \leq k \leq 2j$.

b) $A_1D_1 \sim O_0, A_t \sim D_k \sim O_0,\ t,k \neq 1.$
c) $A_{i+1}D_{j+1} \sim O_0, A_t \sim D_k \sim O_0,\ t \neq i+1, k \neq j+1.$
We discuss the three conditions respectively, as follows.

a) If $j > i$, the number of ways to choose and place $b_1, \cdots, b_j$ is $\binom{n}{j}(j-1)!2^{j-1}$, then the number of ways to choose and place $a_1, \cdots, a_i$ is $\binom{n-j}{i}i!2^i2j$, similar with the argument in the proof of Theorem 3.2, the number of ways of embeddings of the $n-i-j$ untwisted edges is $2^{n-j-i}(n-j-i)!G(n,j+i)$. Hence, for the given i and j, the number of embeddings is $2^n\binom{n}{j}(j-1)!\binom{n-j}{i}i!j(n-j-i)!G(n,j+i) = n!2^nG(n,j+i)$.
For $1 \leq i \leq \lfloor\frac{n}{2}\rfloor, i+1 \leq j \leq n-i$, the number of embeddings of B_n on the Klein bottle is

$$n!2^n\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=i+1}^{n-i}G(n,j+i) = n!2^n\sum_{t=3}^{n}(\lceil\frac{t}{2}\rceil-1)G(n,t).$$

If $j = i$, for the symmetry, the number of embeddings is $\frac{1}{2}n!2^n\sum\limits_{i=1}^{\lfloor\frac{n}{2}\rfloor}G(n,2i)$.
So the number of embeddings of B_n on the Klein bottle in condition a) is

$$\begin{aligned}\tilde{g}_2^{1a}(B_n) &= n!2^n\Big(\sum_{t=3}^{n}(\lceil\frac{t}{2}\rceil-1)G(n,t)+\frac{1}{2}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}G(n,2i)\Big)\\ &= n!2^{n-1}\sum_{t=2}^{n}(t-1)G(n,t).\end{aligned}$$

b) Let $M = \{r \mid r \in A_1, r^{-1} \in D_1\}, m = |M| \geq 1$. The argument is similar with that of condition a). If $j > i$, the number of embeddings is

$$\begin{aligned}& n!2^n\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=i+1}^{n-i-1}\sum_{m=1}^{n-j-i}G(n,j+i+m)\\ =\ & n!2^n\sum_{t=4}^{n}\sum_{k=3}^{t-1}(\lceil\frac{k}{2}\rceil-t-3)G(n,t)\\ =\ & n!2^n\sum_{t=4}^{n}(\lceil\frac{t}{2}\rceil-1)(\lfloor\frac{t}{2}\rfloor-1)G(n,t).\end{aligned}$$

If $j = i$, the number of embeddings is

$$\frac{1}{2}n!2^n\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=1}^{n-2i}G(n,2i+m) = \frac{1}{2}n!2^n\sum_{t=3}^{n}(\lceil\frac{t}{2}\rceil-1)G(n,t).$$

So the number of embeddings of B_n on the Klein bottle in condition $b)$ is

$$\begin{aligned}\tilde{g}_2^{1b}(B_n) &= n!2^n\sum_{t=3}^{n}\Big((\lceil\frac{t}{2}\rceil-1)(\lfloor\frac{t}{2}\rfloor-1)+\frac{1}{2}(\lceil\frac{t}{2}\rceil-1)\Big)G(n,t)\\ &= n!2^{n-2}\sum_{t=3}^{n}(t-1)(t-2)G(n,t).\end{aligned}$$

For the symmetry, the number of embeddings of B_n on the Klein bottle in condition $c)$ is $\tilde{g}_2^{1c}(B_n)=\tilde{g}_2^{1b}(B_n)$.

So the number of embeddings of B_n on the Klein bottle in Case 1 is

$$\begin{aligned}\tilde{g}_2^{1}(B_n) &= \tilde{g}_2^{1a}(B_n)+\tilde{g}_2^{1b}(B_n)+\tilde{g}_2^{1c}(B_n)\\ &= n!2^{n-1}\sum_{t=1}^{n}(t-1)^2G(n,t)=(n-1)!2^{n-1}\sum_{t=1}^{n}t(t-1)^2\binom{2n}{n-t},\end{aligned}$$

the theorem is obtained. □

Theorem 4.4 *The number of embeddings of B_n on the Klein bottle in Case 2 is*

$$\tilde{g}_2^{2}(B_n)=(n-1)!2^{n-1}\sum_{t=1}^{n}t(t-1)\binom{2n}{n-t}.$$

Proof: The number of ways to choose and place $a_1,\cdots,a_k$ is $\binom{n}{k}(k-1)!2^{k-1}$, then choose one of the k conditions in Theorem 4.2. Suppose we choose $A_kA_{2k}^{-1}\sim N_1$, then $A_kA_{2k}^{-1}$ has the form as $\tilde{A}_1t_1\cdots\tilde{A}_{|\Gamma|}t_{|\Gamma|}\tilde{A}_{|\Gamma|+1}t_1\cdots\tilde{A}_{2|\Gamma|}t_{|\Gamma|}$, according to Theorem 3.1, in which $\Gamma=\{t\mid t\in A_k \text{ and } t^{-1}\in A_{2k}\}$. Let $m=|\Gamma|$, then $1\le m\le n-k$, the number of ways to place the m untwisted edges is $\binom{n-k}{m}m!2^m$. And the number of ways to put the $2(n-k-m)$ semi-edges, corresponding to the left $n-k-m$ untwisted edges, into $A_j,\tilde{A}_i$ such that $A_j\sim\tilde{A}_i\sim O_0, j\in[1,k-1]\cup[k+1,2k-1], 1\le i\le 2m$ is $2^{n-k-m}(n-k-m)!G(n,k+m)$. Hence, the number of embeddings is

$$n!2^{n-1}\sum_{k=1}^{n-1}\sum_{m=1}^{n-k}G(n,k+m)=n!2^{n-1}\sum_{t=2}^{n}(t-1)G(n,t)=(n-1)!2^{n-1}\sum_{t=1}^{n}t(t-1)\binom{2n}{n-t},$$

the theorem is obtained. □

Theorem 4.5 *The number of embeddings of B_n on the Klein bottle is*

$$\tilde{g}_2(B_n)=n!2^{n-1}\left(\frac{(2n-1)!}{\big((n-1)!\big)^2}-4^{n-1}\right)$$

Proof: The number of embeddings of B_n on the Klein bottle is $\tilde{g}_2(B_n) = \tilde{g}_2^1(B_n) + \tilde{g}_2^2(B_n)$, according to Theorems 4.3, 4.4, $\tilde{g}_2(B_n) = (n-1)!2^{n-1} \sum_{t=1}^{n} t^2(t-1)\binom{2n}{n-t}$. By calculation and certain simplification, one can get that

$$\sum_{t=1}^{n} t^2(t-1)\binom{2n}{n-t} = n\left(\frac{(2n-1)!}{\left((n-1)!\right)^2} - 4^{n-1}\right),$$

thus the theorem follows. $\square$

Examples

According to Theorems 3.2, 4.5, one can calculate the numbers of embeddings of B_n on the projective plane and Klein bottle easily, give these numbers $\tilde{g}_1(B_n), \tilde{g}_2(B_n), 1 \le n \le 8$ as follows:

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
$\tilde{g}_1(B_n)$	1	10	176	4464	148224	6090240	298414080	16987944960
$\tilde{g}_2(B_n)$	0	8	336	14592	718080	40273920	2553384960	181129052160

Note that the numbers in the table are coincident with those in Table 1 given in [7] by Kwak and Lee.

5 Applications

A *map* is an embedding of a graph on a surface, the graph is called the *underlying graph* of the map. The enumeration of maps on surfaces have been developed and deepened by people, based on the initial works by W.T.Tutte in the 1960s. The reader is referred to the monograph [12] for further background about enumerative theory of maps. For a given graph Γ, the relations between its genus distribution of rooted maps and genus distribution of embeddings on orientable and nonorientable surfaces are given in [13]. We adopt the same notations as that in [13], where $r[\Gamma](x)(\tilde{r}[\Gamma](x))$ and $g[\Gamma](x)(\tilde{g}[\Gamma](x))$ denote rooted orientable (nonorientable) map polynomial on genus and orientable (nonorientable) genus polynomial of Γ, respectively.

Theorem 5.1[13] *For a connected graph* Γ

$$|Aut_{\frac{1}{2}}\Gamma| r[\Gamma](x) = 2\varepsilon(\Gamma) g[\Gamma](x)$$

and

$$|Aut_{\frac{1}{2}}\Gamma| \tilde{r}[\Gamma](x) = 2\varepsilon(\Gamma) \tilde{g}[\Gamma](x)$$

where $Aut_{\frac{1}{2}}\Gamma$ *and* $\varepsilon(\Gamma)$ *denote the semi-arc automorphism group and the size of* Γ.

From [13], $|Aut_{\frac{1}{2}}B_n| = 2^n n!$, by Theorems 3.2, 4.5 and 5.1, one can get the number (of isomorphism classes) $\tilde{r}_i(B_n)$ of rooted maps with the underlying graph B_n on the nonorientable surface of genus i $(i = 1, 2)$.

Corollary 5.1 *The number of rooted maps with the underlying graph B_n on the projective plane is*

$$\tilde{r}_1(B_n) = 2^{2n-1} - \frac{(2n)!}{2(n!)^2}.$$

Corollary 5.2 *The number of rooted maps with the underlying graph B_n on the Klein bottle is*

$$\tilde{r}_2(B_n) = n\left(\frac{(2n-1)!}{\big((n-1)!\big)^2} - 4^{n-1}\right).$$

The rooted maps with the underlying graph B_n is also called one-vertex maps. In [22], the numbers of one-vertex maps on the projective plane and the Klein bottle are also obtained, by enumerating functional equation. The result in Corollary 5.1 is the same to that in [22], and the result in Corollary 5.2 is in much simpler form than that in [22].

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