On the Spectrum of the Derangement Graph

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Submitted: Apr 10, 2007; Accepted: Nov 1, 2007; Published: Nov 28, 2007
Mathematics Subject Classifications: 05C25, 05C50, 05E05, 05E10

Abstract

We derive several interesting formulae for the eigenvalues of the derangement graph and use them to settle affirmatively a conjecture of Ku regarding the least eigenvalue.

Keywords: Cayley Graph, Least Eigenvalue, Derangement Graph, Symmetric Group, Symmetric Function Theory, Complete Symmetric Factorial Functions, Shifted Schur Functions

1 Introduction

Let G be a finite group and $S \subseteq G$ a symmetric subset of generators ($s \in S \Rightarrow s^{-1} \in S$) satisfying $1 \notin S$. The *Cayley graph* $\Gamma(G, S)$ has the elements of G as its vertices, and two elements $u, v \in G$ are joined by an edge provided $vu^{-1} = s$ for some $s \in S$. [1] It is clear that $\Gamma(G, S)$ is regular of vertex degree $|S|$. Let S_n be the symmetric group of permutations of $X = \{1, 2, \ldots, n\}$, and let $\mathcal{D}_n := \{\sigma \in S_n : \sigma(x) \neq x, \forall x \in X\}$ denote

*I would like to thank Cheng Ku for bringing his conjecture to my attention and for many stimulating discussions, David Wales for pointing out a discrepancy in an earlier draft, Rick Wilson and the mathematics department of the California Institute of Technology for their kind hospitality, and the referee for suggesting several improvements in the exposition.

[1] The condition $1 \notin S$ precludes loops, while the symmetry condition allows us to consider the graph as being undirected.

the *derangements* on X, namely the set of fixed point free permutations of S_n. (Note that $\mathcal{D}_n$ is symmetric in the above sense, as the inverse of a derangement is a derangement.) We call $\Gamma_n := \Gamma(S_n, \mathcal{D}_n)$ the *derangement graph* on X. Much is known about this graph:

- Γ_n is connected ($n > 3$). This follows because every permutation can be written as the product of adjacent transpositions $(k, k+1)$, and these, in turn, can be expressed as the product of the two derangements $(1, 2, \ldots, n)^2$ and $(n, n-1, \ldots, 1)^2(k, k+1)$. (If $n = 3$ this fails because the product of two odd permutations is even.) Thus, for $n > 3$ the derangements generate S_n, which means that every vertex of Γ_n can be reached from the identity. To ensure Γ_n is connected we therefore assume $n \geq 4$ in all that follows.

- Γ_n is Hamiltonian. This was first observed by Eggleton and Wallis [7] and subsequently by others (see, *e.g.,* [18]).

- $\alpha(\Gamma_n) = (n-1)!$, where α is the independence number. This was first proved by Deza and Frankl [5], who also observed that the bound is achieved by a coset of the stabilizer of a point. Cameron and Ku [3] (and, independently, Larose and Malvenuto [12]) showed that these are the only such maximum independent sets. (See also [9].)

- $\omega(\Gamma_n) = n$, where ω is the clique number, because a maximum clique in Γ_n is just a Latin square of size n.

- $\chi(\Gamma_n) = n$, where χ is the chromatic number. This follows from a result of Godsil. We say a Cayley graph $\Gamma(G, S)$ is *normal* if S is closed under conjugation. Godsil shows ([8], Corollary 7.1.3) that for any normal Cayley graph, $\chi(\Gamma) = \omega(\Gamma)$ if $\alpha(\Gamma)\omega(\Gamma) = |V(\Gamma)|$, where $|V(\Gamma)|$ is the number of vertices of Γ. The generating set $\mathcal{D}_n$ of Γ_n is a union of conjugacy classes, because a derangement is just a permutation with no cycles of length one, and cycle type is preserved under conjugation. Γ_n has $n!$ vertices, so the claim follows.

Now recall that, for any regular graph of degree k with N vertices, the independence number satisfies the Delsarte-Hoffman bound

$$\alpha \leq N\frac{-\eta}{k-\eta}, \tag{1.1}$$

where η is the least eigenvalue of the adjacency matrix of the graph. Graphs in which equality holds have several interesting properties. For example, in such graphs we have equality between α and the Shannon capacity of the graph. (For an extensive discussion of the Delsarte-Hoffman bound and its implications, see [14].) For the derangement graph $N = n!$ and $k = D_n := |\mathcal{D}_n|$, so we get

$$\eta \leq \frac{-D_n}{n-1}. \tag{1.2}$$

This prompted Cheng Ku to make the following

Conjecture 1.1. [See, *e.g.,* [11]] *The least eigenvalue of the adjacency matrix of the derangement graph is given by*

$$\eta = \frac{-D_n}{n-1}.$$

The main objective of this work is to provide several interesting formulae for the eigenvalues of the derangement graph and to prove Conjecture 1.1. We begin by recalling a result due to Diaconis and Shahshahani on the eigenvalues of a normal Cayley graph. This leads us to a discussion of the characters of the symmetric group. Using a result of Stanley we relate these to symmetric function theory and compute a rough bound on the eigenvalues. We then employ the factorial symmetric functions of Chen and Louck (which are related to the shifted symmetric functions of Okounkov and Olshanski) to derive a remarkable recurrence formula for the eigenvalues of Γ_n. This is the critical tool we need to prove the conjecture.

2 The Standard Representation

The following theorem for the eigenvalues of a normal Cayley graph is due to Diaconis and Shahshahani [6] (for an earlier related result, see [1]; for the version below, see [17]):

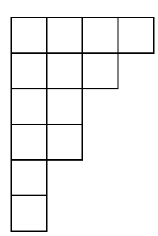
Theorem 2.1. *Let A be the adjacency matrix of a normal Cayley graph $\Gamma(G,S)$. Then the eigenvalues of A are given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Recall that a *partition* λ of n, written $\lambda \vdash n$ or $|\lambda| = n$, is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\sum_i \lambda_i = n$. Its *length* is ℓ and each λ_i is a *part* of the partition. Partitions are represented by Ferrers diagrams:

$$(4,3,2,2,1,1) \qquad \longleftrightarrow$$



and are also written using multiplicity notation

$$(4,3,2,2,1,1) \qquad \longleftrightarrow \qquad 4^1 3^1 2^2 1^2.$$

As is well known (see, *e.g.,* [10, 13, 20]) the irreducible characters χ_λ of S_n are indexed by partitions $\lambda \vdash n$. Also, as the cycle type of a permutation of S_n is just the partition whose parts are the cycle lengths, the conjugacy classes of S_n are also labeled by partitions.

The *standard representation* of S_n corresponding to the partition $\lambda = (n-1, 1)$ plays an important role in the sequel. It is constructed as follows. Let V be an n dimensional inner product space with orthonormal basis $(e_1, e_2, \ldots, e_n)$. Then S_n acts on V by permuting each vector

$$\sigma(e_i) = e_{\sigma(i)}$$

and extending by linearity. One says that V *affords* the defining representation of S_n. It is clear that S_n leaves fixed the one dimensional subspace U generated by the vector $\sum_i e_i$, so U affords the trivial representation of S_n (which is clearly irreducible). The orthogonal complement $W = U^\perp$ also affords an irreducible representation of dimension $n-1$, namely the standard representation, and we have the equivariant decomposition

$$V = U \oplus W.$$

As characters are additive on direct sums, it follows that

$$\chi_W = \chi_V - \chi_U.$$

$\chi_V(\sigma)$ just counts the number of fixed points of σ, so

$$\chi_W(\sigma) = \#\{\text{fixed points of } \sigma\} - 1. \tag{2.1}$$

From (2.1) and Theorem 2.1 the eigenvalue of the derangement graph corresponding to the standard representation W is thus

$$\eta_W = \frac{1}{\chi_W(1)} \sum_{\sigma \in \mathcal{D}_n} \chi_W(\sigma) = \frac{-D_n}{n-1}. \tag{2.2}$$

This is precisely the conjectured least eigenvalue.

In the table below we illustrate the truth of Conjecture 1.1 for the derangement graph Γ_6 by summing the characters of S_6 over the derangements. Notice that the standard representation yields the least eigenvalue ($\eta_{5^1 1^1} = -53$). [2]

[2] The first half of the table is taken from [10] with corrections for the minor typographical errors therein. The second half is obtained by pointwise multiplication of the entries in the first half by the alternating character.

Class	1^6	$2^1 1^4$	$2^2 1^2$	2^3	$3^1 1^3$	$3^1 2^1 1^1$	3^2	$4^1 1^2$	$4^1 2^1$	$5^1 1^1$	6^1	η_λ
# Elts	1	15	45	15	40	120	40	90	90	144	120	
6^1	1	1	1	1	1	1	1	1	1	1	1	+265
$5^1 1^1$	5	3	1	-1	2	0	-1	1	-1	0	-1	-53
$4^1 2^1$	9	3	1	3	0	0	0	-1	1	-1	0	+15
$4^1 1^2$	10	2	-2	-2	1	-1	1	0	0	0	1	+13
3^2	5	1	1	-3	-1	1	2	-1	-1	0	0	-11
$3^1 2^1 1^1$	16	0	0	0	-2	0	-2	0	0	1	0	-5
$3^1 1^3$	10	-2	-2	2	1	1	1	0	0	0	-1	-5
2^3	5	-1	1	3	-1	-1	2	1	-1	0	0	+7
$2^2 1^2$	9	-3	1	-3	0	0	0	1	1	-1	0	+5
$2^1 1^4$	5	-3	1	1	2	0	-1	-1	-1	0	1	+1
1^6	1	-1	1	-1	1	-1	1	-1	1	1	-1	-5

3 The Eigenvalues of the Derangement Graph

Next we derive an explicit formula for the eigenvalues of the derangement graph. We assume the reader has some familiarity with symmetric function theory, but for completeness we recall a few facts here (for more details, see *e.g.,* [10, 13, 20]). Consider the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ of all polynomial functions in n variables over the integers. The symmetric group acts by permuting variables, and the invariant polynomials form the ring of symmetric functions

$$\Lambda_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}.$$

There are many bases for Λ_n. In what follows we will use two: the complete (homogeneous) symmetric functions and the Schur functions. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the *complete symmetric function* h_λ is defined by

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n},$$

where

$$h_k := \sum_{i_1+i_2+\cdots+i_n=k} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

and $i_j \in \mathbb{Z}_{\geq 0}$ for $j = 1, \ldots, n$.

There are many equivalent ways to define the Schur functions. The combinatorial definition is as follows. A *semistandard Young tableau* of shape λ is a Ferrers diagram of λ in which the boxes are filled with numbers that weakly increase across rows and strictly increase down columns. For example,

$$(4,3,2,2,1,1) \qquad \longleftrightarrow \qquad \begin{array}{llll} 1 & 1 & 3 & 3 \\ 2 & 3 & 4 & \\ 3 & 4 & & \\ 6 & 6 & & \\ 7 & & & \\ 8 & & & \end{array} .$$

The *type* of T is a vector giving the multiplicities of each entry in the tableau. In the above example, $\text{type}(T) = (2,1,4,2,0,2,1,1)$. Associated to each tableau is the monomial denoted x^T, defined by raising each variable to its corresponding entry in the type vector. For the above example

$$x^T = x_1^2 x_2 x_3^4 x_4^2 x_6^2 x_7 x_8.$$

A semistandard Young tableau T is *standard* if $\text{type}(T) = \overbrace{(1,1,\ldots,1)}^{n \text{ times}}$, which means that it is filled with the numbers from 1 to $|\lambda|$.

This construction admits a slight generalization. Let $\nu \subseteq \lambda$ (*i.e.*, $\nu_i \leq \lambda_i$ for all i). A *skew* semistandard Young tableau of shape λ/ν and type α is obtained by subtracting the boxes of the Ferrers shape of ν from those of λ and filling in the boxes as before. For example, if $\lambda = (4,3,2,2,1,1)$, $\nu = (3,2,1)$, and $\alpha = (1,0,2,1,1,1,1)$, one such tableau is

$$\begin{array}{llll} \bullet & \bullet & \bullet & \square \\ \bullet & \bullet & \square & \\ \bullet & \square & & \\ \square & \square & & \\ \square & & & \\ \square & & & \end{array} \qquad \longrightarrow \qquad \begin{array}{llll} & & & 7 \\ & & 4 & \\ & 1 & & \\ 3 & 3 & & \\ 5 & & & \\ 6 & & & \end{array} .$$

The tableau monomials are defined as before. Then the *skew Schur function* of shape λ/ν is

$$s_{\lambda/\nu}(x_1, x_2, \ldots, x_n) = \sum_T x^T,$$

where the sum extends over all skew semistandard Young tableau of shape λ/ν. Although it is not obvious from this definition, $s_{\lambda/\nu}$ is a symmetric function (see, *e.g.*, [20], Theorem 7.10.2, p. 311). If $\nu = \emptyset$ then s_λ is the *Schur function* of shape λ.

The canonical (or Hall) inner product on Λ_n can be defined by the requirement that the Schur functions comprise an orthonormal basis:

$$(s_\lambda, s_\nu) = \delta_{\lambda,\nu}.$$

It can be shown ([20], Eq. 7.61) that

$$(s_\lambda, h_\nu) = K_{\lambda,\nu},$$

where $K_{\lambda,\nu}$ is the *Kostka number*, namely the number of semistandard Young tableau of shape λ and type ν.

Following Stanley we define

$$d_\lambda := \sum_{s\in\mathcal{D}_n} \chi_\lambda(s), \tag{3.1}$$

where χ_λ is the irreducible character of the symmetric group corresponding to the partition λ. Stanley shows that this function admits a nice expansion in terms of Schur functions:

Theorem 3.1. [[20], Exercise 7.63, p. 519]

$$\sum_{\lambda\vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n}(-1)^{n-k}(n)_k h_{k1^{n-k}}, \tag{3.2}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ *is the falling factorial function.*

Taking inner products of both sides of (3.2) with s_λ gives

$$d_\lambda = \sum_{k=0}^{n}(-1)^{n-k}(n)_k K_{\lambda,k1^{n-k}}. \tag{3.3}$$

It is clear that $K_{\lambda,k1^{n-k}} = f^{\lambda/k}$ where $f^{\lambda/\mu}$ is the number of standard Young tableau of skew shape λ/μ, because the type $k1^{n-k}$ means that there are k ones in the Young diagram, and these are necessarily all in the top row. The remaining entries must all be distinct. By (3.1) and Theorem 2.1 the eigenvalues of the derangment graph can be written

$$\eta_\lambda := \frac{d_\lambda}{f^\lambda} \tag{3.4}$$

because (see, *e.g.,* [20], Equation 7.79) the dimension $\chi_\lambda(1)$ of the irreducible representation corresponding to the irreducible character χ_λ is simply the number of standard Young tableau of shape λ. Hence we get

Theorem 3.2. *The eigenvalues of the derangement graph are given by*

$$\eta_\lambda = \sum_{k=0}^{n}(-1)^{n-k}(n)_k \frac{f^{\lambda/k}}{f^\lambda}.$$

A more explicit formula for η_λ can be obtained by using Frobenius' formula for the number of standard Young tableau of skew shape ([20], Cor. 7.16.3, p. 344):

$$f^{\lambda/\mu} = |\lambda/\mu|!\ \det\left(\frac{1}{(\lambda_i-\mu_j-i+j)!}\right)_{i,j=1}^{n}, \tag{3.5}$$

where $\lambda \vdash n$ (and $1/x! = 0$ if $x < 0$). The number of skew standard Young tableau of shape λ/k is thus

$$f^{\lambda/k} = (n-k)! \begin{vmatrix} \frac{1}{(\lambda_1-k)!} & \frac{1}{(\lambda_1+1)!} & \frac{1}{(\lambda_1+2)!} & \cdots & \frac{1}{(\lambda_1+\ell-1)!} \\ \frac{1}{(\lambda_2-k-1)!} & \frac{1}{\lambda_2!} & \frac{1}{(\lambda_2+1)!} & \cdots & \frac{1}{(\lambda_2+\ell-2)!} \\ \frac{1}{(\lambda_3-k-2)!} & \frac{1}{(\lambda_3-1)!} & \frac{1}{\lambda_3!} & \cdots & \frac{1}{(\lambda_3+\ell-3)!} \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell\times\ell}, \tag{3.6}$$

where ℓ is the length of λ. Following the usual convention we define the *shifted partition* μ associated to λ

$$\mu_i := \lambda_i + \ell - i. \tag{3.7}$$

In terms of μ we can write

$$f^{\lambda/k} = (n-k)! \begin{vmatrix} \frac{1}{(\mu_1-\ell-k+1)!} & \frac{1}{(\mu_1-\ell+2)!} & \frac{1}{(\mu_1-\ell+3)!} & \cdots & \frac{1}{\mu_1!} \\ \frac{1}{(\mu_2-\ell-k+1)!} & \frac{1}{(\mu_2-\ell+2)!} & \frac{1}{(\mu_2-\ell+3)!} & \cdots & \frac{1}{\mu_2!} \\ \frac{1}{(\mu_3-\ell-k+1)!} & \frac{1}{(\mu_3-\ell+2)!} & \frac{1}{(\mu_3-\ell+3)!} & \cdots & \frac{1}{\mu_3!} \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell\times\ell}. \tag{3.8}$$

Factoring out the terms in the last column gives

$$f^{\lambda/k} = \frac{(n-k)!}{\prod_i \mu_i!} \begin{vmatrix} (\mu_1)_{\ell+k-1} & (\mu_1)_{\ell-2} & (\mu_1)_{\ell-3} & \cdots & 1 \\ (\mu_2)_{\ell+k-1} & (\mu_2)_{\ell-2} & (\mu_2)_{\ell-3} & \cdots & 1 \\ (\mu_3)_{\ell+k-1} & (\mu_3)_{\ell-2} & (\mu_3)_{\ell-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell\times\ell}. \tag{3.9}$$

$(x)_n$ is a monic polynomial of degree n in x, so using column operations on the last $\ell - 1$ columns we get

$$f^{\lambda/k} = \frac{(n-k)!}{\prod_i \mu_i!}|M(\mu)|, \tag{3.10}$$

where

$$M(\mu) := \begin{pmatrix} (\mu_1)_{\ell+k-1} & \mu_1^{\ell-2} & \mu_1^{\ell-3} & \cdots & 1 \\ (\mu_2)_{\ell+k-1} & \mu_2^{\ell-2} & \mu_2^{\ell-3} & \cdots & 1 \\ (\mu_3)_{\ell+k-1} & \mu_3^{\ell-2} & \mu_3^{\ell-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}_{\ell\times\ell}. \tag{3.11}$$

Combining Theorem 3.2, Equations (3.10) and (3.11), and the well-known *degree formula* (*cf.*, [10], (11.6))

$$f^{\lambda} = \frac{n!}{\prod_i \mu_i!}\prod_{i<j}(\mu_i - \mu_j) \tag{3.12}$$

yields

Theorem 3.3. *The eigenvalues of the derangement graph are given by*

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} \frac{|M(\mu)|}{\prod_{i<j}(\mu_i - \mu_j)}. \tag{3.13}$$

4 Derangement Numbers and an Approximation Scheme

One approach to evaluating (3.13) is to sum the determinants. To this end, for all $m \geq 0$ we define a *shifted derangement number*

$$b(r;m) := \sum_{k=0}^{r} (-1)^{r-k} (r)_{k+m}. \tag{4.1}$$

The ordinary derangement number D_r is $b(r;0)$ (see, *e.g.*, [21], p. 67).

Lemma 4.1. *The derangement numbers satsify the following properties:*

i. The first six derangement numbers are $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, *and* $D_5 = 44$.

ii. $D_n = [n!/e]$, *where* $[x]$ *is the nearest integer to* x. *In particular, the derangement numbers are monotonic increasing for* $n \geq 1$.

iii. For $n \geq 1$ *the derangement numbers satisfy the following recursions:*

$$D_n = nD_{n-1} + (-1)^n \tag{4.2}$$

and

$$D_n = (n-1)(D_{n-1} + D_{n-2}). \tag{4.3}$$

Proof. These properties all follow easily from the definition of D_n. □

Theorem 4.2. *The eigenvalues of the derangement graph are given by*

$$\eta_\lambda = \frac{(-1)^n}{\prod_{i<j}(\mu_i - \mu_j)} \begin{vmatrix} (-1)^{\mu_1} b(\mu_1; \ell-1) & \mu_1^{\ell-2} & \mu_1^{\ell-3} & \cdots & 1 \\ (-1)^{\mu_2} b(\mu_2; \ell-1) & \mu_2^{\ell-2} & \mu_2^{\ell-3} & \cdots & 1 \\ (-1)^{\mu_3} b(\mu_3; \ell-1) & \mu_3^{\ell-2} & \mu_3^{\ell-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell\times\ell}, \tag{4.4}$$

where μ *is defined as in* (3.7).

Proof. This follows immediately from (3.13), (4.1), and the multilinearity of the determinant. □

We can use Theorem 4.4 to approximate η_λ. First we need

Lemma 4.3. *The shifted derangement number satisfies*

$$b(r;m) = (-1)^m (r)_m D_{r-m}. \tag{4.5}$$

Proof. Use (4.1) and the fact that

$$(r)_{k+m} = \frac{r!}{(r-m-k)!} = \frac{r!}{(r-m)!}\frac{(r-m)!}{(r-m-k)!} = (r)_m (r-m)_k.$$

□

From Lemma 4.1 and Lemma 4.3 we get

$$b(r;m) = (-1)^m \frac{r!}{(r-m)!} D_{r-m} \approx (-1)^m \frac{r!}{e}, \tag{4.6}$$

whence

$$\eta_\lambda \approx \frac{(-1)^{n+\ell-1}}{e\prod_{i<j}(\mu_i - \mu_j)} \begin{vmatrix} (-1)^{\mu_1}\mu_1! & \mu_1^{\ell-2} & \mu_1^{\ell-3} & \cdots & 1 \\ (-1)^{\mu_2}\mu_2! & \mu_2^{\ell-2} & \mu_2^{\ell-3} & \cdots & 1 \\ (-1)^{\mu_3}\mu_3! & \mu_3^{\ell-2} & \mu_3^{\ell-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell\times\ell}. \tag{4.7}$$

When μ_1 is much greater than $\mu_2, \mu_3, \ldots$ the first term in the Laplace expansion of the determinant by the first column dominates the other terms, and we have

$$\begin{aligned} \eta_\lambda &\approx \frac{(-1)^{n+l-1}}{e}(-1)^{\mu_1}\frac{\mu_1!}{(\mu_1-\mu_2)(\mu_1-\mu_3)\cdots(\mu_1-\mu_l)} && (4.8)\\ &\approx \frac{1}{e}(-1)^{n+l-1+\mu_1}\frac{\mu_1!}{\mu_1^l}. && (4.9)\end{aligned}$$

This gives a rough estimate for the eigenvalues when $\lambda_1 \gg \lambda_2$.

5 Complete Factorial Symmetric Functions

Instead of summing the determinants, we investigate the structure of the summands more carefully. Begin again with (3.13). It follows from results of Chen and Louck [4] that the summands can be expressed in terms of what they call *complete factorial symmetric functions* $w_k(z_1, z_2, \ldots, z_\ell)$. [3] We recall some of their results here. The key is their Lemma 2.1 (which they attribute to Verde-Star [22]):

[3]In [4] Chen and Louck also define what they call *factorial Schur functions*, which were generalized by Macdonald [13], and subsequently generalized further by Okounkov and Olshanski [16] to what they called *shifted Schur functions*. The summands are special cases of shifted Schur functions. We will point out some connections of our results to those of Okounkov and Olshanski as we proceed.

Lemma 5.1. *The divided difference of the falling factorial function is*

$$\frac{(x)_{m+1} - (y)_{m+1}}{x-y} = \sum_{0 \le k \le m} (x)_k (y-k-1)_{m-k}.$$

Using this lemma iteratively gives ([4], Proposition 2.2 and Equation 2.5)

$$\frac{1}{\prod_{1 \le i < j \le \ell}(z_i - z_j)} \begin{vmatrix} (z_1)_{\ell+k-1} & z_1^{\ell-2} & z_1^{\ell-3} & \cdots & 1 \\ (z_2)_{\ell+k-1} & z_2^{\ell-2} & z_2^{\ell-3} & \cdots & 1 \\ (z_3)_{\ell+k-1} & z_3^{\ell-2} & z_3^{\ell-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \end{vmatrix}_{\ell \times \ell} = w_k(z_1, z_2, \ldots, z_\ell), \tag{5.1}$$

where

$$w_k(z_1, z_2, \ldots, z_\ell) = \sum_{i_1+i_2+\cdots+i_\ell = k} \prod_{1 \le j \le \ell} (z_j - i_1 - i_2 - \cdots - i_{j-1} - j + 1)_{i_j}, \tag{5.2}$$

or, equivalently,

$$w_k(z_1, z_2, \ldots, z_\ell) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le \ell} (z_{i_1} - i_1 + 1)(z_{i_2} - i_2)(z_{i_3} - i_3 - 1) \cdots (z_{i_k} - i_k - k + 2). \tag{5.3}$$

Observe that these functions, while symmetric, are inhomogeneous. [4, 5] They are related to the ordinary complete symmetric functions by

$$w_k(z_1, z_2, \ldots, z_\ell) = h_k(z_1, z_2, \ldots, z_\ell) + \text{lower order terms}.$$

Of particular importance for our purposes are the following two properties of the complete factorial symmetric functions. The first is a *stabilization property*: [6]

Theorem 5.2.

$$w_k(z_1, z_2, \ldots, z_{\ell-1}, 0) = w_k(z_1, z_2, \ldots, z_{\ell-1}) \tag{5.4}$$

Proof. This follows directly from (5.1) by factoring out the product $z_1 z_2 \ldots z_{\ell-1}$ from both the numerator and denominator. □

The second is a *recurrence property*:

Theorem 5.3. [[4], Proposition 2.4]

$$w_k(z_1, z_2, \ldots, z_\ell) = w_k(z_2 - 1, z_3 - 1, \ldots, z_\ell - 1) + z_1 w_{k-1}(z_1 - 1, z_2 - 1, \ldots, z_\ell - 1). \tag{5.5}$$

We will use both of these results in the next section.

[4] The symmetry follows immediately from (5.1) and the well known fact that the quotient of an alternating polynomial by the Vandermonde determinant is a symmetric polynomial.

[5] Okounkov and Olshanski also define what they refer to as *complete shifted functions* $h_k^*(x_1, x_2, \ldots, x_\ell)$, but these are neither symmetric nor homogeneous. They become symmetric when reexpressed in terms of the shifted variables $z_i = x_i - i + \text{constant}$. By picking the constant judiciously we get $h_k^*(x_1, x_2, \ldots, x_\ell) = w_k(z_1, z_2, \ldots, z_\ell)$. Notice that the relationship between the two sets of variables is precisely the relationship between λ and μ (*cf.*, (3.7)).

[6] This result is implicit in the work of [4] and also follows from the corresponding stabilization property of the shifted Schur functions given in [16].

6 A Recurrence Relation for the Eigenvalues of the Derangement Graph

Comparing Theorem 3.3 and (5.1) we obtain

Theorem 6.1. *The eigenvalues of the derangement graph are given by*

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} w_k(\mu_1, \mu_2, \ldots, \mu_\ell), \tag{6.1}$$

where μ is the shifted partition associated to λ (defined in (3.7)*) and $n = |\lambda|$.*

Remark 6.2. *It follows from Theorem 6.1 that the spectrum of the derangement graph is* integral*. This result can also be shown directly from Theorem 2.1 by observing that central characters are algebraic integers (see,* e.g.*, [10] (3.2) or [19], Corollary 1 of Proposition 16, p. 52) and that the characters of the symmetric group are integral valued.*

From Theorem 3.2 and Theorem 6.1 we obtain the useful identification [7]

$$w_k(\mu_1, \mu_2, \ldots, \mu_\ell) = (n)_k \frac{f^{\lambda/k}}{f^\lambda}. \tag{6.2}$$

Next we combine Theorem 5.3 and Theorem 6.1 to derive a powerful recurrence relation for the eigenvalues of the derangement graph. To this end we observe that the w_ks satisfy a *vanishing property*: [8]

Theorem 6.3. $w_k(\mu_1, \mu_2, \ldots, \mu_\ell) = 0$ *whenever $k > \lambda_1$.*

Proof. From (3.7), $\mu_i = \lambda_i + \ell - i$. Thus, if $k > \lambda_1$ then $k + \ell - 1 > \lambda_1 + \ell - 1 = \mu_1$. But, by construction, the parts λ_i are weakly decreasing, so the parts μ_i are strongly decreasing. Hence $k + \ell - 1 > \mu_i$ for all i, and the entire first column of the matrix in the numerator in (5.1) (with $z_i = \mu_i$) vanishes. □

Remark 6.4. *In the light of* (6.2) *the vanishing theorem merely formalizes the intuitive idea that we cannot remove more than λ_1 boxes from the first row of a Young tableau.*

To state the main result of this section we need some terminology. To any tableau of shape λ we may assign xy-coordinates to each of the boxes by defining the upper-left-most box to be $(1, 1)$, with the x axis increasing to the right and the y axis increasing downwards. Then the *hook* through the box (x, y) is the union of the boxes (x', y) and (x, y'), where $x' \geq x$ and $y' \geq y$. We will call the hook through the box $(1, 1)$ the *principal hook* (of λ). By abuse of notation, we let h denote either the principal hook itself or its cardinality. (Its meaning will be clear from context.) We also let $\lambda - h$ denote the partition

[7] This is a special case of Theorem 8.1 in [16].

[8] This is a special case of the vanishing theorem of Okounkov and Olshanski ([16], Theorem 3.1).

obtained from λ by removing the principal hook. We call the first column of the tableau of shape λ the *principal ladder* (of λ), and define $\lambda - 1$ as the partition obtained from λ by removing its principal ladder. (The latter notation makes more sense if one thinks of $\lambda - 1$ as the vector subtraction $(\lambda_1, \lambda_2, \ldots, \lambda_\ell) - (1, 1, \ldots, 1)$.) By successively removing the principal hooks and ladders of λ we obtain the key result.

Theorem 6.5. *For any partition λ the eigenvalues of the derangement graph satisfy the following recurrence:*

$$\eta_\lambda = (-1)^h \left(\eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1}\right) \tag{6.3}$$

with initial condition $\eta_\emptyset = 1$.

Proof. By Theorem 6.1 and the vanishing theorem we can write

$$\eta'_\lambda = \sum_{k=0}^{\infty} (-1)^k w_k(\mu_1, \mu_2, \ldots, \mu_\ell), \tag{6.4}$$

where

$$\eta'_\lambda := (-1)^{|\lambda|} \eta_\lambda. \tag{6.5}$$

Using (5.5) in (6.4) yields

$$\eta'_\lambda = \sum_{k=0}^{\infty} (-1)^k \left[w_k(\mu_2 - 1, \mu_3 - 1, \ldots, \mu_\ell - 1) + \mu_1 w_{k-1}(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_\ell - 1)\right]. \tag{6.6}$$

From (3.7) we know that $\mu_i = \lambda_i + \ell - i$, so subtracting one from each part of μ is the same as subtracting one from each part of λ. Thus, the partition $(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_\ell - 1)$ is associated to the partition $\lambda - 1$. Similarly, the partition $(\mu_2 - 1, \mu_3 - 1, \ldots, \mu_\ell - 1)$ is associated to the partition $\lambda - h$. The second term in (6.6) can be written

$$-\mu_1 \sum_{k=0}^{\infty} (-1)^{k-1} w_{k-1}(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_\ell - 1) = -\mu_1 \sum_{k=0}^{\infty} (-1)^k w_k(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_\ell - 1), \tag{6.7}$$

because by convention $w_{-1} = 0$. Thus (6.6) becomes

$$\eta'_\lambda = \eta'_{\lambda-h} - \mu_1 \eta'_{\lambda-1}. \tag{6.8}$$

Now, $\mu_1 = \lambda_1 + \ell - 1 = h$. Moreover, $|\lambda| = n$ implies that $|\lambda - h| = n - h$ and $|\lambda - 1| = n - \ell$. Hence, from (6.5) and (6.8)

$$\begin{aligned} (-1)^n \eta_\lambda &= (-1)^{n-h} \eta_{\lambda-h} - (-1)^{n-\ell} h \eta_{\lambda-1} && (6.9) \\ &= (-1)^{n-h} \left(\eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1}\right), && (6.10) \end{aligned}$$

from which the stated recurrence follows. Finally, Equation (6.1) and the fact that $w_0 = 1$ gives $\eta_\emptyset = 1$. (Note that we have implicitly used the stabilization property of complete factorial symmetric functions (Theorem 5.2).) $\square$

Theorem 6.5 allows us to compute the eigenvalues of the derangement graph corresponding to certain simple shapes. For example, for hook shapes we have

Corollary 6.6. *Let $\lambda = j1^{n-j}$ denote the hook shape with first part j and remaining parts 1. Then for $1 \leq j \leq n$,*

$$\eta_{j1^{n-j}} = (-1)^n \left(1 + (-1)^j n D_{j-1}\right). \tag{6.11}$$

Proof. From Theorem 6.5

$$\eta_{j1^{n-j}} = (-1)^n \left(\eta_\emptyset + (-1)^j \cdot n \cdot \eta_{j-1}\right). \tag{6.12}$$

Furthermore, either from the recurrence itself or directly from Theorem 2.1 we see that $\eta_r = \sum_{\sigma \in \mathcal{D}_r} 1 = D_r$. □

Remark 6.7. *This result also follows with a little work from the properties of derangement numbers given in Lemma 4.1 and a result of Okazaki ([15]; see [20], Exercise 7.63, p. 469).*

7 Proof of the Conjecture

Theorem 7.1. *Conjecture 1.1 holds for all λ with $n \geq 4$, and, moreover, for $n \geq 5$ the standard representation gives the unique minimum eigenvalue.*

The proof is divided into four parts. First we show that the maximum eigenvalue is obtained by the trivial representation. Next, we show the conjecture holds for hooks, then that it holds for all *near hooks*, that is, partitions of the form $\lambda = j21^{n-j-2}$ where $2 \leq j \leq n-2$. Finally, we show that it holds for all other partitions. First note that, by direct computation using Theorem 6.5, we have, for $n = 4$: $\eta_{1^4} = -3$, $\eta_{21^2} = 1$, $\eta_{2^2} = 3$, $\eta_{31} = -3$, $\eta_4 = 9$. Clearly, in this case the least eigenvalue is achieved by the standard representation $\lambda = 31$, but it is not unique, as the minimum is also realized by the alternating representation.

Lemma 7.2. *The maximum eigenvalue of the derangement graph Γ_n is $\eta_n = D_n$.*

Proof. This can be shown using Theorem 6.5, but it follows most easily from the fact that k is the largest eigenvalue of any regular graph of degree k. (See, *e.g.*, [2], Proposition 3.1.) □

Lemma 7.3. *Conjecture 1.1 holds for hooks.*

Proof. From (6.11)

$$\eta_{(n-1,1)} = (-1)^n \left(1 + (-1)^{n-1} n D_{n-2}\right) = (-1)^n - n D_{n-2}, \tag{7.1}$$

so

$$
\begin{aligned}
\alpha &:= \eta_{j1^{n-j}} - \eta_{(n-1,1)} = (-1)^n \left(1 + (-1)^j n D_{j-1}\right) - (-1)^n + n D_{n-2} \\
&= n \left(D_{n-2} + (-1)^{n+j} D_{j-1}\right).
\end{aligned}
$$

We must show that $\alpha > 0$ for $j < n-1$. (By Lemma 7.2 we already know the trivial representation corresponding to $j = n$ is positive–indeed, it is the largest eigenvalue.) If $n+j$ is even then $\alpha > 0$ and there is nothing to prove, so assume $n+j$ is odd. As $n \geq 5$, $D_{n-2} > D_{j-1}$ for all $1 \leq j \leq n-2$ by Lemma 4.1 (ii), whence it follows that $\alpha > 0$. □

Lemma 7.4. *Conjecture 1.1 holds for near hooks.*

Proof. Applying Theorem 6.5 twice gives

$$
\begin{aligned}
\eta_{j21^{n-j-2}} &= (-1)^{n-1} \left(\eta_1 + (-1)^j (n-1) \eta_{(j-1,1)}\right) \\
&= (-1)^{n+j-1} (n-1) \eta_{(j-1,1)} \\
&= (-1)^{n-1} (n-1) \left(1 + (-1)^{j-1} j D_{j-2}\right) \\
&= (n-1) \left((-1)^{n-1} + (-1)^{n+j} j D_{j-2}\right). \qquad (7.2)
\end{aligned}
$$

Thus, from (7.1) and (7.2),

$$
\begin{aligned}
\beta &:= \eta_{j21^{n-j-2}} - \eta_{(n-1,1)} \\
&= (n-1) \left((-1)^{n-1} + (-1)^{n+j} j D_{j-2}\right) + n D_{n-2} + (-1)^{n-1} \\
&= n D_{n-2} + (-1)^{n+j} (n-1) j D_{j-2} + (-1)^{n-1} n. \qquad (7.3)
\end{aligned}
$$

We must show $\beta > 0$. But, regardless of the parity of $n+j$,

$$
\beta \geq n D_{n-2} - (n-1)(n-2) D_{n-4} + (-1)^{n-1} n, \qquad (7.4)
$$

because $D_{j-2} \leq D_{n-4}$ for $j \leq n-2$ and $n \geq 5$ by Lemma 4.1 (ii). Using Lemma 4.1 (iii) we can write

$$
D_{n-2} = (n-2) D_{n-3} + (-1)^{n-2} = (n-2)(n-3) D_{n-4} + (n-2)(-1)^{n-3} + (-1)^{n-2}, \quad (7.5)
$$

so

$$
\begin{aligned}
\beta &\geq [n(n-2)(n-3) - (n-1)(n-2)] D_{n-4} + n(n-2)(-1)^{n-1} \\
&= [n-2] \left[(n^2 - 4n + 1) D_{n-4} + (-1)^{n-1} n\right] > 0, \qquad (7.6)
\end{aligned}
$$

because both terms in square brackets are positive for $n \geq 5$. □

Proof (of Theorem 7.1). We have already verified the result in the case $n = 4$, so assume $n \geq 5$. Now let $\lambda \vdash n$. By Lemmas 7.3 and 7.4 we may assume λ is neither a hook (in which case we would have $n = h$), nor a near hook (in which case we would have

$n = h+1$). So we may safely assume $n \geq h+2$ and $h > \ell \geq 2$. Then we get the following chain of equalities and inequalities:

$$\begin{aligned}
|\eta_\lambda| &= |\eta_{\lambda-h} + (-1)^{\lambda_1} h\eta_{\lambda-1}| && \text{(Theorem 6.5)}\\
&\leq |\eta_{\lambda-h}| + h|\eta_{\lambda-1}| \\
&\leq D_{n-h} + hD_{n-\ell} && \text{(Lemma 7.2)}\\
&< (1+h)D_{n-\ell} && \text{(Lemma 4.1 (ii) and } h > \ell)\\
&\leq (n-1)D_{n-\ell} \\
&\leq (n-1)D_{n-2} \\
&= D_{n-1} + (-1)^n && \text{(Lemma 4.1 (iii))}\\
&\leq D_{n-1} + 1 \\
&< D_{n-1} + D_{n-2} \\
&= \frac{D_n}{n-1} && \text{(Lemma 4.1 (iii))}\\
&= |\eta_{(n-1,1)}| && \text{(Equation (2.2)).}
\end{aligned}$$

□

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