Bijective counting of tree-rooted maps and shuffles of parenthesis systems

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Abstract

The number of tree-rooted maps, that is, rooted planar maps with a distinguished spanning tree, of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$ where $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^th Catalan number. We present a (long awaited) simple bijection which explains this result. Then, we prove that our bijection is isomorphic to a former recursive construction on shuffles of parenthesis systems due to Cori, Dulucq and Viennot.

1 Introduction

In the late sixties, Mullin published an enumerative result concerning planar maps on which a spanning tree is distinguished [3]. He proved that the number of rooted planar maps with a distinguished spanning tree, or *tree-rooted maps* for short, of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$ where $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. This means that tree-rooted maps of size n are in one-to-one correspondence with pairs of plane trees of size n and $n+1$ respectively. But although Mullin asked for a bijective explanation of this result, no natural mapping was found between tree-rooted maps and pairs of trees. Twenty years later, Cori, Dulucq and Viennot exhibited one such mapping while working on Baxter permutations [1]. More precisely, they established a bijection between pairs of trees and *shuffles of two parenthesis systems*, that is, words on the alphabet $a, \overline{a}, b, \overline{b}$, such that the subword consisting of the letters $a, \overline{a}$ and the subword consisting of the letters $b, \overline{b}$ are parenthesis systems. It is known that tree-rooted maps are in one-to-one correspondence with shuffles of two parenthesis systems [3, 6], hence the bijection of Cori *et al.* somehow answers Mullin's question. But this answer is quite unsatisfying in the world of maps. Indeed, the bijection of Cori *et al.* is recursively defined on the set of prefixes of shuffles of parenthesis systems and it was not understood how this bijection could be interpreted on maps. The purpose of this paper is to fill this gap. This is done by defining a natural, non-recursive, bijection between tree-rooted maps of size n and pairs made of a tree of size n and a non-crossing partition of size $n+1$. The description of this bijection and the corresponding proofs occupy the first half of this paper. Then, we show that our bijection is isomorphic to the construction of Cori *et al.* via the encoding of tree-rooted maps by shuffles of parenthesis systems.

Tree-rooted maps, or alternatively shuffles of parenthesis systems, are in one-to-one correspondence with square lattice walks confined in the quarter plane (we describe this correspondence in the next section). Therefore, our bijection can also be seen as a way of counting these walks. Some years ago, Guy, Krattenthaler and Sagan worked on walks in the plane [2] and exhibited a number of nice bijections. However, they advertised the result of Cori *et al.* as being *considerably harder* to prove bijectively. We believe that the encoding in terms of tree-rooted maps makes this result more natural.

The outline of this paper is as follows. In Section 2, we recall some definitions and preliminary results on tree-rooted maps. In Section 3, we present our bijection between tree-rooted maps of size n and pairs consisting of a tree and a non-crossing partition of size n and $n+1$ respectively. This simple bijection explains why the number of tree-rooted maps of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$. In Section 4, we prove that our bijection is isomorphic to the construction of Cori *et al.*

Our study requires us to introduce a large number of mappings; we refer the reader to Figure 18 which summarizes our notations.

2 Preliminary results

We begin by some preliminary definitions on planar maps. A *planar map*, or *map* for short, is a two-cell embedding of a connected planar graph into the oriented sphere considered up to orientation preserving homeomorphisms of the sphere. Loops and multiple edges are allowed. A *rooted* map is a map together with a half-edge called the *root*. A rooted map is represented in Figure 1. The vertex (resp. the face) incident to the *root* is called the *root-vertex* (resp. *root-face*). When representing maps in the plane, the root-face is usually taken as the infinite face and the root is represented as an arrow pointing on the root-vertex (see Figure 1). Unless explicitly mentioned, all the maps considered in this paper are rooted.

A *planted plane tree*, or *tree* for short, is a rooted map with a single face. A vertex v is an *ancestor* of another vertex v' in a tree T if v is on the (unique) path in T from v' to the root-vertex of T. When v is the first vertex encountered on that path, it is the *father* of v'. A *leaf* is a vertex which is not a father. Given a rooted map M, a submap of M is a *spanning tree* if it is a tree containing all vertices of M. (The spanning tree inherit its root from the map.) We now define the main object of this study, namely tree-rooted maps. A *tree-rooted map* is a rooted map together with a distinguished spanning tree. Tree-rooted maps shall be denoted by symbols like M_T where it is implicitly assumed that M is the underlying map and T the spanning tree. Graphically, the distinguished spanning tree will be represented by thick lines (see Figure 5). The *size* of a map, a tree, a tree-rooted map, is the number of edges.

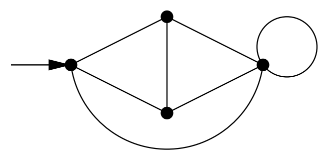


Figure 1: A rooted map.

A number of classical bijections on trees are defined by following the border of the tree. Doing the *tour of the tree* means following its border in counterclockwise direction starting and finishing at the root (see Figure 4). Observe that the tour of the tree induces a linear order, the order of appearance, on the vertex set and on the edge set of the tree. For tree-rooted maps, the tour of the spanning tree T also induces a linear order on half-edges not in T (any of them is encountered once during a tour of T). We shall say that a vertex, an edge, a half-edge *precedes* another one *around* T.

Our constructions lead us to consider *oriented maps*, that is, maps in which all edges are oriented. If an edge e is oriented from u to v, the vertex u is called the *origin* and v the *end*. The half-edge incident to the origin (resp. end) is called the *tail* (resp. *head*). The root of an oriented map will always be considered and represented as a head.

Figure 2: Half-edges and endpoints.

We now recall a well-known correspondence between tree-rooted maps and shuffles of two parenthesis systems [3, 6]. We derive from it the enumerative result mentioned above: the number of tree-rooted maps of size n (i.e. with n edges) is $\mathcal{C}_n\mathcal{C}_{n+1}$. For this purpose, we introduce some notations on words. A word w on a set A (called the alphabet) is a finite sequence of elements (letters) in A. The length of w (that is, the number of letters in w) is denoted $|w|$ and, for a in A, the number of occurrences of a in w is denoted $|w|_a$. A word w on the two-letter alphabet $\{a, \overline{a}\}$ is a *parenthesis system* if $|w|_a = |w|_{\overline{a}}$ and for all prefixes w', $|w'|_a \geq |w'|_{\overline{a}}$. For instance, $aa\overline{a}a\overline{a}\overline{a}$ is a parenthesis system. A *shuffle of two parenthesis systems*, or *parenthesis-shuffle* for short, is a word on the alphabet $\{a, \overline{a}, b, \overline{b}\}$ such that the subword of w consisting of letters in $\{a, \overline{a}\}$ and the subword consisting of letters in $\{b, \overline{b}\}$ are parenthesis systems. For instance $aba\overline{b}\overline{a}ba\overline{a}\overline{b}\overline{a}$ is a parenthesis-shuffle.

Parenthesis-shuffles can also be seen as walks in the quarter plane. Consider walks made of steps *North, South, East, West,* confined in the quadrant $x \geq 0$, $y \geq 0$. The parenthesis-shuffles of size n are in one-to-one correspondence with walks of length $2n$

starting and returning at the origin. This correspondence is obtained by considering each letter a (resp. $\overline{a}, b, \overline{b}$) as a *North* (resp. *South, East, West*) step. For instance, we represented the walk corresponding to $abb\overline{a}baa\overline{b}\overline{b}\overline{a}\overline{a}\overline{b}$ in Figure 3. The fact that the subword of w consisting of letters in $\{a, \overline{a}\}$ (resp. $\{b, \overline{b}\}$) is a parenthesis system implies that the walk stays in the half-plane $y \geq 0$ (resp. $x \geq 0$) and returns at $y = 0$ (resp. $x = 0$).

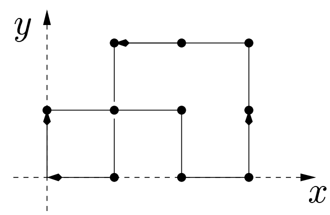


Figure 3: A walk in the quarter plane.

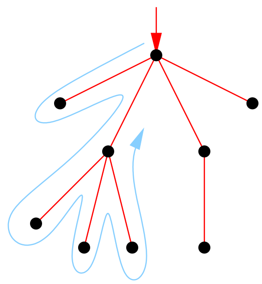
The size of a parenthesis system, or a parenthesis-shuffle, is half its length. For instance, the parenthesis-shuffle $aba\overline{b}\overline{a}ba\overline{a}\overline{b}\overline{a}$ has size 5. It is well known that the number of parenthesis systems of size n is the n^{th} Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. From this, a simple calculation proves that the number of parenthesis-shuffles of size n is $\mathcal{S}_n = \mathcal{C}_n\mathcal{C}_{n+1}$. Indeed, there are $\binom{2n}{2k}$ ways to shuffle a parenthesis system of size k (on $\{a, \overline{a}\}$) with a parenthesis system of size $n-k$ (on $\{b, \overline{b}\}$). And summing on k gives the result:

$$\begin{aligned} \mathcal{S}_n &= \sum_{k=0}^{n} \binom{2n}{2k} \mathcal{C}_k \mathcal{C}_{n-k} = \frac{(2n)!}{(n+1)!^2} \sum_{k=0}^{n} \binom{n+1}{k}\binom{n+1}{n-k} \\ &= \frac{(2n)!}{(n+1)!^2}\binom{2n+2}{n} = \mathcal{C}_n\mathcal{C}_{n+1}. \end{aligned}$$

Note, however, that this calculation involves the Chu-Vandermonde identity.

It remains to show that tree-rooted maps of size n are in one-to-one correspondence with parenthesis-shuffles of size n. We first recall a very classical bijection between trees and parenthesis systems. This correspondence is obtained by making the tour of the tree. Doing so and writing a the first time we follow an edge and $\overline{a}$ the second time we follow that edge (in the opposite direction) we obtain a parenthesis system. This parenthesis system is indicated for the tree of Figure 4. Conversely, any parenthesis system can be seen as a code for constructing a tree.

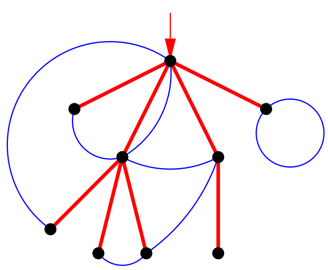
Now, consider a tree-rooted map. During the tour of the spanning tree we cross edges of the map that are not in the spanning tree. In fact, each edge not in the spanning tree will be crossed twice (once at each half-edge). Hence, making the tour of the spanning tree and writing a the first time we follow an edge of the tree, $\overline{a}$ the second time, b the first time



$a\overline{a}aa\overline{a}a\overline{a}a\overline{a}\overline{a}aa\overline{a}\overline{a}a\overline{a}$

Figure 4: A tree and the associated parenthesis system.

we cross an edge not in the tree and $\overline{b}$ the second time, we obtain a parenthesis-shuffle. We shall denote by Ξ this mapping from tree-rooted maps to parenthesis-shuffles. We applied the mapping Ξ to the tree-rooted map of Figure 5.



$\xrightarrow{\Xi}$ $bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab\overline{a}a\overline{b}b\overline{a}bb\overline{a}\overline{b}a\overline{b}\overline{b}a\overline{a}\overline{a}ab\overline{b}\overline{a}$

Figure 5: A tree-rooted map and the associated parenthesis-shuffle.

The reverse mapping can be described as follows: given a parenthesis-shuffle w we first create the tree corresponding to the subword of w consisting of letters $a, \overline{a}$ (this will give the spanning tree) then we glue to this tree a head for each letter b and a tail for each letter $\bar{b}$. There is only one way to connect heads to tails so that the result is a planar map (that is, no edges intersect). Note that, if the map M has size n, the corresponding parenthesis-shuffle w has size n since $|w|_a$ is the number of edges in the tree and $|w|_b$ is the number of edges not in the tree.
This encoding due to Walsh and Lehman [6] establishes a one-to-one correspondence between tree-rooted maps of size n and parenthesis-shuffles of size n. Hence, there are $\mathcal{C}_n\mathcal{C}_{n+1}$ tree-rooted maps of size n.

Such an elegant enumerative result is intriguing for combinatorists since Catalan numbers have very nice combinatorial interpretations. We have just seen that these numbers count parenthesis systems and trees. In fact, Catalan numbers appear in many other contexts (see for instance Ex. 6.19 of [5] where 66 combinatorial interpretations are listed). We now give another classical combinatorial interpretation of Catalan numbers, namely *non-crossing partitions*. A non-crossing partition is an equivalence relation $\sim$ on a linearly ordered set S such that no elements $a < b < c < d$ of S satisfy $a \sim c$, $b \sim d$ and

$a \not\sim b$. The equivalence classes of non-crossing partitions are called *parts*. Non-crossing partitions have been extensively studied (see [4] and references therein).

Non-crossing partitions can be represented as cell decompositions of the half-plane. If the set S is $\{s_1, \ldots, s_n\}$ with $s_1 < s_2 < \cdots < s_n$, we associate with s_i the vertex of coordinates $(i, 0)$ and with each part we associate a connected region of the lower half-plane $y \leq 0$ incident to the vertices of that part. The existence of a cell decomposition with no intersection between cells is precisely the definition of non-crossing partitions. A non-crossing partition of size 8 is represented in Figure 6. The only non-trivial parts of this non-crossing partition are $\{1, 4, 5\}$ and $\{6, 8\}$.

Non-crossing partitions of size n (i.e. on a set of size n) are in one-to-one correspondence with trees of size n. One way of seeing this is to draw the dual of the cell-representation of the partition, that is, to draw a vertex in each part and each *anti-part* (connected cells complementary to parts in the half-plane decomposition) and connect vertices corresponding to adjacent cells by an edge. The root is chosen in the infinite cell as indicated in Figure 6. In the sequel, this mapping between non-crossing partitions and trees is denoted Υ. It is a bijection between non-crossing partitions of size n and trees of size n. It proves that the number of non-crossing partitions of size n is $\mathcal{C}_n$.

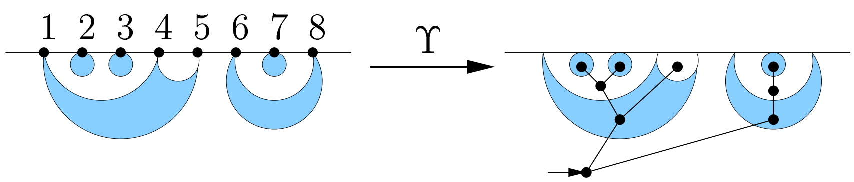


Figure 6: A non-crossing partition and the associated tree.

3 Bijective decomposition of tree-rooted maps

We begin with the presentation of our bijection between tree-rooted maps and pairs consisting of a tree and a non-crossing partition. This bijection has two steps: first we orient the edges of the map and then we disconnect its vertices.

Map orientation: Let M_T be a tree-rooted map. We denote by $\vec{M}^T$ the oriented map obtained by orienting the edges of M according to the following rules:
• edges in the tree T are oriented from the root to the leaves,
• edges not in the tree T are oriented in such a way that their head precedes their tail around T.
As always in this paper, the root is considered as a head.

In the sequel, the mapping $M_T \mapsto \vec{M}^T$ is denoted δ. We applied this mapping to the tree-rooted map of Figure 7. Note that any vertex of $\vec{M}^T$ is incident to at least one head

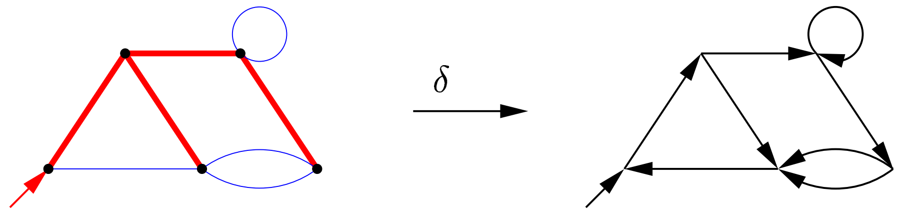


Figure 7: A tree-rooted map M_T and the corresponding oriented map $\vec{M}^T$.

(since the spanning tree is oriented from the root to the leaves).

Vertex explosion: We replace each vertex v of the oriented map $\vec{M}^T$ by as many vertices as heads incident to v and we suppress some adjacency relations between half-edges incident to v according to the rule represented in Figure 8. That is, each tail t becomes adjacent to exactly one head which is the first head encountered in counterclockwise direction around v starting from t.

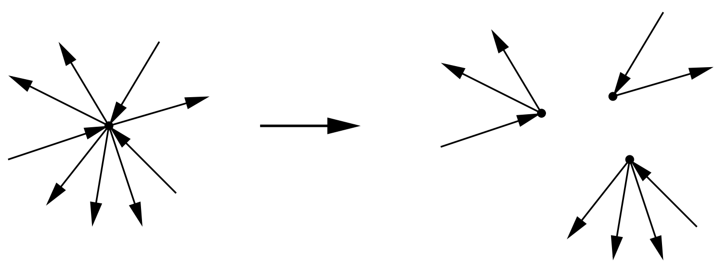


Figure 8: Local rule for suppressing the adjacency relations.

We shall prove (Lemma 11) that this suppression of some adjacency relations in $\vec{M}^T$ produces a tree denoted $\varphi_0(\vec{M}^T)$. Observe that this tree has the same number of edges, say n, as the original map M. Hence, its vertex set S has size $n+1$. This set is linearly ordered by the order of appearance around the tree $\varphi_0(\vec{M}^T)$. We define an equivalence relation $\varphi_1(\vec{M}^T)$ on S: two vertices are equivalent if they come from the same vertex of $\vec{M}^T$. We will prove (Lemma 12) that the equivalence relation $\varphi_1(\vec{M}^T)$ is a non-crossing partition on the set S. The mapping $\vec{M}^T \mapsto (\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ is called the *vertex explosion process* and is denoted φ.

Therefore, with any tree-rooted map M_T of size n we associate a tree $\varphi_0(\vec{M}^T)$ of size n and a non-crossing partition $\varphi_1(\vec{M}^T)$ of size $n+1$. The following theorem states that this correspondence is one-to-one.

Theorem 1 *Let Φ be the mapping associating the ordered pair $(\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ with the tree-rooted map M_T. This mapping is a bijection between the set of tree-rooted maps*

of size n and the Cartesian product of the set of trees of size n and the set of non-crossing partitions of size $n+1$.
It follows that the number of tree-rooted maps of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$.

Graphically, the bijection Φ is best represented by keeping track of the underlying non-crossing partition during the vertex explosion process. This is done by creating for each vertex of M a connected cell representing the corresponding part of the non-crossing partition. The graphical representation of the vertex explosion process φ becomes as indicated in Figure 9. For instance, we applied the mapping φ to the oriented map of Figure 10.

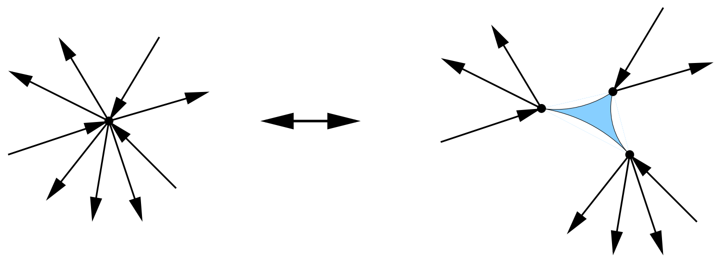


Figure 9: The vertex explosion process and a part of the non-crossing partition.

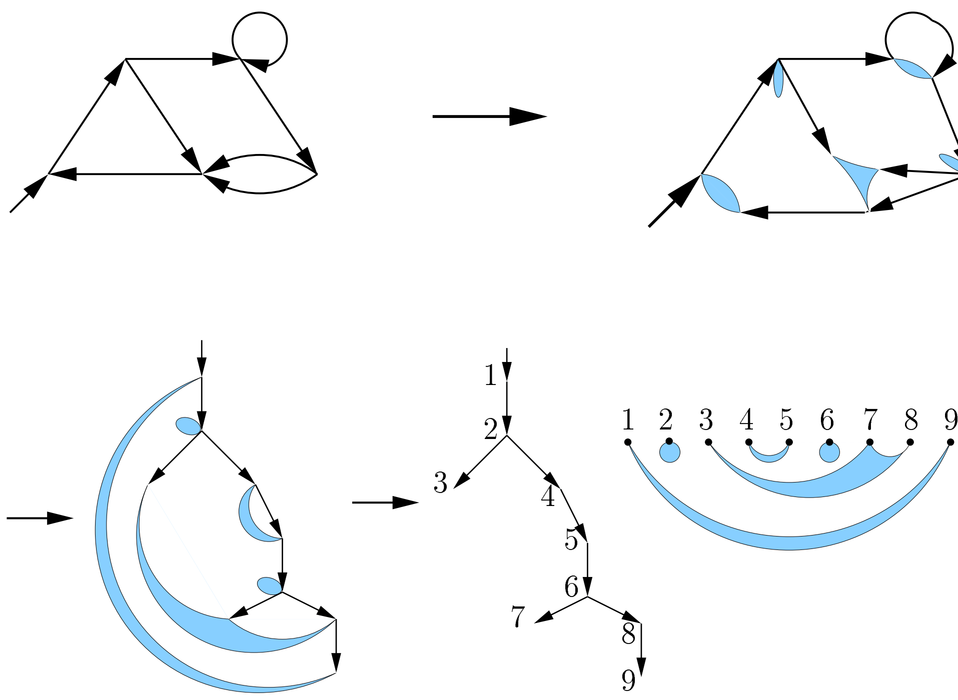


Figure 10: The vertex explosion process φ.

The rest of this section is devoted to the proof of Theorem 1. We first give a characterization of the set of oriented maps, called *tree-oriented maps*, associated to tree-rooted maps by the mapping δ. We also define the reverse mapping γ. Then we prove that the vertex explosion process φ is a bijection between tree-oriented maps (of size n) and pairs made of a tree and a non-crossing partition (of size n and $n+1$ respectively).

3.1 Tree-rooted maps and tree-oriented maps

In this subsection, we consider certain orientations of maps called *tree-orientations* (Definition 2). We prove that the mapping $\delta : M_T \mapsto \vec{M}^T$ restricted to any given map M induces a bijection between spanning trees and tree-orientations of M. The key property explaining why the mapping δ is injective is that during a tour of a spanning tree T, the tails of edges in T are encountered before their heads whereas it is the contrary for the edges not in T. Using this property we will define a procedure γ for recovering spanning trees from tree-orientations of M (Definition 5). We will prove that δ and γ are reverse mappings that establish a one-to-one correspondence between tree-rooted maps and tree-oriented maps (Proposition 3).

We begin with some definitions concerning cycles and paths in oriented maps. A simple cycle (resp. simple path) is *directed* if all its edges are oriented consistently. A simple cycle defines two regions of the sphere. The *interior region* (resp. *exterior region*) of a directed cycle is the region situated at its left (resp. right) as indicated in Figure 11. We call *positive cycle* a directed cycle having the root in its exterior region. Graphically, positive cycles appear as counterclockwise directed cycles when the map is projected on the plane with the root in the infinite face.

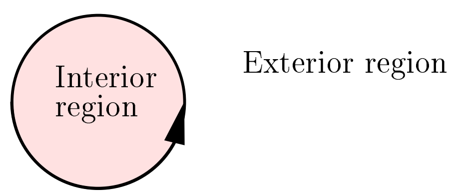


Figure 11: Interior and exterior regions of a directed cycle.

Definition 2 *A* tree-orientation *of a map is an orientation without a positive cycle such that any vertex can be reached from the root by a directed path. A* tree-oriented map *is a map with a tree-orientation.*

We will prove that the images of tree-rooted maps by the mapping δ are tree-oriented maps. More precisely, we have the following proposition.

Proposition 3 *For any given map M, the mapping $\delta : M_T \mapsto \vec{M}^T$ induces a bijection between spanning trees and tree-orientations of M.*

We first prove the following lemma.

Lemma 4 *For all tree-rooted maps M_T, the map $\vec{M}^T$ is tree-oriented.*

Proof: For any vertex v, there is a path in T from the root to v. This path is oriented from the root to v in $\vec{M}^T$. It remains to prove that there is no positive cycle. Suppose the contrary and consider a positive cycle C. By definition, the root is in the exterior region of C. Since C is a cycle there are edges of C which are not in T. Consider the first such edge e encountered during the tour of T. When we first cross e we enter for the first time the interior region of C. Given the orientation of C, the half-edge of e that we first cross is its tail (see Figure 12). But, by definition of $\vec{M}^T$, the half-edge of e that we first cross should be its head. This gives a contradiction. □

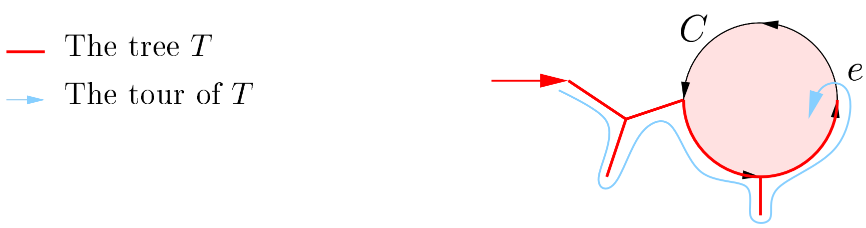


Figure 12: Entering the cycle C.

We now define a procedure γ constructing a spanning tree T on a tree-oriented map $\vec{M}$.

Algorithm 5

***Procedure* γ:**

1. *At the beginning, the submap T is consists only of the root and root-vertex.*
2. *We make the tour of T (starting from the root) and apply the following rule.*
   When the tail of an edge e is encountered and its head has not been encountered yet, we add e to T (together with its end).
   Then we continue the tour of T, that is, if e is in T we follow its border, otherwise we cross e.
3. *We stop when arriving at the root and return the submap T.*

We now prove the correctness of the procedure γ.

Lemma 6 *The mapping γ is well defined (terminates) on tree-oriented maps and returns a spanning tree.*

Proof:

• *At any stage of the procedure, the submap T is a tree.*

Suppose not, and consider the first time an edge e creating a cycle is added to T. We denote by T_0 the tree T just before that time. The edge e is added to T_0 when its tail t is encountered. At that time, its head h has not been encountered but is incident to T_0 (since adding e creates a cycle). We know that, when e is added, the border of T_0 from the root to t has been followed but not the border of T_0 from t to the root. Moreover, the head h lies after t around T_0 (since h has not been encountered yet). Observe that the right border of any edge of T_0 has been followed (just after this edge was added to T_0). Thus, the border of T_0 from t to h is made of the left borders of some edges $e_1, e_2, \ldots, e_k$. Hence, these edges form a directed path from h to t and $e, e_1, e_2, \ldots, e_k$ form a directed cycle C. Since h lies after t around T_0, the root is in the exterior region of C (see Figure 13). Therefore, the cycle C is positive which is impossible.

Figure 13: The submap T remains a tree.

• *The procedure γ terminates.*

The set T remains a tree connected to the root. Hence, it is impossible to follow the same border of the same edge twice without encountering the root.

• *At the end of the procedure γ, the tree T is spanning.*

At the end of the procedure, the whole border of T has been followed. Hence, any half-edge incident to T has been encountered. Now, suppose that a vertex v is not in T and consider a directed path from the root to v. (This path exists by definition of tree-orientations.) There is an edge of this path with its origin in T and its end out of T. Therefore, its tail is incident to T but not its head. Thus, it should have been added to T (with its end) when its tail was encountered. This is a contradiction. $\square$

We continue the proof of Proposition 3. We proved that the mapping δ associates a tree-orientation of a map to any spanning tree of that map (Lemma 4). We proved that the mapping γ associates a spanning tree of a map to any tree-orientation of that map (Lemma 6). It remains to prove that $\delta \circ \gamma$ and $\gamma \circ \delta$ are identity mappings.

Lemma 7 *Let $\vec{M}$ be a tree-oriented map and T be the spanning tree constructed by the procedure γ. The edges in T are oriented from the root to the leaves and the edges not in T are oriented in such a way that their heads precede their tails around T.*

Proof:

• *Edges in T are oriented from the root to the leaves.* An edge e is added to T when its tail is encountered. At that time the end of e is not in T or adding e would create a cycle. The property follows by induction.

• *Edges not in T are oriented in such a way that their head precedes their tail around T.* If an edge breaks this rule it should have been added to T when its tail was encountered.

□

Corollary 8 *The mapping $\delta \circ \gamma$ is the identity mapping on tree-oriented maps.*

Proof: Let $\vec{M}$ be a tree-oriented map and T be the tree constructed by the procedure γ. By Lemma 7, the edges in T are oriented from the root to the leaves and the edges not in T are oriented in such a way that their head precedes their tail around T. By definition of δ, this is also the case in $\delta \circ \gamma(\vec{M})$. Thus, $\delta \circ \gamma$ is the identity mapping on tree-oriented maps. □

Lemma 9 *The mapping $\gamma \circ \delta$ is the identity mapping on tree-rooted maps.*

Proof: Let M_T be a tree-rooted map. Suppose the spanning tree T' constructed by the procedure $\gamma(\delta(M_T))$ differs from T. We consider the order of edges induced by the tour of T. Let e be the smallest edge in the symmetric difference of T and T'. The tours of T and T' must coincide until a half-edge h of e is encountered. We distinguish the head and the tail of e according to its orientation in $\delta(M_T)$. If e is in T, its tail is encountered before its head around T (by definition of $\delta(M_T)$). In this case, h is a tail. If e is not in T', its head is encountered before its tail around T' (by Lemma 7). In this case, h is a head. Therefore, e cannot be in $T \setminus T'$. Similarly, e cannot be in $T' \setminus T$ since e being in T' implies that h is a head and e not being in T implies that h is a tail. We obtain a contradiction. □

This completes the proof of Proposition 3: tree-oriented maps are in one-to-one correspondence with tree-rooted maps. □

3.2 The vertex explosion process on tree-oriented maps

This subsection is devoted to the proof of the following proposition.

Proposition 10 *The mapping $\varphi : \vec{M} \mapsto (\varphi_0(\vec{M}), \varphi_1(\vec{M}))$ is a bijection between tree-oriented maps of size n and ordered pairs consisting of a tree of size n and a non-crossing partition of size $n+1$.*

We start with a lemma concerning the mapping φ_0.

Lemma 11 *The image of any tree-oriented map $\vec{M}$ by φ_0 is a tree (oriented from the root to the leaves).*

Proof: Let $\vec{M}$ be a tree-oriented map. Any vertex is incident to at least one head (there is a directed path from the root to any vertex), hence the mapping φ_0 is well defined. The image $\varphi_0(\vec{M})$ has the same number of edges, say n, as $\vec{M}$. The map $\vec{M}$ has $n+1$ heads (one per edge plus one for the root). Since any vertex in $\varphi_0(\vec{M})$ is incident to exactly one head, the image $\varphi_0(\vec{M})$ has $n+1$ vertices. Thus, it is sufficient to prove that $\varphi_0(\vec{M})$ has no cycle (connectivity then follows).
Suppose $\varphi_0(\vec{M})$ contains a simple cycle C. Since any vertex in C is incident to exactly one head, the edges of C are oriented consistently. We identify the edges of $\vec{M}$ and the edges of $\varphi_0(\vec{M})$. The edges of C form a cycle in $\vec{M}$ but this cycle might not be simple. We consider a directed path P in $\vec{M}$ from the root to a vertex v (of $\vec{M}$) incident with an edge of C. We suppose (without loss of generality) that v is the only vertex of P incident with an edge of C. Let h be the head in P incident with v and t' be the first tail in C following h in counterclockwise direction around v. We can construct a directed simple cycle C' (in $\vec{M}$) made of edges in C and containing t' (see Figure 14). Let h' be the head of C' incident with v. Since C' is a directed cycle of the tree-oriented map $\vec{M}$, it contains the root in its interior region. Since v is the only vertex of P incident with an edge in C', the head h is in the interior region of C'. Therefore, in counterclockwise direction around v we have h, h' and t' (and possibly some other half-edges). We consider the tail t following h in the cycle C (considered as a directed simple cycle of $\varphi_0(\vec{M})$). By the choice of t' we know that t is between t' and h in counterclockwise direction around v (t and t' may be distinct or not). Hence, in counterclockwise direction around v we have h, h' and t. Hence, h' is not the first head encountered in counterclockwise direction around v starting from t. Therefore, by definition of the vertex explosion process, h' and t are not adjacent in $\varphi_0(\vec{M})$. We reach a contradiction.

□

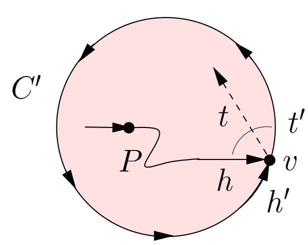


Figure 14: The cycle C' in $\vec{M}$.

We now study the properties of the mapping φ_1. Two consecutive half-edges around a vertex define a *corner*. A vertex has as many corners as incident half-edges. Let T be a tree and v be a vertex of T. The *first corner* of the vertex v is the first corner of v encountered around T. If the tree is oriented from the root to the leaves, the first corner of v is at the right of the head incident to v as shown in Figure 15.

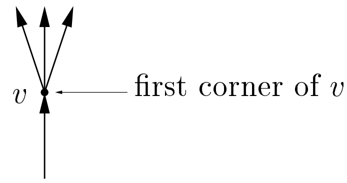


Figure 15: The first corner of a vertex.

We compare the vertices of the tree $\varphi_0(\vec{M})$ according to their order of appearance around this tree. We write $u < v$ if u precedes v (i.e. the first corner of u precedes the first corner of v) around the tree.

Lemma 12 *For any tree-oriented map $\vec{M}$, the equivalence relation $\varphi_1(\vec{M})$ on the set of vertices of the tree $\varphi_0(\vec{M})$ ordered by their order of appearance around this tree is a non-crossing partition.*

Proof: The proof relies on the graphical representation of the equivalence relation $\sim = \varphi_1(\vec{M})$ given by Figure 9. During the vertex explosion process, we associate a connected cell C_v with each vertex v of $\vec{M}$, that is, with each equivalence class of the relation $\sim$. The cell C_v can be chosen to be incident only with the first corners of the vertices in its class but not otherwise incident with the tree. Moreover the cells can be chosen so that they do not intersect.
Suppose $v_1 < v_2 < v_3 < v_4$, $v_1 \sim v_3$ and $v_2 \sim v_4$. One can draw a path from the first corner of v_1 to the first corner of v_3 staying in a cell C and a path from the first corner of v_2 to the first corner of v_4 staying in a cell C'. It is clear that these two paths intersect (see Figure 16). Thus $C = C'$ and $v_1 \sim v_2$. $\square$

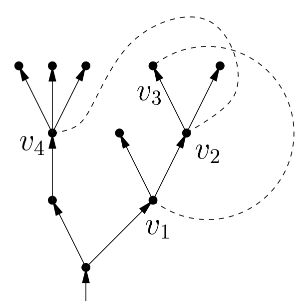


Figure 16: The two paths intersect.

We have proved that the application $\varphi : \vec{M} \mapsto (\varphi_0(\vec{M}), \varphi_1(\vec{M}))$ associates a tree of size n and a non-crossing partition of size $n+1$ with any tree-oriented map of size n. Conversely, we define the mapping ψ.

Definition 13 *Let T be a tree of size n and $\sim$ be a non-crossing partition on a linearly ordered set S of size $n+1$. We identify S with the set of vertices of T ordered by the order of appearance around T. We construct the oriented map $\psi(T,\sim)$ as follows. First we orient the tree T from the root to the leaves. With each part $\{v_1, v_2, \ldots, v_k\}$ of the partition, we associate a simply connected cell incident to the first corner of v_i, $i = 1 \ldots k$ but not otherwise incident with T. Since $\sim$ is a non-crossing partition, these cells can be chosen without intersections. Then we contract each cell into a vertex in such a way no edges of T intersect.*

We first prove the following lemma.

Lemma 14 *For any tree T of size n and any non-crossing partition $\sim$ of size $n+1$, the oriented map $\psi(T,\sim)$ is tree-oriented.*

Proof: Every vertex of $\vec{M} = \psi(T,\sim)$ is connected to the root by a directed path (since it is the case in T). It remains to show that there is no positive cycle.
Let C be a positive cycle of $\vec{M}$ and e an edge of C. We consider the directed path P of T from the root to e (the root and e included). By definition, the root is in the exterior region of C. Let h be the last head of P contained in the exterior region of C and t the tail following h in P (the tail t exists since the last edge e of P is in C). By definition, the tail t is either in C or in its interior region. Let v be the end of h (i.e the origin of t) in $\vec{M}$ and h' the head of C incident with v (see Figure 17). In counterclockwise direction around v, we have h, t and h' (and possibly some other half-edges). The vertex v is obtained by contracting a cell C_v of the partition $\sim$ corresponding to some vertices of T. Each of these vertices is incident to one head in T, hence h and h' were incident to two distinct vertices, say v_1 and v_2, of T. The cell C_v is incident to the first corner of v_1 which is situated between h and t in counterclockwise direction around v_1. Therefore, after the cell C_v is contracted, the half-edges of v_2 are situated between h and t in counterclockwise direction around v. Thus, in counterclockwise direction around v, we have h, h' and t (and possibly some other half-edges). We obtain a contradiction. $\square$

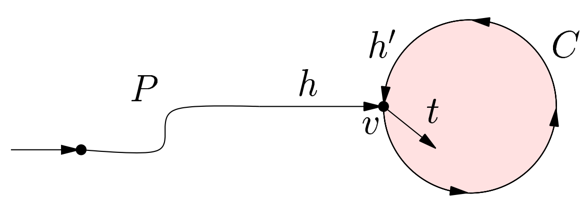


Figure 17: The map $\vec{M} = \psi(T,\sim)$ has no positive cycle.

We now conclude the proof of Theorem 1.

• Let $\vec{M}$ be a tree-oriented map. We know from Lemma 11 that $T = \varphi_0(\vec{M})$ is a tree oriented from the root to the leaves. Moreover, we know from Lemma 12 that the partition $\sim = \varphi_0(\vec{M})$ of the vertex set of T is non-crossing. Let u be a vertex of T. Let $\{v_1, \ldots, v_k\}$ be a part of the partition $\sim$ corresponding to a vertex v of $\vec{M}$. The cell C_v associated to v during the vertex explosion process is incident to the corner of v_i, $i = 1 \ldots k$ at the right of the head incident with v_i (see Figure 9). Since T is oriented from the root to the leaves, this corner is the first corner of v_i. Therefore, by definition of ψ, we have $\psi \circ \varphi(\vec{M}) = \vec{M}$. Thus, $\psi \circ \varphi$ is the identity mapping on tree-oriented maps.

• Let T be a tree of size n and $\sim$ be a non-crossing partition on a linearly ordered set S of size $n+1$. We know from Lemma 14 that $\vec{M} = \psi(T, \sim)$ is a tree-oriented map. We think of the tree T as being oriented from the root to the leaves and we identify the set S with the vertex set of T. Let v be a vertex of $\vec{M}$ corresponding to the part $\{v_1, \ldots, v_k\}$ of the partition $\sim$. The vertex v is obtained by contracting a cell C_v incident with the first corner of v_i, $i = 1 \ldots k$, that is, the corner at the right of the head h_i incident with v_i. Therefore, if t is a tail incident with v_i in T, then, h_i is the first head encountered in counterclockwise direction around v starting from t (in $\vec{M}$). Given the definition of the vertex explosion process, the adjacency relations between the half-edges incident with v that are preserved by the vertex explosion process are exactly the adjacency relations in the tree T. Thus, the trees $\varphi_0(\vec{M})$ and T are the same. Moreover, the part of the partition $\varphi_1(\vec{M})$ associated to the vertex v is $\{v_1, \ldots, v_k\}$. Thus, the partitions $\varphi_1(\vec{M})$ and $\sim$ are the same. Hence, $\varphi \circ \psi$ is the identity mapping on pairs made of a tree of size n and a non-crossing partition of size $n+1$.

Thus, the mapping φ is a bijection between tree-oriented maps of size n and pairs made of a tree of size n and a non-crossing partition of size $n+1$. This completes the proof of Proposition 10 and Theorem 1.

□

4 Correspondence with a bijection due to Cori, Dulucq and Viennot

In this section, we prove that our bijection Φ is isomorphic to a former bijection due to Cori, Dulucq and Viennot defined on parenthesis-shuffles [1]. We know that tree-rooted maps are in one-to-one correspondence with parenthesis-shuffles by the mapping Ξ defined in Section 2. Our bijection $\Phi : M_T \mapsto (\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ associates with any tree-rooted map M_T of size n, a tree $\varphi_0(\vec{M}^T)$ of size n and a non-crossing partition $\varphi_1(\vec{M}^T)$ of size $n+1$. The bijection $\Lambda : w \mapsto (\lambda'_0(w), \lambda'_1(w))$ of Cori *et al.* associates with any parenthesis-shuffle w of size n, a tree $\lambda'_0(w)$ of size n and a binary tree $\lambda'_1(w)$ of size $n+1$. We shall prove that these two bijections are isomorphic via the encoding of tree-rooted maps by parenthesis-shuffles. That is, we shall prove that there exist two independent bijections Ω and Θ such that, if $w = \Xi(M_T)$, then $\varphi_0(\vec{M}^T) = \Omega(\lambda'_0(w))$ and $\varphi_1(\vec{M}^T) = \Theta(\lambda'_1(w))$.

In fact, we have adjusted some definitions from [1] so that Ω is the identity mapping on trees. This situation is represented in Figure 18.

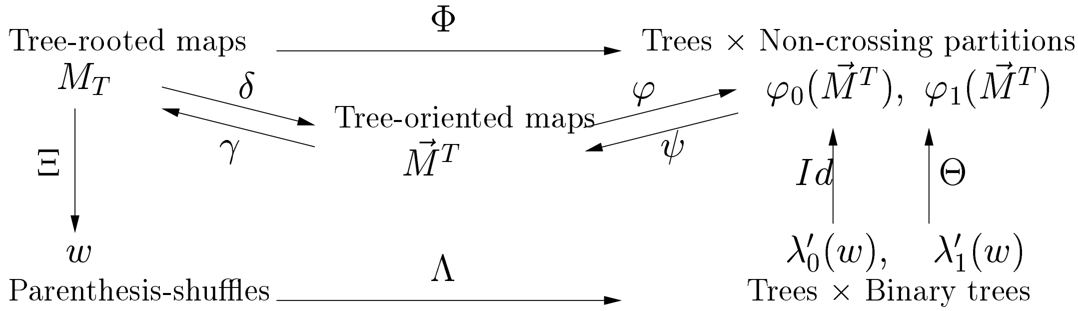


Figure 18: The bijection diagram.

4.1 The bijection Λ of Cori, Dulucq and Viennot

We begin with a presentation of the bijection Λ of Cori *et al.* For the sake of simplicity, the presentation given here is not *completely* identical to the one of the original article [1]. But, whenever our definitions differ there is an obvious equivalence via a composition with a simple, well-known bijection. The interested reader can look for more details in the original article. In this article, Cori *et al.* defined recursively two mappings λ_0 and λ_1 on the set of *prefix-shuffles*. A prefix-shuffle is a word w on the alphabet $\{a, \overline{a}, b, \overline{b}\}$ such that, for all prefixes w' of w, we have $|w'|_a \geq |w'|_{\overline{a}}$ and $|w'|_b \geq |w'|_{\overline{b}}$. Note that the set of prefix-shuffles is the set of prefixes of parenthesis-shuffles. The mappings λ_0 and λ_1 both eventually return trees. In the original paper [1], the trees returned by λ_0 and λ_1 were called the *leaf code* and the *tree code* respectively.

We first define the mapping λ_0. It involves the mapping σ that associates the tree $\sigma(T_1, T_2)$ represented in Figure 19 with the ordered pair of trees (T_1, T_2).

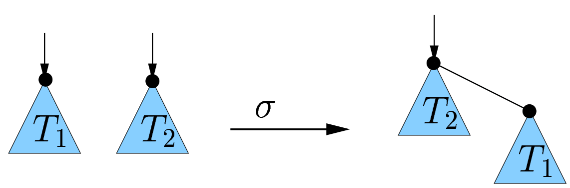
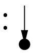


Figure 19: The mapping σ on ordered pairs of trees.

We consider the alphabet $\mathbf{U} = \{u, v\}$ and the infinite alphabet $\mathbf{T}$ consisting of all trees. A word s on the alphabet $\mathbf{U} \cup \mathbf{T}$ is a *tree-sequence* if $s = ut_1u \ldots t_{i-1}ut_ivt_{i+1} \ldots t_kv$ where $1 \leq i \leq k$ and $t_1, \ldots, t_k$ are trees. The mapping λ_0 associates tree-sequences with prefix-shuffles.

Definition 15 *The mapping λ_0 is recursively defined on prefix-shuffles by the following rules:*

- *If $w = \epsilon$ is the empty word, $\lambda_0(w)$ is the tree-sequence $u\tau v$ where τ is the tree reduced to a root and a vertex.*

  τ : 

- *If $w = w'a$, the tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the last occurrence of u by $u\tau v$.*

- *If $w = w'b$, the tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the first occurrence of v by $u\tau v$.*

- *If $w = w'\overline{a}$, we consider the first occurrence of v in $\lambda_0(w')$ and the trees T_1 and T_2 directly preceding and following it. The tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the subword T_1vT_2 by the tree $\sigma(T_1, T_2)$.*

- *If $w = w'\overline{b}$, we consider the last occurrence of u in $\lambda_0(w')$ and the trees T_1 and T_2 directly preceding and following it. The tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the subword T_1uT_2 by the tree $\sigma(T_1, T_2)$.*

We applied the mapping λ_0 to the word $w = ba\overline{a}a\overline{b}\overline{a}$. The different steps are represented in Figure 20.

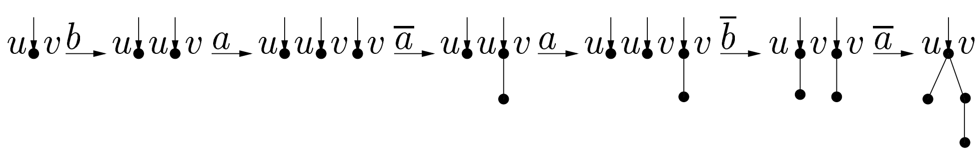


Figure 20: The mapping λ_0 applied to the prefix-shuffle $w = ba\overline{a}a\overline{b}\overline{a}$.

It is easily seen by induction that the number of v (resp. u) in $\lambda_0(w)$ is $|w|_a - |w|_{\overline{a}} + 1$ (resp. $|w|_b - |w|_{\overline{b}} + 1$). Hence, the mapping λ_0 is well defined on prefix-shuffles. Moreover, the first letter u and last letter v are never replaced by anything. Observe also (by induction) that the letters u always precede the letters v in $\lambda_0(w)$. Thus, $\lambda_0(w)$ is indeed a tree-sequence. If w is a parenthesis-shuffle, there is exactly one letter u and one letter v in $\lambda_0(w)$, hence $\lambda_0(w)$ is a three letter word uTv.

Definition 16 *The mapping λ_0' associates with a parenthesis-shuffle w the unique tree T in the tree-sequence $\lambda_0(w) = uTv$.*

Observe that, for any prefix-shuffle w, the total number of edges in the trees $t_1, \ldots, t_k$ of the tree-sequence $\lambda_0(w) = ut_1u \ldots t_{i-1}ut_ivt_{i+1} \ldots t_kv$ is $|w|_{\overline{a}} + |w|_{\overline{b}}$. Hence, if w is parenthesis-shuffle of size n, the tree $\lambda_0'(w)$ has size n.

We now define the mapping λ_1 which associates *binary trees* with prefix-shuffles. A binary tree is a (planted plane) tree for which each vertex is either a *node* of degree 3 or a *leaf* of degree 1. The size of a binary tree is defined as the number of its nodes. It is well-known that binary trees of size n (i.e. with n nodes) are in one-to-one correspondence with trees of size n (i.e. with n edges).

In a binary tree, the two sons of a node are called *left son* and *right son*. In counter-clockwise order around a node we find the father (or the root), the left son and the right son (see Figure 21). A *left leaf* (resp. *right leaf*) is a leaf which is a left son (resp. right son). As before, we compare vertices according to their order of appearance around the tree and we shall talk about the *first* and *last* leaf. Moreover, a leaf will be either *active* or *inactive*. Graphically, active leaves will be represented by circles and inactive ones by squares.

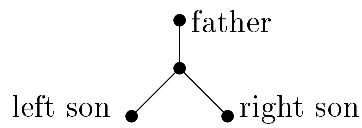
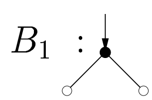


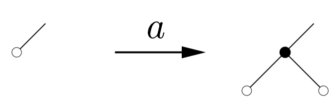
Figure 21: Left and right son of a node

Definition 17 *The mapping λ_1 is recursively defined on prefix-shuffles by the following rules:*

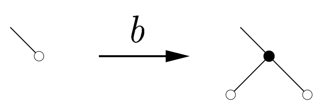
- *If $w = \epsilon$ is the empty word, $\lambda_1(w)$ is the binary tree B_1 consisting of a root, a node and two active leaves.*



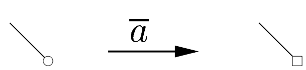
- *If $w = w'a$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by replacing the last active left leaf by B_1.*



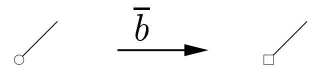
- *If $w = w'b$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by replacing the first active right leaf by B_1.*



- *If $w = w'\overline{a}$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by inactivating the first active right leaf.*



- *If $w = w'\overline{b}$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by inactivating the last active left leaf.*



We applied the mapping λ_1 to the word $w = ba\overline{a}a\overline{b}\overline{a}$. The different steps are represented in Figure 22.

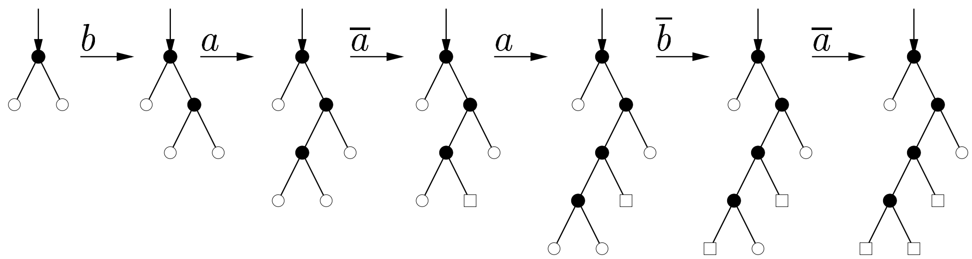


Figure 22: The mapping λ_1 on the word $w = ba\overline{a}a\overline{b}\overline{a}$.

It is easily seen by induction that the number of active right leaves (resp. left leaves) in $\lambda_1(w)$ is $|w|_a - |w|_{\overline{a}} + 1$ (resp. $|w|_b - |w|_{\overline{b}} + 1$). Hence, the mapping λ_1 is well defined on prefix-shuffles. Observe that the binary tree $\lambda_1(w)$ has $|w|_a + |w|_b + 1$ nodes. Observe also (by induction) that active left leaves always precede active right leaves in $\lambda_1(w)$. Moreover, if w is a parenthesis-shuffle, only the first left leaf and the last right leaf are active (since they can never be inactivated).

Definition 18 *The mapping λ'_1 associates with a parenthesis-shuffle w of size n the binary tree of size $n+1$ obtained from $\lambda_1(w)$ by inactivating the two active leaves.*

We now make some informal remarks explaining why the mapping $w \mapsto (\lambda_0(w), \lambda_1(w))$ is injective. It is, of course, possible to decide from $(\lambda_0(w), \lambda_1(w))$ if w is the empty word. Indeed, w is the empty word iff $\lambda_1(w) = B_1$ (equivalently iff $\lambda_0(w) = \tau$). Otherwise, the remarks below show that the last letter α of $w = w'\alpha$ can be determined as well as $\lambda_0(w')$ and $\lambda_1(w')$. So any prefix-shuffle w can be entirely recovered from $(\lambda_0(w), \lambda_1(w))$.

Remarks:

- For any prefix-shuffle w, the number of letters u (resp. v) in the tree-sequence $\lambda_0(w)$ is equal to the number of active left leaves (resp. right leaves) in the binary tree $\lambda_1(w)$. Furthermore, it can be shown by induction that the size of the tree t_i lying between the i^{th} and $i+1^{th}$ letters u, v in $\lambda_0(w)$ is the number of inactive leaves lying between the i^{th} and $i+1^{th}$ active leaves in $\lambda_1(w)$.
- The three following statements are equivalent:

- the word w is not empty and the last letter α of $w = w'\alpha$ is in $\{a, b\}$,
- there is a sequence $u\tau v$ in $\lambda_0(w)$,
- there is an active left leaf and an active right leaf which are siblings.

In this case, $\lambda_1(w')$ is obtained from $\lambda_1(w)$ by deleting the two actives leaves and making the father an active leaf ℓ. Moreover, $\alpha = a$ (resp. $\alpha = b$) if ℓ is a left leaf (resp. right leaf) in $\lambda_1(w')$ in which case $\lambda_0(w')$ is obtained from $\lambda_0(w)$ by replacing the subword $u\tau v$ by u (resp. v).

• If the last letter α of $w = w'\alpha$ is in $\{\overline{a}, \overline{b}\}$, we know from the above remark that the tree T lying between the last letter u and the first letter v in the tree-sequence $\lambda_0(w)$ has size $k > 0$. Since $k > 0$, the tree T admits a (unique) preimage (T_1, T_2) by the mapping σ. Let k' be the size of the tree T_1. Then $k' < k$. We know that there are k inactive leaves lying between the last active left leaf and the first active right leaf in $\lambda_1(w)$. The binary tree $\lambda_1(w')$ is obtained from $\lambda_1(w)$ by activating the $k' + 1^{th}$ leaf ℓ encountered when following the border of the tree starting from the last active left leaf. Moreover, $\alpha = \overline{a}$ (resp. $\alpha = \overline{b}$) if ℓ is a right leaf (resp. left leaf), in which case the tree-sequence $\lambda_0(w')$ is obtained from $\lambda_0(w)$ by replacing T by $T_1 v T_2$ (resp. $T_1 u T_2$).

From these remarks, we see that the mapping $w \mapsto (\lambda_0(w), \lambda_1(w))$ is injective. It can be shown, with the same ideas, that it is bijective on the set of pairs consisting of a tree-sequence S and a binary tree B with active and inactive leaves satisfying the following conditions:

- the active left leaves precede the active right leaves in B,
- the number of active left leaves (resp. right leaves) in B is the same as the number of u (resp. v) in S,
- the number of inactive leaves lying between the i^{th} and $i + 1^{th}$ active leaves in B is the size of the tree lying between the i^{th} and $i + 1^{th}$ letters u, v in S.

We now define the mapping Λ of Cori *et al.* on parenthesis-shuffles.

Definition 19 *The mapping $w \mapsto (\lambda'_0(w), \lambda'_1(w))$ defined on parenthesis-shuffles is denoted Λ.*

We know that Λ associates with a parenthesis-shuffle of size n a pair consisting of a tree of size n and a binary tree of size $n + 1$. The remarks above should convince the reader that the mapping Λ is a bijection between these two sets of objects.

4.2 The bijections Φ and Λ are isomorphic

We now return to our business and prove that the bijection Λ of Cori *et al.* and our bijection Φ are isomorphic. Before stating precisely this result, we define a (non-classical) bijection θ between binary trees and trees. By composition, this allows us to define a bijection Θ between binary trees and non-crossing partitions.

Let e be an edge of a binary tree. The edge e is said to be branching if one of its vertices is a right son and the other is a left son or the root-vertex. Intuitively, this means that the edge e is non-parallel to its parent-edge. For instance, the branching edges of the binary tree in Figure 23 are indicated by thick lines.

Definition 20 *Let B be a binary tree. The tree $\theta(B)$ is obtained by contracting every non-branching edge. The non-crossing partition $\Theta(B)$ is the image of $\theta(B)$ by the mapping Υ^{-1} (see Figure 6).*

We applied the mapping Θ to the binary tree of Figure 23.

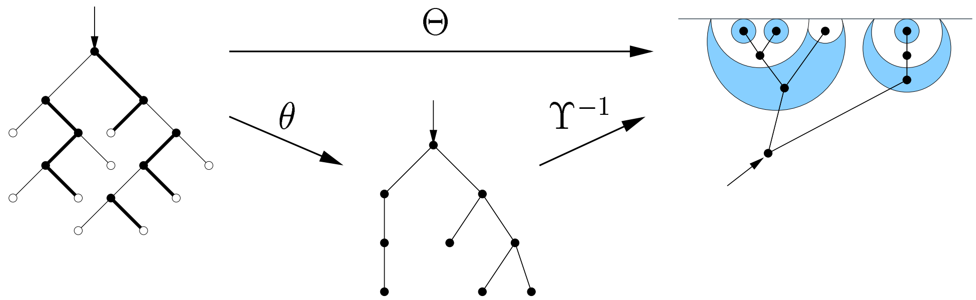


Figure 23: The mappings θ and Θ.

The mapping Θ is a bijection between binary trees of size n (n nodes) and trees of size n (n edges). The proof is omitted here since we will not use this property.
We now state the main result of this section.

Theorem 21 *Let M_T be a tree-rooted map and $w = \Xi(M_T)$ its associated parenthesis-shuffle. Let $\varphi_0(\vec{M}^T)$ and $\varphi_1(\vec{M}^T)$ be the tree and the non-crossing partition obtained from M_T by the mapping Φ. Let $\lambda'_0(w)$ and $\lambda'_1(w)$ be the tree and binary tree obtained from w by the mapping Λ. Then $\varphi_0(\vec{M}^T) = \lambda'_0(w)$ and $\varphi_1(\vec{M}^T) = \Theta(\lambda'_1(w))$.*

This relation between the mappings Λ and Φ is represented by Figure 18. As an illustration, we applied the mapping Φ to the tree-rooted map M_T of Figure 24 and we applied the mapping Λ to $w = \Xi(M_T) = ba\overline{a}a\overline{b}\overline{a}$. The rest of this section is devoted to the proof of Theorem 21.

4.3 Prefix-maps

The mappings λ'_0 and λ'_1 are defined on parenthesis-shuffles from the more general mappings λ_0 and λ_1 defined on prefix-shuffles. In order to relate $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ (resp. $\varphi_1(\vec{M}^T)$ and $\lambda'_1(w)$) we need to define the *prefix-maps* which are in one-to-one correspondence with prefix-shuffles. As we will see, prefix-maps are tree-oriented maps together with some *dangling* heads in the root-face. In Subsections 4.4 and 4.5 we shall extend the mappings φ_0 and φ_1 defined in Section 3 to prefix-maps.

For any prefix-shuffle w we denote by w_a (resp. w_b) the subword of w consisting of the letters $a, \overline{a}$ (resp. $b, \overline{b}$). The words w_a and w_b are prefixes of parenthesis systems. We say

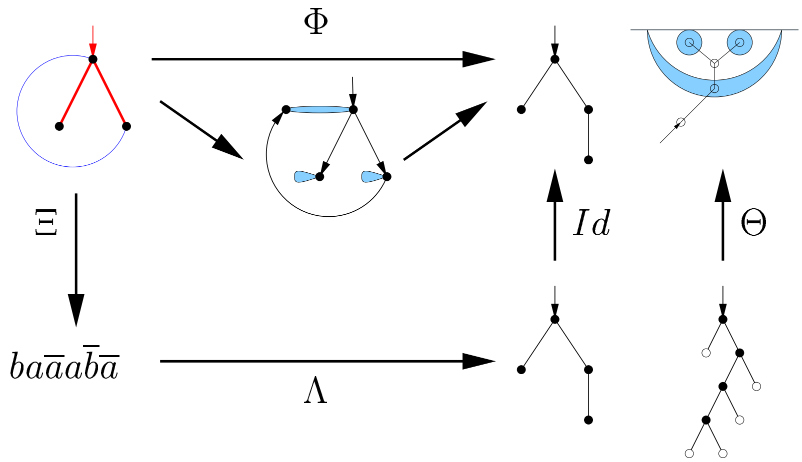


Figure 24: The isomorphism between Λ and Φ.

that an occurrence of a letter $c = a, b$ is *paired* with an occurrence of $\overline{c}$ if the subword of w_c lying between these two letters is a parenthesis system. There are $|w|_a - |w|_{\overline{a}}$ non-paired letters a and $|w|_b - |w|_{\overline{b}}$ non-paired letters b in w. We denote by w_a^+ the parenthesis system obtained from w_a by adding $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$ at the end of this word.

Let w be a prefix-shuffle. We define T_w as the tree associated to the parenthesis system w_a^+, that is, T_w is such that, making the tour of T_w and writing a the first time we follow an edge and $\overline{a}$ the second time, we obtain w_a^+. We orient the edges of T_w from the root to the leaves. Then, we add half-edges to T_w by looking at the position of the letters b and $\overline{b}$ in w. More precisely, we read the word w and while making the tour of T according to the letters $a, \overline{a}$, we insert heads for the letters b and tails for the letters $\overline{b}$. If an occurrence of b and an occurrence of $\overline{b}$ are paired in w we connect the corresponding head and tail. We obtain an oriented map together with some heads called *dangling heads* corresponding to non-paired letters b of w. In the tree T_w, the edges corresponding to non-paired letters a are called *active* while the others are called *inactive*. The *prefix-map associated with* w, denoted by M_w, is the oriented map (with dangling heads and active edges) obtained. For instance, the prefix-map associated with $bab\overline{a}a\overline{b}ab\overline{b}\overline{a}ab$ has been represented in Figure 25 (the active edges are dashed).

Observe that T_w is a spanning tree of the prefix-map M_w. The orientation of M_w is the tree-orientation associated to the spanning tree T_w by the mapping δ defined in Section 3. In particular, when w is a parenthesis-shuffle, the prefix-map M_w is a map (i.e. it has no active edge and no dangling head except for the root) which is tree-oriented. More precisely, if $w = \Xi(M_T)$, the tree-oriented map M_w is $\vec{M}^T \equiv \delta(M_T)$.

Let w be a prefix-shuffle. The heads of active edges in the prefix map M_w are called *rooting heads*, and their ends are called *rooting vertices*. By convention, the root is considered as a rooting head. As before, we compare active edges (resp. rooting vertices,

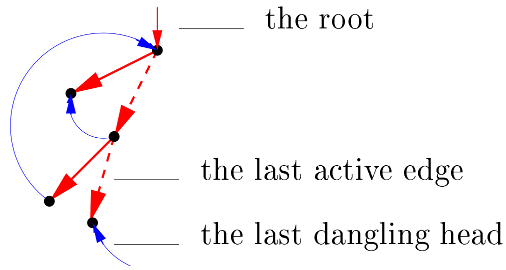


Figure 25: The prefix-map associated to $bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab$.

dangling heads) of M_w according to their order of appearance around T_w. By convention, the root is considered as the first rooting head.

Let w^+ be the word w followed by $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$. We obtain w^+ by making the tour of the tree T_w and writing a the first time we follow an edge of the tree, $\overline{a}$ the second time, b when we cross a head not in the tree and $\overline{b}$ when we cross a tail not in the tree. Each prefix of w^+ corresponds to a given time in this journey. In particular, w corresponds to a given corner c of a vertex v. The $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$ at the end of w^+ correspond to the left border of active edges followed from c to the root. Thus, the active edges are the edges on the directed path of T_w from the root to v. Note that an active edge precedes another one if it appears before on the path from the root to v. Therefore, v is the last rooting vertex and c is the corner at the left of the last rooting head. Moreover, active edges are directed from a rooting vertex to the next one (for the appearance order). In particular, the next-to-last rooting vertex (if it exists) is the origin of the last active edge.

We now explore the relation between M_w and $M_{w\alpha}$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$.

Lemma 22 *Let c be the corner at the left of the last rooting head of M_w.*

- *M_{wa} is obtained from M_w by adding an edge e in the corner c. It is oriented from this corner to a vertex not present in M_w. The edge e is the last active edge of M_{wa}.*
- *M_{wb} is obtained from M_w by adding a dangling head h in the corner c. The head h is the last dangling head of M_{wb}.*
- *$M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. The origin of e becomes the last rooting vertex.*
- *$M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner c and connecting it to the last dangling head.*

In any case, the appearance order on the edges, half-edges and vertices present in M_w is the same in $M_{w\alpha}$.

Proof: As mentioned above, the corner c is the corner reached when the word w is written during the tour of T_w in M_w.

• Case $\alpha = a$. The letter a added to w is not paired. Therefore, it corresponds to a new active edge e added to T_w. This new edge is added in the corner c. The edge e is oriented from c to a new vertex (since it is leaf of T_{wa}). All active edges of M_w are encountered before c around the spanning tree T_w. Therefore, e is the last active edge of M_{wa}.

• Case $\alpha = b$. The letter b added to w is not paired. Therefore, it corresponds to a new dangling head h. This new head is added in the corner c. All dangling heads of M_w are encountered before c around the spanning tree T_w. Therefore, h is the last dangling head of M_{wb}.

• Case $\alpha = \overline{a}$. The last letter a of w is paired with the letter $\overline{a}$ added to w. This last letter a corresponds to the last active edge. Therefore, the last active edge e of M_w is inactivated. We know that the next-to-last rooting vertex of M_w is the origin v of the last active edge e. Therefore, v becomes the last rooting vertex.

• Case $\alpha = \overline{b}$. The last letter b of w is paired with the letter $\overline{b}$ added to w. This last letter b corresponds to the last dangling head h'. Hence, $M_{w\overline{b}}$ is obtained from M_w by adding a tail h in the corner c and connecting it to h'. $\square$

This completes our study of prefix-maps. We are now ready to extend the mappings φ_0 and φ_1 to prefix maps and to prove Theorem 21.

4.4 The trees $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ are the same

In this subsection, we prove that, when $w = \Xi(M_T)$, the trees $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ are the same.

Let w be a prefix-shuffle and M_w the corresponding prefix-map. Note that any vertex of M_w is incident to at least one head. The *prefix-forest* of w, denoted by F_w, is obtained by deleting the tails of active edges and then applying the vertex explosion process of Figure 9 (we forget about the cells corresponding to the parts of the non-crossing partition). We will prove that the prefix-forest is indeed a forest (i.e. a collection of trees) in Proposition 23. For instance, we represented the prefix-forest of $w = bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab$ in Figure 26.

Note that, if $w = \Xi(M_T)$ is a parenthesis-shuffle, the prefix-map M_w is $\vec{M}^T$ and no edge is active. Thus, in this case, the prefix-forest F_w is the tree $\varphi_0(\vec{M}^T)$. We now prove a relation between the prefix-forest F_w and the tree-sequence $\lambda_0(w)$.

Proposition 23 *Let w be a prefix-shuffle. Let $h_1 < \cdots < h_k$ be the dangling heads and $h'_1 < \cdots < h'_l$ be the rooting heads of the prefix-map M_w (linearly ordered by the appearance order). The prefix-forest F_w is a collection of $k + l$ trees $t_1, \ldots, t_k, t'_1, \ldots, t'_l$. The root of the tree $t_i, i = 1, \ldots, k$ is h_i and the root of the tree $t'_i, i = 1, \ldots, l$ is h'_i. Moreover, the tree-sequence $\lambda_0(w)$ is $ut_1u \ldots ut_kut'_lv \ldots vt'_1v$.*

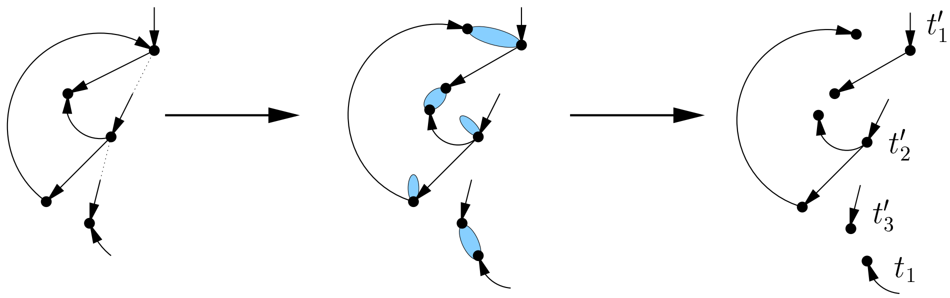


Figure 26: The prefix-forest F_w (on the right).

Proof: We use Lemma 22 and prove the property by induction on the length of w.
If w is the empty word, the prefix-map M_w is the tree τ reduced to a vertex and a root. Hence, the prefix-forest F_w is reduced to a single tree $\tau = t'_1$. The tree-sequence $\lambda_0(w')$ is equal to $u\tau v$ thus the property is satisfied. If $w' = w\alpha$, we suppose the lemma true for w, we write $\lambda_0(w) = ut_1u \dots ut_kut'_lv \dots vt'_1v$ and study separately the four possible cases.
• Case $\alpha = a$. The prefix-map M_{wa} is obtained from M_w by adding an edge e incident to the last rooting vertex. The edge e is the last active edge of M_{wa}. It is oriented toward a new vertex v not present in M_w. The tail of e is deleted in the construction of F_{wa} and its head $h = h'_{l+1}$ is only incident to v. Therefore, F_{wa} is obtained from F_w by adding the tree $\tau = t'_{l+1}$ (the tree reduced to a root and a vertex) rooted on the last rooting head h. By definition, $\lambda_0(wa) = ut_1u \dots ut_ku\tau vt'_lv \dots vt'_1v$, so we observe that the property is satisfied by wa.
• Case $\alpha = b$. The prefix-map M_{wb} is obtained from M_w by adding a dangling head $h = h_{k+1}$ in the corner at the left of the last rooting head h'_l. Therefore, during the vertex explosion process h "steals" the tree t'_l rooted on h'_l in F_w (see Figure 27). That is, in F_{wb} the tree rooted on h'_l is reduced to a vertex and the tree rooted on h is t'_l. The head h is the last dangling head of M_{wb}.
By definition, $\lambda_0(wb) = ut_1u \dots ut_kut'_lu\tau vt'_{l-1} \dots vt'_1v$, so we observe that the property is satisfied by wb.
• Case $\alpha = \overline{a}$. The prefix-map $M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. The origin v of e is the next-to-last rooting vertex of M_w. Moreover, e is the first edge encountered in clockwise order around v starting from h'_{l-1}. In $F_{w\overline{a}}$, the head h'_l is part of the edge e which links the tree t'_l to the tree t'_{l-1} rooted on h'_{l-1} (see Figure 28). Therefore, the tree rooted on h'_{l-1} in $F_{w\overline{a}}$ is $t = \sigma(t'_l, t'_{l-1})$.
By definition, $\lambda_0(w\overline{a}) = ut_1u \dots ut_kutvt'_{l-2} \dots vt'_1v$, so we observe that the property is satisfied by $w\overline{a}$.
• Case $\alpha = \overline{b}$. The prefix-map $M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner at the left of the last rooting head h'_l and connecting it to the last dangling head h_k. In $F_{w\overline{b}}$, the head h_k is part of an edge e which links the tree t_k to the tree t'_l rooted on h'_l. Therefore, the tree rooted on h'_l in $F_{w\overline{b}}$ is $t = \sigma(t_k, t'_l)$. The illustration would be the same as Figure 28 except $h'_{l-1}, h'_l, t'_{l-1}, t'_l$ would be replaced by h'_l, h_k, t'_l, t_k respectively.

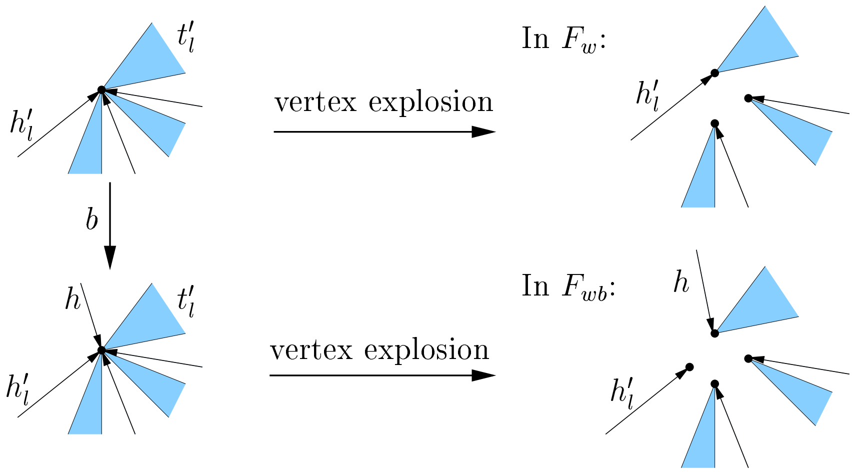


Figure 27: The case $\alpha = b$.

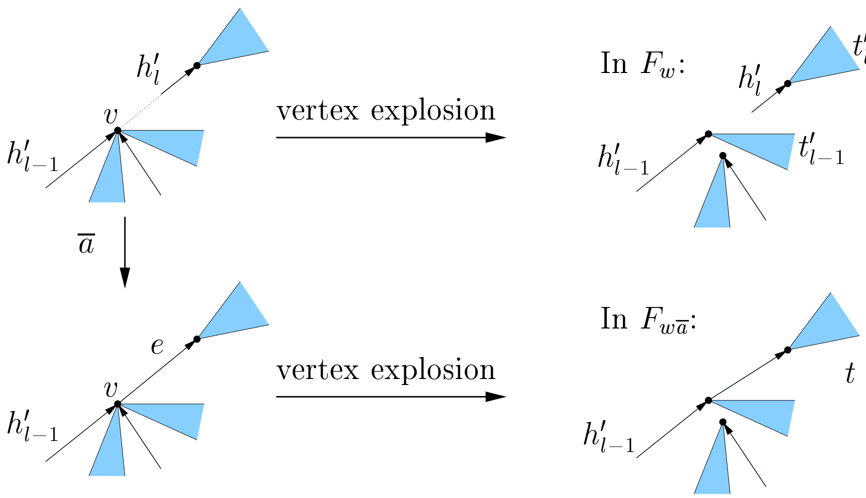


Figure 28: The case $\alpha = \overline{a}$.

By definition, $\lambda_0(w\bar{b}) = ut_1u \dots t_{k-1}utvt'_{l-1}v \dots vt'_1v$, so we observe that the property is satisfied by $w\bar{b}$.

$\square$

As mentioned above, when w is a parenthesis-shuffle $w = \Xi(M_T)$, the prefix-map M_w is the tree-oriented map $\vec{M}^T$ and the prefix-forest F_w is the tree $\varphi_0(\vec{M}^T)$. Therefore, Proposition 23 implies that the tree-sequence $\lambda_0(w)$ is equal to $u\varphi_0(\vec{M}^T)v$. Thus, the trees $\lambda'_0(w)$ and $\varphi_0(\vec{M}^T)$ are the same.

$\square$

4.5 The partitions $\varphi_1(\vec{M}^T)$ and $\Theta \circ \lambda'_1(w)$ are the same

In this subsection, we prove that, when $w = \Xi(M_T)$, the non-crossing partition $\varphi_1(\vec{M}^T)$ is the image of the binary tree $\lambda'_1(w)$ by the mapping Θ defined in Definition 20.

Let M_T be a tree-rooted map. The *partition-tree* of M_T is the tree $P = \Upsilon \circ \varphi_1(\vec{M}^T)$ (the mapping Υ is represented in Figure 6). Observe that the tree P can be drawn directly on the map obtained after the vertex explosion process of Figure 9. To do so, one keeps the cells corresponding to the vertices of M_T (these cells are glued to the first corner of the vertices of the tree $\varphi_0(\vec{M}^T)$). Then, one draws a vertex in each face of M_T and in each cell corresponding to a vertex of M_T: this gives the vertices of P. The edges of P join vertices in adjacent cells and faces. The tree is rooted canonically. In particular, the root-vertex of P lies in the root-face of M_T. This construction is illustrated in Figure 29.

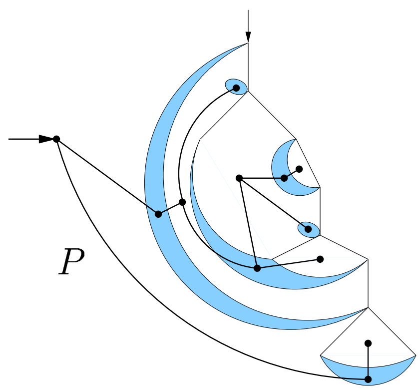


Figure 29: The partition-tree of a tree-rooted map.

We want to extend this construction to prefix-maps. We need some extra vocabulary. Consider a prefix-shuffle w and the corresponding prefix map M_w. We denote by $M_w^\star$ the map obtained after the vertex explosion process when one keeps the cells corresponding to the vertices of M_w. A face of $M_w^\star$ is said *white* if it corresponds to a face of M_w and *black* if it corresponds to a vertex of M_w. For instance, the map $M_w^\star$ in Figure 30 has 2 white faces and 4 black faces. The edges of $M_w^\star$ that correspond to edges of M_w are called *regular*. The edges of $M_w^\star$ that separate black and white faces are called *permeable*. The map $M_w^\star$ inherits the root of M_w. In particular, it has the same root-face. The map $M_w^\star$ has $k = |w|_b - |w|_{\bar{b}}$ dangling heads which are all in the root-face. We can compare these heads according to their order of appearance *around* the root-face, that is, when following its border in counterclockwise direction starting from the root. We denote by $h_1, \ldots, h_k$ the heads of $M_w^\star$ encountered in this order around the root-face.

We define the *partition-tree* P_w of the prefix-map M_w as follows. (We shall prove later that the partition-tree is indeed a tree.) We draw a vertex in each face of $M_w^\star$.

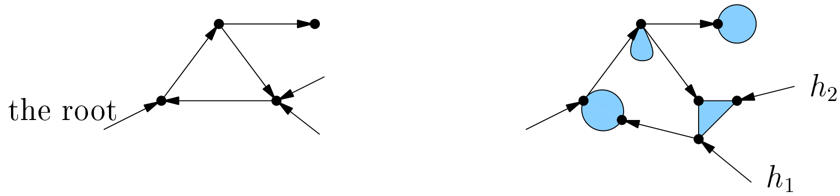


Figure 30: The prefix-map associated to $w = baa\bar{b}bb\bar{a}a$ and the map $M_w^\star$.

The vertex v_0 drawn in the root-face is called the *exterior vertex*. We draw k additional vertices $v_1, \ldots, v_k$ in the root-face, each associated to a dangling head (v_i is associated to h_i). These are the vertices of P_w. The edges of P_w are the *duals* of permeable edges. We need to be more precise. If e is a permeable edge that is not incident to the root-face, its dual joins the vertices drawn in the incident black and white faces. If e is a permeable edge incident to the root-face and a black face f, its dual joins the vertex drawn in f to v_i if h_i is the last dangling head encountered before e around the root-face, or to v_0 if no dangling head precedes e. Note that the partition-tree P_w can be drawn in such a way that no edge of P_w intersects another. For instance, the partition-tree associated to $w = baa\bar{b}bb\bar{a}a$ is shown in Figure 31.

Moreover the vertices of the partition-tree have a color and an *activity*. The vertices of P_w corresponding to *white* and *black* faces of $M_w^\star$ are called *white* and *black* vertices respectively. The *active* white vertices are $v_0, \ldots, v_k$. The *active* black vertices are the vertices corresponding to rooting vertices of M_w (see Subsection 4.3 where the notion of *rooting vertex* is introduced). The other vertices are said to be *inactive*.

It remains to define the root of the partition-tree. Consider the first edge e followed around the root-face of $M_w^\star$. It is a permeable edge. Its dual e^* in P_w joins the exterior vertex v_0 to the vertex drawn in the black face corresponding to the root-vertex of M_w. The root of P_w is incident to v_0 and follows e^* in counterclockwise direction around v_0. This root is indicated in Figure 31.

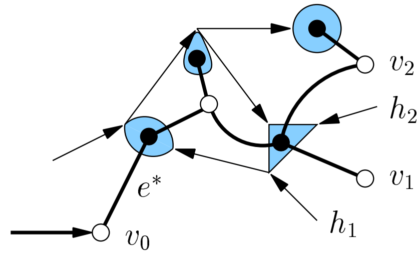


Figure 31: The partition-tree P_w (thick lines) drawn on $M_w^\star$ ($w = baa\bar{b}bb\bar{a}a$).

Observe that, when $w = \Xi(M_T)$ is a parenthesis-shuffle, the map $M_w = \vec{M}^T$ has no dangling heads and the partition-tree P_w is $\Upsilon \circ \varphi_1(\vec{M}^T)$.

We now relate the partition-tree P_w to the binary tree $\lambda_1(w)$.

Proposition 24 *For all prefix-shuffles w, the partition-tree P_w is equal to $\theta \circ \lambda_1(w)$ where $\lambda_1(w)$ is the binary tree defined in Definition 17 and θ is the mapping defined in Definition 20.*

Proposition 24 implies that for any parenthesis-shuffle $w = \Xi(M_T)$ we have $P_w = \theta \circ \lambda_1'(w)$. Given that $P_w = \Upsilon \circ \varphi_1(\vec{M}^T)$, we obtain $\varphi_1(\vec{M}^T) = \Theta \circ \lambda_1'(w)$.

The rest of this subsection is devoted to the proof of Proposition 24. We first describe a recursive construction of the partition-tree P_w. That is, we describe how to obtain $P_{w\alpha}$ from P_w when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$ (Lemma 25). Then we describe a recursive construction of $\theta \circ \lambda_1(w)$ (Lemma 26). We conclude the proof by induction on the length of w.

4.5.1 Recursive construction of the partition-tree P_w

The recursive description of the partition-tree requires us to define an order on active vertices. Let w be a prefix-shuffle and M_w be the associated prefix-map. The rooting vertices of M_w can be compared by their order of appearance around the spanning tree T_w of M_w. The active black vertices inherit their order from the rooting vertices. The black vertex of P_w corresponding to the root-vertex of M_w is the first element for this order. We can also compare the dangling heads $h_1, \dots, h_k$ of M_w according to their order of appearance around T_w. This order *is the same* as the order of appearance around the root-face of $M_w^\star$. Indeed, the order of appearance around the root-face of $M_w^\star$ is also the order of appearance around the root-face of M_w. Furthermore, the deletion of an edge of M_w not in T_w does not modify this order. By deleting all the edges not in T_w we obtain the appearance order around T_w. The active white vertices inherit their order from the dangling heads. The exterior vertex v_0 is considered the first element. That is, v_i *precedes* v_j for $0 \leq i \leq j \leq k$.

Let v be a vertex of a tree which is not a leaf. The son of v following (resp. preceding) the father of v (or the root if v is the root-vertex) in counterclockwise direction around v is called *leftmost son* (resp. *rightmost son*); see Figure 32.

We are now ready to describe the relation between the partition-tree P_w and the partition-tree $P_{w\alpha}$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$.

Lemma 25 *The partition-tree P_w is a tree. Moreover,*

- *the partition-tree P_{wa} is obtained from P_w by adding a new leaf which becomes the last active black vertex. This leaf is the leftmost son of the last active white vertex,*

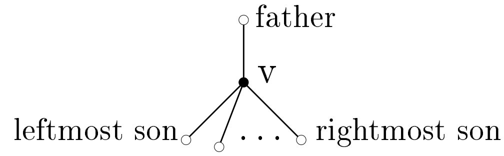


Figure 32: A vertex and its leftmost and rightmost sons.

- *the partition-tree P_{wb} is obtained from P_w by adding a new leaf which becomes the last active white vertex. This leaf is the rightmost son of the last active black vertex,*
- *the partition-tree $P_{w\overline{a}}$ is obtained from P_w by inactivating the last active black vertex,*
- *the partition-tree $P_{w\overline{b}}$ is obtained from P_w by inactivating the last active white vertex.*

To illustrate this lemma we have represented the evolution of a partition-tree in Figure 33. Active vertices are represented by circles and inactive ones by squares. The white (resp. black) active vertices are denoted $v_0, v_1, \ldots$ (resp. $r_1, r_2, \ldots$).

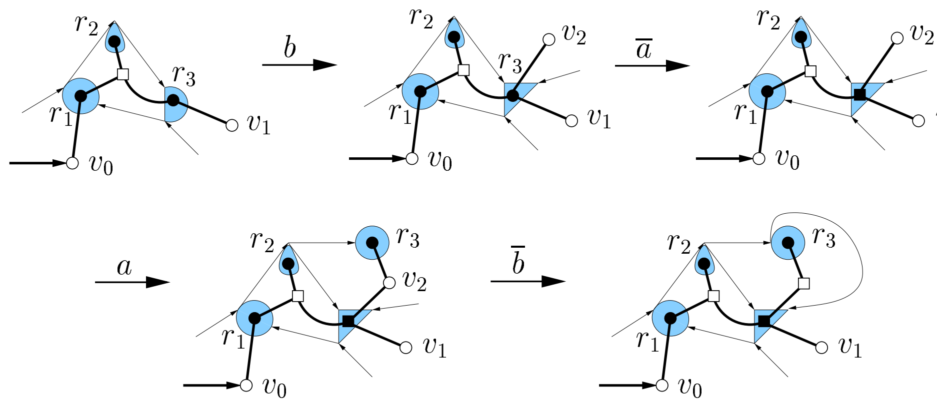


Figure 33: Evolution of the partition-tree from $w = baa\overline{b}b$ to $w = baa\overline{b}bb\overline{a}a\overline{b}$.

Before we embark on the proof, we need to define a correspondence E (resp. V) between the heads of M_w and the edges (resp. vertices distinct from v_0) of P_w. The correspondences E and V are represented in Figure 34.
Consider a head h of M_w and its end v in $M_w^\star$. The edge following h in counterclockwise direction around v is a permeable edge. The dual of this edge in the partition-tree P_w is denoted $E(h)$. The correspondence E between heads of M_w and edges of P_w is one-to-one. The edge $E(h)$ is incident to a white and to a black vertex. If h is in the tree T_w (in particular, if h is the root), we define $V(h)$ as the black vertex incident to $E(h)$. Else $V(h)$ is the white vertex incident to $E(h)$. The correspondence V is a bijection between heads of M_w and vertices of P_w distinct from v_0. Indeed, black vertices of P_w correspond to vertices of M_w which are in one-to-one correspondence with heads in T_w, white vertices

distinct from $v_0, \ldots, v_k$ correspond to faces of M_w which are in one-to-one correspondence with heads not in T_w (a face f is associated with the head we cross when we first enter f during the tour of T_w), and the vertices $v_1, \ldots, v_k$ are in one-to-one correspondence with the dangling heads $h_1, \ldots, h_k$.

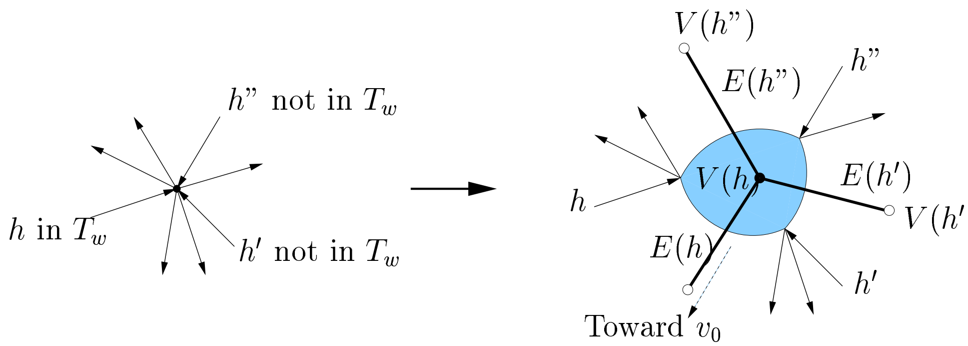


Figure 34: Left: a typical vertex of the prefix map M_w incident with three heads: h in T_w and h' , h'' not in T_w. Right: the correspondence E (resp. V) between heads of M_w and edges (resp. vertices) of P_w.

Proof: We prove the lemma by induction on the length of w. If w is the empty word, P_w is a tree. Suppose now, by induction hypothesis, that P_w is a tree. We first show the following property: *for any head h of M_w, the edge $E(h)$ links $V(h)$ to its father in P_w.* The mapping $V \circ E^{-1}$ is a bijection from the edges of P_w to the vertices of P_w distinct from its root-vertex v_0. Moreover an edge e of P_w is always incident to the vertex $V \circ E^{-1}(e)$ in P_w. Since P_w is a tree, the only possibility is that any edge e of P_w links the vertex $V \circ E^{-1}(e)$ to its father in P_w.

We are now ready to study separately the different cases $\alpha = a, \overline{a}, b, \overline{b}$. We use Lemma 22 and denote by c the corner of M_w at the left of the last rooting head of M_w.

• Case $\alpha = a$.

- The prefix-map M_{wa} is obtained from M_w by adding a new edge e in the corner c oriented away from c. Let h be the head of e and s its end. The vertex s is the last rooting vertex in M_{wa}. The partition-tree P_{wa} is obtained from P_w by adding the edge $E(h)$ and the black vertex $V(h)$ to P_w (see Figure 35). By definition, the vertex $V(h)$ is the last active black vertex in P_{wa}.

- By definition, the corner c is situated after any dangling head around T_w. Hence, it is situated after any dangling head around the root-face of $M_w^\star$. Therefore, the edge $E(h)$ joins $V(h)$ to the last active white vertex v_k. Moreover, since $V(h)$ is only incident to $E(h)$ and P_w is a tree, we check that P_{wa} is a tree and $V(h)$ a leaf.

- It remains to show that $V(h)$ is the leftmost son of v_k. By definition, the permeable edges that have their dual incident to v_k are situated between h_k (or the root h_0 of $M_w^\star$ if $k = 0$) and c around the root-face of $M_w^\star$. The dual of the first of these permeable edge is $E(h_k)$ and the dual of the last of them is $E(h)$. If $k \neq 0$, we know that $E(h_k)$ links $v_k = V(h_k)$ to its father in P_w. Therefore, $V(h)$ is the leftmost son of v_k. If $k = 0$, we know (by definition) that the root of P_w follows $E(h_0)$ in counterclockwise direction around v_0. Therefore, $V(h)$ is the leftmost son of v_0.

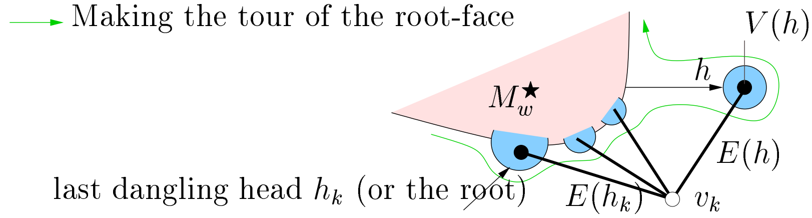


Figure 35: The new vertex $V(h)$ is the leftmost son of v_k.

• Case $\alpha = b$.
We denote by h and v the last rooting head and vertex.
- The prefix-map M_{wb} is obtained from M_w by adding a dangling head h_{k+1} in the corner c. It is the last dangling head of M_{wb}. The partition-tree P_{wb} is obtained by adding the vertex $v_{k+1} = V(h_{k+1})$ and the edge $E(h_{k+1})$ to P_w (see Figure 36). By definition, v_{k+1} is the last active white vertex of P_{wb}.
- The dangling head h_{k+1} is incident to v in M_{wb}. Hence, the edge $E(h_{k+1})$ joins v_{k+1} to the last active black vertex $V(h)$ of P_w. Moreover, since v_{k+1} is only incident to $E(h_{k+1})$ and P_w is a tree, P_{wb} is a tree and v_{k+1} a leaf.
- It remains to prove that v_{k+1} is the rightmost son of $V(h)$. By definition, $E(h_{k+1})$ and $E(h)$ are respectively the dual of the permeable edges preceding and following the head h in counterclockwise direction around its end. Therefore, $E(h)$ follows $E(h_{k+1})$ in counterclockwise direction around $V(h)$. Given that $E(h)$ links $V(h)$ to its father, v_{k+1} is the rightmost son of $V(h)$.

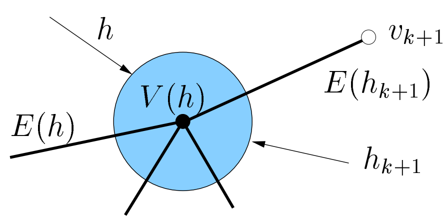


Figure 36: The new vertex v_{k+1} is the rightmost son of $V(h)$.

• Case $\alpha = \overline{a}$.
The prefix-map $M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. Thus, $P_{w\overline{a}}$ is obtained from P_w by inactivating the last active black vertex.
• Case $\alpha = \overline{b}$.
The prefix-map $M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner c and connecting it to the last dangling head h_k. This creates a new face of M_w (hence of $M_w^\star$) and lowers by one the number of dangling heads. The last active white vertex v_k is trapped in the

new face of $M_{w\bar{b}}$. Hence, $P_{w\bar{b}}$ is obtained from P_w by inactivating the last active black vertex v_k. $\square$

4.5.2 Recursive construction of the tree $\theta \circ \lambda_1(w)$.

We continue the proof of Proposition 24. We now describe the relation between the trees $\theta \circ \lambda_1(w)$ and $\theta \circ \lambda_1(w\alpha)$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$ (the mapping λ_1 is defined in Definition 17).

We first need to define a correspondence between the leaves of a binary tree B and the vertices of the tree $\theta(B)$. An edge of B is said *left* (resp. *right*) if it links a node to its left son (resp. right son). We consider a leaf l of B. If l is a left (resp. right) leaf, the path from l to the root begins with a non-empty sequence of left (resp. right) edges. By definition, only the last edge $e(l)$ of this sequence is branching except if l is the first left leaf in which case no edge is branching. We associate the first left leaf of B with the root-vertex of $\theta(B)$ and we associate any other leaf l with the son of the branching edge $e(l)$ in $\theta(B)$. This correspondence is one-to-one. For instance, the leaves $l_1, \dots, l_6$ of the binary tree B in Figure 37 are associated with the vertices $v_1, \dots, v_6$ of the tree $\theta(B)$.

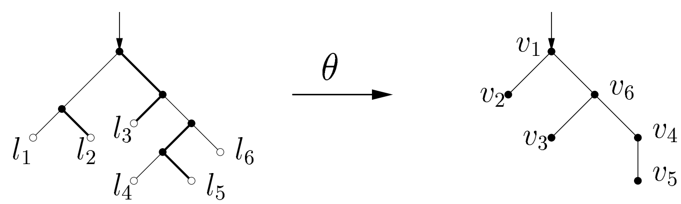


Figure 37: Correspondence between leaves of B and vertices of $\theta(B)$.

Consider a prefix-shuffle w. In the binary tree $\lambda_1(w)$, leaves are either active or inactive. We say that a vertex of $\theta \circ \lambda_1(w)$ is *left*, *right*, *active* or *inactive*, if the associated leaf of $\lambda_1(w)$ is so. Moreover, the leaves of the binary tree $\lambda_1(w)$ can be compared by their order of appearance around this tree. The vertices of $\theta \circ \lambda_1(w)$ inherit this order. For instance, the root-vertex of $\theta \circ \lambda_1(w)$ is the first active left vertex (recall that the first left leaf of $\lambda_1(w)$ is always active).

Our final lemma is the counterpart of Lemma 25.

Lemma 26 *Let $\mathcal{T}$ be the tree $\theta \circ \lambda_1(w)$ and $\mathcal{T}_\alpha = \theta \circ \lambda_1(w\alpha)$ for α in $\{a, b, \overline{a}, \overline{b}\}$.*

- *The tree $\mathcal{T}_a$ is obtained from $\mathcal{T}$ by adding a new leaf which becomes the first active right vertex. This leaf is the leftmost son of the last active left vertex.*
- *The tree $\mathcal{T}_b$ is obtained from $\mathcal{T}$ by adding a new leaf which becomes the last active left vertex. This leaf is the rightmost son of the first right vertex.*

- *The tree $\mathcal{T}_{\overline{a}}$ is obtained from $\mathcal{T}$ by inactivating the first active right vertex.*
- *The tree $\mathcal{T}_{\overline{b}}$ is obtained from $\mathcal{T}$ by inactivating the last active left vertex.*

Proof: We study separately the four cases $\alpha = a, b, \overline{a}, \overline{b}$.
• Case $\alpha = a$. By definition of the mapping λ_1 (Definition 17), the binary tree $\lambda_1(wa)$ is obtained from $\lambda_1(w)$ by replacing the last active left leaf l by a node with two leaves l_l and l_r. The left leaf l_l replaces l as the last left leaf. The right leaf l_r becomes the first right leaf. The edge from l to l_r is branching. The other branching edges are unchanged. Therefore, $\mathcal{T}_a$ is obtained from $\mathcal{T}$ by adding a new leaf. This leaf is associated with l_r hence becomes the first active right vertex. The father of this leaf was associated with l in $\mathcal{T}$ and is associated with l_l in $\mathcal{T}_a$. Therefore, it was and remains the last active left vertex. It is easily seen that the new leaf becomes its leftmost son.
• The case $\alpha = b$ is symmetric to the case $\alpha = a$. We do not detail it.
• Case $\alpha = \overline{a}$. The binary tree $\lambda_1(w\overline{a})$ is obtained from $\lambda_1(w)$ by inactivating the first active right leaf. Therefore, $\mathcal{T}_{\overline{a}}$ is obtained from $\mathcal{T}$ by inactivating the first active right vertex.
• The case $\alpha = \overline{b}$ is symmetric to the case $\alpha = \overline{a}$. □

4.5.3 Recursive proof of Proposition 24.

We want to show that, for any prefix-shuffle w, the partition-tree P_w is the tree $\theta \circ \lambda_1(w)$. We show by induction the following more precise property: for any prefix-shuffle w,
- the partition-tree P_w is equal to $\theta \circ \lambda_1(w)$,
- the active and inactive vertices of P_w and $\theta \circ \lambda_1(w)$ are the same,
- the white (resp. black) vertices of P_w correspond to left (resp. right) vertices of $\theta \circ \lambda_1(w)$,
- the order on white (resp. black) vertices of P_w is equal (resp. inverse) to the order on left (resp. right) vertices of $\theta \circ \lambda_1(w)$.

Suppose that w is the empty word. The partition-tree P_w has one edge, an active white vertex which is its root-vertex and an active black vertex. Similarly, $\theta \circ \lambda_1(w)$ has one edge, an active left vertex which is its root-vertex and an active right vertex. Hence, we check that the property is true.

In view of Lemma 25 and Lemma 26, it is clear that the property is true by induction on the set of prefix-shuffles. □

This concludes the proof of Proposition 24 and Theorem 21. □

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