The maximum number of perfect matchings in graphs with a given degree sequence

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Abstract

We show that the number of perfect matchings in a simple graph G with an even number of vertices and degree sequence $d_1, d_2, \dots, d_n$ is at most $\prod_{i=1}^{n}(d_i!)^{\frac{1}{2d_i}}$. This bound is sharp if and only if G is a union of complete balanced bipartite graphs.

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1 Introduction

Let $G = (V, E)$ be an undirected simple graph. For a vertex $v \in V$, let $\deg v$ denote its degree. Assume that $|V|$ is even, and let $\operatorname{perfmat} G$ denote the number of perfect matchings in G. The main result of this short note is:

Theorem 1.1

$$\operatorname{perfmat} G \le \prod_{v \in V}((\deg v)!)^{\frac{1}{2\deg v}}, \tag{1.1}$$

where $0^{\frac{1}{0}} = 0$. If G has no isolated vertices then equality holds if and only if G is a disjoint union of complete balanced bipartite graphs.

For bipartite graphs the above inequality follows from the Bregman-Minc Inequality for permanents of $(0, 1)$ matrices, mentioned below.

The inequality (1.1) was known to Kahn and Lovász, c.f. [2, (7)], but their proof was never published, and it was recently stated and proved independently by the second author in [3]. Here we show that it is a simple consequence of the Bregman-Minc Inequality.

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2 The proof

Let A be an $n \times n$ $(0,1)$ matrix, i.e. $A = [a_{ij}]_{i,j=1}^n \in \{0,1\}^{n\times n}$. Denote $r_i = \sum_{j=1}^n a_{ij}$, $i = 1, \ldots, n$. The celebrated Bregman-Minc inequality, conjectured by Minc [4] and proved by Bregman [1], states

$$\operatorname{perm} A \leq \prod_{i=1}^n (r_i!)^{\frac{1}{r_i}}, \tag{2.1}$$

where equality holds (if no r_i is zero) iff up to permutation of rows and columns A is a block diagonal matrix in which each block is a square all-1 matrix.

Proof of Theorem 1.1: The square of the number of perfect matchings of G counts ordered pairs of such matchings. We claim that this is the number of spanning 2-regular subgraphs H of G consisting of even cycles (including cycles of length 2 which are the same edge taken twice), where each such H is counted 2^s times, with s being the number of components (that is, cycles) of H with more than 2 vertices. Indeed, every union of a pair of perfect matchings M_1, M_2 is a 2-regular spanning subgraph H as above, and for every cycle of length exceeding 2 in H there are two ways to decide which edges came from M_1 and which from M_2.

The permanent of the adjacency matrix A of G also counts the number of spanning 2-regular subgraphs H' of G, where now we allow odd cycles and cycles of length 2 as well. Here, too, each such H' is counted 2^s times, where s is the number of cycles of H' with more than 2 vertices, (as there are 2 ways to orient each such cycle as a directed cycle and get a contribution to the permanent). Thus the square of the number of perfect matchings is at most the permanent of the adjacency matrix, and the desired inequality follows from Bregman-Minc by taking the square root of (2.1), where the numbers r_i are the degrees of the vertices of G.

It is clear that if G is a vertex-disjoint union of balanced complete bipartite graphs then equality holds in (1.1). Conversely, if G has no isolated vertices and equality holds, then equality holds in (2.1), and no r_i is zero. Therefore, after permuting the rows and columns of the adjacency matrix of G it is a block diagonal matrix in which every block is an all-1 square matrix, and as our graph G has no loops, this means that it is a union of complete balanced bipartite graphs, completing the proof. $\Box$

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