Bijective Proofs of Identities from Colored Binary Trees

Sherry H.F. Yan

Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China

hfy@zjnu.cn

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Abstract

This note provides bijective proofs of two combinatorial identities involving generalized Catalan number $C_{m,5}(n) = \frac{m}{5n+m}\binom{5n+m}{n}$ recently proposed by Sun.

1 Introduction

Recently, by using generating functions and Lagrange inversion formula, Sun [2] deduced the following identity involving generalized Catalan number $C_{m,5}(n) = \frac{m}{5n+m}\binom{5n+m}{n}$, i.e.,

$$\sum_{p=0}^{\lfloor n/4 \rfloor} \frac{m}{5p+m}\binom{5p+m}{p}\binom{n+p+m-1}{n-4p} = \sum_{p=0}^{\lfloor n/2 \rfloor} (-1)^p \frac{m}{m+n}\binom{m+n+p-1}{p}\binom{m+2n-2p-1}{n-2p}, \tag{1.1}$$

which, in the case $m = 1$, leads to

$$\sum_{p=0}^{\lfloor n/4 \rfloor} \frac{1}{5p+1}\binom{5p+1}{p}\binom{n+p}{n-4p} = \sum_{p=0}^{\lfloor n/2 \rfloor} (-1)^p \frac{1}{n+1}\binom{n+p}{p}\binom{2n-2p}{n-2p}. \tag{1.2}$$

In this note, we give a parity reversing involution on colored binary trees which leads to a combinatorial interpretation of Formula (1.2). We make a simple variation of the bijection between colored ternary trees and binary trees proposed by Sun [2] and find a correspondence between a certain class of binary trees and the set of colored 5-ary trees. The generalization of the parity reversing involution and the bijection to forests of colored binary trees and forests of colored 5-ary trees leads to a bijective proof of Formula (1.1).

2 A parity reversing involution on colored binary trees

In this section, we provide a parity reversing involution on a class of colored binary trees. Before introducing the involution, we recall some definitions and notations. An *internal vertex* of a binary tree is a vertex that has children. Let $\mathcal{B}_n$ denote the set of full binary trees with n internal vertices. Let $B \in \mathcal{B}_n$ and $P = v_0 v_1 \dots v_k$ be a path of length k of B (viewing from the root). P is called a *L-path* if v_i is a left child of v_{i-1} for $1 \le i \le k$. P is called a *maximal L-path* if there exists no vertex such that uP or Pu forms a L-path. Suppose that $P = v_0 v_1 \dots v_k$ is a maximal L-path, then v_0 is called an *initial vertex* of B and P is called the *associate path* of v_0. Denote by $l(v)$ the length of the associate L-path of v. A *colored binary tree* is a binary tree in which each initial vertex v is assigned a color $c(v)$ such that $0 \le c(v) \le \lfloor l(v)/2 \rfloor$. The *color number* of a colored binary tree B, denoted by $c(B)$, is equal to the sum of all the colors of initial vertices of B. A colored binary tree B with $c(B) = 1$ is illustrated in Figure 1.

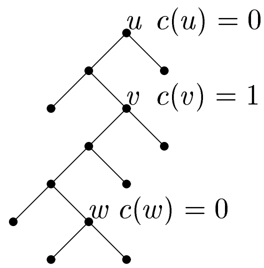

Figure 1: A colored binary tree B with $c(B) = 1$.

Let $\mathcal{CB}_{n,p} = \{B | B \in B_n \text{ with } c(B) = p\}$ and $F_{n,p}$ be its cardinality. Define $\mathcal{CB}_n = \bigcup_{p=0}^{\lfloor n/2 \rfloor} \mathcal{CB}_{n,p}$. Let $F(x,y) = \sum_{n \ge 0} \sum_{p=0}^{\lfloor n/2 \rfloor} f_{n,p} y^p x^n$ be the ordinary generating function for $f_{n,p}$ with the assumption $f_{0,0} = 1$. Given a colored binary tree $B \in \mathcal{CB}_n$, let v be the root of B. Suppose that $c(v) = k$, then $l(v) \ge 2k$. Then the generating function for the number of colored binary trees whose root v has color k is equal to $\sum_{k \ge 0} y^k \sum_{r \ge 2k} x^r F^r(x,y)$, where r is the length of the maximal L-path from the root. Summing over all the possibilities for $k \ge 0$, we arrive at

$$F(x,y) = \frac{1}{(1 - yx^2 F^2(x,y))(1 - xF(x,y))}.$$

By applying Lagrange inversion formula, we get

$$F^m(x,y) = \sum_{n \ge 0} \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{m}{n+m} \binom{m+n+p-1}{p} \binom{m+2n-2p-1}{n-2p} y^p x^n,$$

which, in the case of $m = 1$, reduces to

$$F(x,y) = \sum_{n \ge 0} \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{n+1} \binom{n+p}{p} \binom{2n-2p}{n-2p} y^p x^n. \tag{2.1}$$

Let $B \in \mathcal{CB}_n$ and v be an initial vertex of B. If $c(v) = 2k$ and $l(v) = 4k$ or $4k+1$ for some nonnegative integer k, then we say v is a *proper* vertex, otherwise it is said to be *improper*. A colored binary tree B is called *proper* if all the initial vertices are proper; otherwise, B is said to be *improper*. Denote by $\mathcal{CB}'_n$ the set of all proper binary trees with n internal vertices. Define $\mathcal{ECB}_n$ and $\mathcal{OCB}_n$ to be the sets of colored binary trees with n internal vertices whose color numbers are even and odd, respectively.

Theorem 2.1 *There is a parity reversing involution ϕ on the set of improper colored binary trees with n internal vertices. It follows that,*

$$|\mathcal{ECB}_n| - |\mathcal{OCB}_n| = |\mathcal{CB}'_n|.$$

Proof. Let B be an improper colored binary trees with n internal vertices with $c(B) = p$. Traverse the binary tree B by depth first search, let v be the first improper vertex traversed. If $c(v)$ is even, then let $\phi(B)$ be a colored binary tree obtained from B by coloring v by $c(v)+1$. If $c(v)$ is odd, then let $\phi(B)$ be a colored binary tree obtained from B by coloring v by $c(v) - 1$. Obviously, the obtained binary tree $\phi(B)$ is an improper colored binary tree in both cases. Furthermore, in the former case the color number of $\phi(B)$ is $p+1$, while in the latter case, the color number is $p-1$. Hence ϕ is an involution on the set of improper colored binary trees with n internal vertices, which reverses the parity of the color number. Since each initial vertex v of a proper binary tree is colored by an even number, it is clear that $\mathcal{CB}'_n \subseteq \mathcal{ECB}_n$. Hence, we have

$$|\mathcal{ECB}_n| - |\mathcal{OCB}_n| = \mathcal{CB}'_n.$$

∎

From Theorem 2.1 and Formula 2.1, we see that the right side summand of Formula (1.2) counts the number of proper colored binary trees, that is,

$$|\mathcal{CB}'_n| = \sum_{p=0}^{\lfloor n/2 \rfloor} (-1)^p \frac{1}{n+1} \binom{n+p}{p} \binom{2n-2p}{n-2p}.$$

3 The bijective proof

A (complete) *k-ary tree* is an ordered tree in which each internal vertex has k children. The number of k-ary trees with n internal vertices is counted by generalized Catalan number $C_{1,k}(n) = \frac{1}{kn+1}\binom{kn+1}{n}$ [1]. A *colored* 5-ary tree is a 5-ary tree in which each vertex is assigned a nonnegative integer called *color number*, denoted by c_v. Let $\mathcal{T}_{n,p}$ denote the set of colored 5-ary trees with p internal vertices such that the sum of all the color numbers of each tree is $n - 4p$. Denote by $\mathcal{T}_n = \bigcup_{p=0}^{\lfloor n/4 \rfloor} \mathcal{T}_{n,p}$. Let $B \in \mathcal{CB}'_n$ be a proper binary tree with n internal vertices. Since each initial vertex v of B is colored by $\lfloor l(v)/2 \rfloor$, we can discard all the colors of vertices of B and B can be viewed as a full binary tree with n internal vertices in which each maximal L-path has length $\equiv 0$ or 1 (mod 4).

Now we construct a map σ from $\mathcal{T}_n$ to $\mathcal{CB}'_n$ as follows:

1. For each vertex v of $T \in \mathcal{T}_n$ with color number $c_v = k$, remove the color number and add a path $P = v_1 v_2 \ldots v_k$ to v such that v is a right child of v_k and v_1 is a child of the father of v, and annex a left leaf to v_i for $1 \leq i \leq k$. See Figure 2(a) for example.

2. Let T^* be the tree obtained from T by Step 1. For any internal vertex v of T^* which has 5 children, let T_1, T_2, T_3, T_4, T_5 be the five subtrees of v. Annex a path $P = v_1 v_2 v_3$ to v such that v_1 is a left child of v, then take T_1 and T_2 as the left and right subtree of v_3, take T_3 as the right subtree of v_2, and take T_4 as the subtree of v_1. See Figure 2(b) for example.

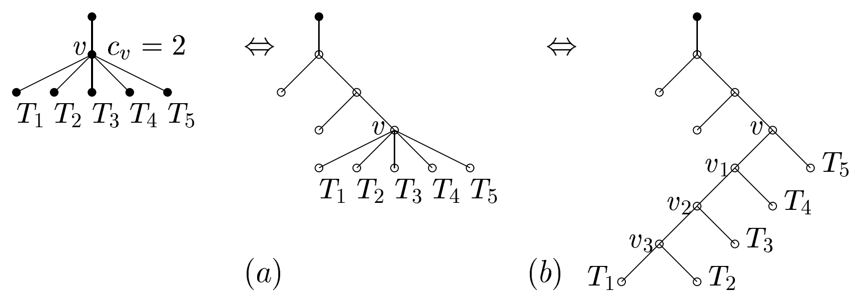

Figure 2: The bijection σ.

It is clear that the obtained tree $\sigma(T)$ is a proper binary tree with n internal vertices.

Conversely, we can obtain a colored 5-ary tree from a proper binary tree by a similar procedure from binary trees to colored ternary trees given by Sun [2]. We omit the reverse map of σ here.

Theorem 3.1 *The map σ is a bijection between $\mathcal{T}_n$ and $\mathcal{CB}'_n$.*

A 5-ary tree with p internal vertices has $5p+1$ vertices altogether. Given such a tree T, choose $n-4p$ of its $5p+1$ vertices, repetition allowed– $\binom{n+p}{n-4p}$ choices– and define the color number of each vertex to be the number of times it was chosen. Thus there are $\binom{n+p}{n-4p}$ colored 5-ary trees in $\mathcal{T}_n$ whose underlying trees are T. Since there are $\frac{1}{5p+1}\binom{5p+1}{p}$ 5-ary trees with p internal vertices, the involution ϕ and the bijection σ lead to a combinatorial proof of Formula (1.2).

Recall that an *n-Dyck path* is a lattice path from $(0,0)$ to $(2n,0)$ that does not go below the x-axis and consists of up steps $(1,1)$ and down steps $(1,-1)$. Note that there is a standard bijection from full binary trees on $2n$ edges to n-Dyck paths. Given a full binary tree, walk counterclockwise around the tree starting at the root and process in turn each edge that has not been traversed such that a left edge corresponds to an up step and a right edge corresponds to a down step. An *ascent* of a Dyck path is a maximal sequence of contiguous up steps and its length is the number of up steps in it. From the bijection σ, we have the following result.

Corollary 3.2 *The left side of (1.2) counts n-Dyck paths in which each ascent A of an n-Dyck path D has length $\equiv 0$ or 1 (*mod 4) by the statistic $\sum_{A\in D}\lfloor \text{length(A)}/4\rfloor$.

In order to prove Formula (1.1), we consider the forest of colored binary trees $F = (B_1, B_2, \ldots, B_m)$ with n internal vertices and m components where $B_i \in \mathcal{CB}_{n_i}$ and $n_1 + n_2 + \ldots + n_m = n$. Define the *color number* of F as the sum of all the color numbers of B_i, $1 \leq i \leq m$. It is easy to check that the number of forests of colored binary trees with n internal vertices and m components, whose color number equals p, is counted by

$$[y^p x^n]F^m(x,y) = \frac{m}{n+m}\binom{m+n+p-1}{p}\binom{m+2n-2p-1}{n-2p}.$$

F is said to be a *proper* forest if each B_i is proper; otherwise, F is said to be *improper*. Now we can modify the involution ϕ as follows: suppose that B_k is the leftmost improper binary tree, then let $\phi(F)$ be the forest of colored binary trees obtained from F by changing B_k to $\phi(B_k)$. From Theorem 2.1, we see $\phi(F)$ is an improper forest of colored binary trees with n internal vertices and m components. Hence the modified involution ϕ is an involution on the set of improper forests of colored binary trees, which reverses the parity of the color numbers of the forests. Hence the right side summand of Formula (1.1) counts the number of proper forests of colored binary trees with n internal vertices and m components.

Let $F = (T_1, T_2, \ldots, T_m)$ be a forest of 5-ary trees such that $T_i \in \mathcal{T}_{n_i}$ and $n_1 + n_2 + \ldots + n_m = p$. Define $\sigma(F) = (\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_m))$. From Theorem 3.1, it is clear that $\sigma(F)$ is a proper forest of binary trees. Note that there are totally $m + 5p$ vertices in a forest F of 5-ary trees with p internal vertices and m components, so there are $\binom{n+p+m-1}{n-4p}$ forests of colored 5-ary trees with p internal vertices and m components, in which the sum of the color numbers of each forest equals to $n - 4p$. Hence the modified involution ϕ and the modified bijection σ lead to a bijective proof of Formula (1.1).

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