A sharp bound for the reconstruction of partitions

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Abstract

Answering a question of Cameron, Pretzel and Siemons proved that every integer partition of $n \geq 2(k+3)(k+1)$ can be reconstructed from its set of k-deletions. We describe a new reconstruction algorithm that lowers this bound to $n \geq k^2+2k$ and present examples showing that this bound is best possible.

Analogues and variations of Ulam's notorious graph reconstruction conjecture have been studied for a variety of combinatorial objects, for instance words (see Schützenberger and Simon [2, Theorem 6.2.16]), permutations (see Raykova [4] and Smith [5]), and compositions (see Vatter [6]), to name a few.

In answer to Cameron's query [1] about the partition context, Pretzel and Siemons [3] proved that every partition of $n \geq 2(k+3)(k+1)$ can be reconstructed from its set of k-deletions. Herein we describe a new reconstruction algorithm that lowers this bound, establishing the following result, which Negative Example 2 shows is best possible.

Theorem 1. *Every partition of $n \geq k^2+2k$ can be reconstructed from its set of k-deletions.*

We begin with notation. Recall that a *partition of n*, $\lambda = (\lambda_1, \dots, \lambda_\ell)$, is a finite sequence of nonincreasing integers whose sum, which we denote $|\lambda|$, is n. The *Ferrers diagram* of λ, which we often identify with λ, consists of ℓ left-justified rows where row i contains λ_i cells. An *inner corner* in this diagram is a cell whose removal leaves the diagram of a partition, and we refer to all other cells as *interior cells*.

We write $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all i; another way of stating this is that $\mu \leq \lambda$ if and only if μ is contained in λ (here identifying partitions with their diagrams). If $\mu \leq \lambda$, we write λ/μ to denote the set of cells which lie in λ but not in μ. We say that the partition μ is a *k-deletion* of the partition of λ if $\mu \leq \lambda$ and $|\lambda/\mu| = k$.

Recall that this order defines a lattice on the set of all finite partitions, known as *Young's lattice*, and so every pair of partitions has a unique *join* (or *least upper bound*)

$$\mu \vee \lambda = (\max\{\mu_1, \lambda_1\}, \max\{\mu_2, \lambda_2\}, \dots)$$

and *meet*

$$\mu \wedge \lambda = (\min\{\mu_1, \lambda_1\}, \min\{\mu_2, \lambda_2\}, \dots).$$

Finally, recall that the *conjugate* of a partition λ is the partition λ' obtained by flipping the diagram of λ across the NW-SE axis; it follows that λ'_i counts the number of entries of λ which are at least i.

Before proving Theorem 1 we show that it is best possible:

Negative Example 2. For $k \geq 1$, consider the two partitions

$$\begin{aligned}\mu &= (\underbrace{k+1, \dots, k+1}_{k}, k-1) \text{ and}\\ \lambda &= (\underbrace{k+1, \dots, k+1}_{k-1}, k, k).\end{aligned}$$

Note that no k-deletion of μ can contain the cell $(k, k+1)$ and that no k-deletion of λ can contain the cell $(k+1, k)$. Therefore every k-deletion of μ and of λ is actually a $(k-1)$-deletion of

$$\mu \wedge \lambda = (\underbrace{k+1, \dots, k+1}_{k-1}, k, k-1),$$

so μ and λ cannot be differentiated by their sets of k-deletions.

We are now ready to prove our main result.

Proof of Theorem 1. Suppose that we are given a positive integer k and a set Δ of k-deletions of some (unknown) partition λ of $n \geq k^2 + 2k$. Our goal is to determine λ from this information. We begin by setting $\mu = \bigvee_{\delta \in \Delta} \delta$, noting that we must have $\lambda \geq \mu$. Hence if $|\mu| = n$ then we have $\lambda = \mu$ and we are immediately done, so we will assume that $|\mu| < n$.

First consider the case where μ has less than k rows. Let r denote the bottommost row of μ which contains at least k cells (r must exist because μ has less than k rows and $|\mu| \geq k^2 + k$). Thus the rth row of λ contains at least k cells as well, so there are k-deletions of λ in which the removed cells all lie in or below row r. Hence the first $r-1$ rows of λ and μ agree. Now note that λ has more than $2k$ cells to the right of column k, so there are k-deletions of λ in which the removed cells all lie to the right of column k, and thus the first k columns of λ and μ agree. This implies that λ and μ agree on all rows below r (since these rows have less than k cells in μ) and so all cells of λ/μ must lie in row r, uniquely determining λ, as desired. The case where μ has less than k columns follows by symmetry.

We may now assume that μ has at least k rows and k columns. Let r (resp. c) denote the bottommost row (resp. rightmost column) containing at least k cells. Both r and c exist because μ has at least k rows and columns. Therefore both λ and μ can be divided into three quadrants, 1, 2, and 3, as shown in Figure 1.

As before, we see that the first $r-1$ rows and $c-1$ columns of λ and μ agree. We consider three cases based on whether and where r and c intersect.

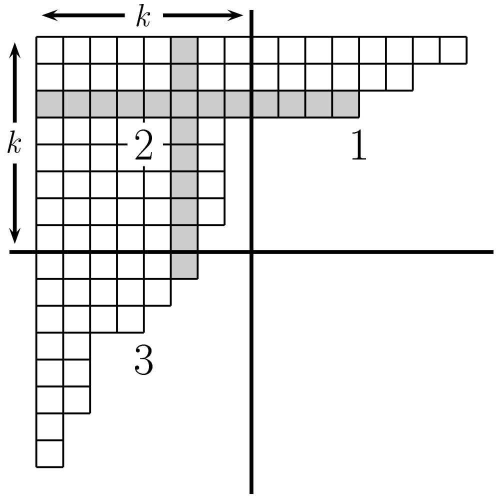

Figure 1: An example partition μ from Case 1 of the proof of Theorem 1, divided into three quadrants. Here $k = 8$, and r and c appear shaded.

Case 1: r and c intersect at an interior cell of μ. Suppose that r and c intersect at the cell (i, j). It follows from the maximality of r and c that $i, j < k$, and thus the cell (k, k) does not lie in μ. Were the cell (k, k) to lie in λ then, because $|\lambda| \geq k^2 + 2k$, λ must contain at least $2k$ cells to the right of or below (k, k) and thus λ would contain a k-deletion with the cell (k, k), a contradiction; thus λ also does not contain (k, k).

Hence Quadrant 2 of λ contains less than k^2 cells, so λ must have more than k cells in quadrant 1 or 3. Hence there are k-deletions of λ with more than k cells in quadrant 1 or 3; suppose by symmetry that λ and μ both have more than k cells in quadrant 1.

There are then k-deletions of λ in which the removed cells are all chosen from quadrant 1, so λ and μ agree on all cells in quadrants 2 and 3. This shows that r is also the bottommost row of λ with at least k cells, and so λ/μ contains no cells below row r in quadrant 1. As we already know that λ and μ agree on their first $r - 1$ rows, we can therefore conclude that all cells of λ/μ lie in row r, which allows us to reconstruct λ and complete the proof of this case.

Case 2: r and c intersect at an inner corner of μ. Then this inner corner must be the rightmost cell of row r and the bottom cell of column c. It follows that $r, c \geq k$. Because λ and μ agree to the left of column c and above row r, all cells of λ/μ must lie below or to the right of (r, c). However, the cell $(r + 1, c + 1)$ cannot lie in λ because if it did then one could form a k-deletion of λ by removing only points lying to the right of column c, which would leave at least k cells in row $r + 1$ and contradict the definition of r. This leaves only two possibilities for λ/μ: the cells $(r, c + 1)$ and $(r + 1, c)$. However, only one of these cells can be added to μ to produce a partition; if both could be added then row $r + 1$ and column $c + 1$ of λ would each contain at least k cells, implying the existence of k-deletions of λ in which each contain at least k cells and thus contradicting the choice of r and c. This case therefore reduces to checking which one of the cells $(r, c + 1)$ and $(r + 1, c)$ can be added to μ to produce a partition.

Case 3: r and c do not intersect. Suppose that the rightmost cell in row r is (r, j) and the bottommost cell in column c is (i, c). If $j < c - 1$ then because λ and μ agree to the left of column c, λ/μ cannot contain any cells in or below row r, and we already have that λ and μ agree above row r, so we are left with the conclusion that $\lambda = \mu$. By symmetry we are also done if $i < r - 1$, leaving us to consider the case where $i = r - 1$ and $j = c - 1$. Again using the fact that λ and μ agree above row r and to the left of column c (and the definitions of r and c) we see that the only possibility for λ/μ is (r, c), completing the proof of this case and the theorem. $\square$

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