Partitions and Edge Colourings of Multigraphs

Alexandr V. Kostochka* and Michael Stiebitz†

Submitted: May 23, 2007; Accepted: Jul 1, 2008; Published: Jul 6, 2008

Abstract

Erdős and Lovász conjectured in 1968 that for every graph G with $\chi(G) > \omega(G)$ and any two integers $s, t \geq 2$ with $s + t = \chi(G) + 1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. Except for a few cases, this conjecture is still unsolved. In this note we prove the conjecture for line graphs of multigraphs.

1 Introduction

It was conjectured by Erdős and Lovász (see Problem 5.12 in [2]) that for every graph G with $\chi(G) > \omega(G)$ and any two integers $s, t \geq 2$ with $s + t = \chi(G) + 1$, there is a partition (S, T) of the vertex set $V(G)$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. The only settled cases of this conjecture that we know are $(s, t) \in \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)\}$ (see [1, 3, 5, 6]). In this note we prove for the line graphs of multigraphs the following slightly stronger statement.

Theorem 1 *Let s and t be arbitrary integers with $2 \leq s \leq t$. If the line graph $L(G)$ of some multigraph G has chromatic number $s + t - 1 > \omega(L(G))$, then it contains a clique Q of size s such that $\chi(L(G) - Q) \geq t$.*

It will be convenient to prove the theorem in the language of edge colorings of multigraphs. Every *multigraph* in this note is finite, undirected and has no loops.

The *edge set* and the *vertex set* of G is denoted by $V(G)$ and $E(G)$ respectively. For a vertex v of G, the *degree*, $d(v)$, of v in G is the number of edges incident with v. The set N_v of all neighbours of v in G may have much smaller size than $d(v)$.

*Department of Mathematics, University of Illinois, Urbana, IL 61801 and Institute of Mathematics, Novosibirsk 630090, Russia. E-mail address: kostochk@math.uiuc.edu. This material is based upon work supported by NSF Grants DMS-0400498 and DMS-06-50784 and grant 06-01-00694 of the Russian Foundation for Basic Research.

†Institute of Mathematics, Technische Universität Ilmenau, D-98684 Ilmenau, Germany. E-mail address: Michael.Stiebitz@tu-ilmenau.de.

The *chromatic index* of G, denoted by $\chi'(G)$, is the chromatic number of its line graph $L(G)$; in other words, it is the smallest number of colours with which the edges of G may be coloured so that no two adjacent edges receive the same colour.

A *triangle* in G is a set of three mutually adjacent vertices in G, and the edges of a triangle are those edges in $E(G)$ joining the vertices of the triangle. The maximum number of edges in a triangle in G will be denoted by $\tau(G)$. Furthermore, let $\Delta(G)$ denote the *maximum degree* of G, and let $\omega'(G) = \max\{\tau(G), \Delta(G)\}$. Clearly, $\omega'(G)$ is the clique number of the line graph of G and hence $\chi'(G) \geq \omega'(G)$.

2 Proof of Theorem 1

For given $2 \leq s \leq t$, suppose that G is a counterexample with the fewest vertices. Then G is connected. Since $\chi'(G) > \omega'(G) \geq \tau(G)$, G contains at least four vertices. By Shannon's theorem [4], $\chi'(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$. Consequently, $s \leq \Delta(G)$.

By an *s-star* of G we mean a pair (E', v) such that $E' \subseteq E(G)$ is a set of s edges incident with the vertex v. For an s-star (E', v), let $X(E', v)$ denote the set of all vertices of G joined by an edge of E' with v.

Let (E', v) be an arbitrary s-star of G. The set E' forms an s-clique in $L(G)$. Since G is a counterexample to our theorem, we have $\chi'(G - E') \leq t - 1$. Let $G' = G - E'$, and let $\varphi : E(G') \longrightarrow \{1, \ldots, t-1\}$ be a $(t-1)$-edge-colouring of G'. For each vertex x of G, let

$$\varphi(x) = \{\varphi(e) |\, e \in E(G') \text{ is incident with}\, x\} \text{ and } \bar{\varphi}(x) = \{1, \ldots, t-1\} \setminus \varphi(x).$$

Since $s + t - 1 = \chi'(G) > \omega'(G) \geq \Delta(G)$ and all s edges of E' are incident with v, the degree of v in $G' = G - E'$ is at most $t - 2$ and, therefore,

(a) $\bar{\varphi}(v) \neq \emptyset$.

Next, we claim that

(b) *for every colour $\alpha \in \bar{\varphi}(v)$ and for any two distinct vertices $x, y \in X(E', v)$, there is an edge $e \in E(G')$ joining x and y with $\varphi(e) = \alpha$. Consequently, $|X(E', v)| \leq 2$.*

Proof. Suppose to the contrary that no edge joining x and y is colored with α. For $u \in \{x, y\}$, there is an edge $e_u \in E'$ joining u and v. Colour the $s - 1$ edges of $E' \setminus \{e_x\}$ with colours $t, t+1, \ldots, t+s-2$, so that e_y is coloured with t. If $\alpha \in \bar{\varphi}(x)$, we can colour the edge e_x with α. Otherwise, there is an edge $e \in E(G) \setminus E'$ incident with x colored with α. Since e is not incident with y, we can recolour e with colour t and then colour e_x with α. In both cases we obtain a $(t + s - 2)$-edge-colouring of G, a contradiction to $s + t - 1 = \chi'(G)$. □

(c) *Let w be a vertex of G with $d(w) \geq s$. Then, for the neighbourhood N_w of w in G, we have $|N_w| \geq 2$, and any two vertices of N_w are adjacent in G. Furthermore, if $s \geq 3$, then $|N_w| = 2$.*

Proof. If N_w consists only of a single vertex w', then $d(w') \geq d(w) \geq s$. Since G is connected and has at least four vertices, w' has a neighbour $x \neq w$. Hence there is an s-star (E', w') of G with $w, x \in X(E', w')$. From (a) and (b) it then follows that x and w are adjacent in G, a contradiction to $|N_w| = 1$. This proves that $|N_w| \geq 2$. If x, y are two distinct neighbours of w, then there is an s-star (E', w) with $x, y \in X(E', w)$. Then (a) and (b) imply that x and y are adjacent. If $s \geq 3$ and $|N_w| \geq 3$, then there is an s-star (E', w) such that $|X(E', w)| \geq 3$, a contradiction to (b). Hence (c) is proved. □

To complete the proof of Theorem 1, we consider two cases.

Case 1: $s \geq 3$. Since $s \leq \Delta(G)$, there is a vertex u in G with $d(u) \geq s$. By (c), N_u consists of two vertices, say x and y, and these two vertices are adjacent in G. Since G is a connected graph with at least four vertices, either N_x or N_y contains more than two vertices, say $|N_x| \geq 3$. Then (c) implies that $d(x) < s$. Let E_1 denote the set of all edges of G joining x with u or y. Furthermore, let E_2 denote the set of all edges of G joining u with y. Then $2 \leq |E_1| < s$ and $|E_1| + |E_2| \geq s$. Hence, there is a nonempty subset E_2' of E_2 such that $E' = E_1 \cup E_2'$ contains exactly s edges. Since E' is an s-clique in $L(G)$, by the choice of G, we have $\chi'(G - E') \leq t - 1$. Let $G' = G - E'$, and let $\varphi : E(G') \longrightarrow \{1, \ldots, t-1\}$ be any $(t-1)$-edge-colouring of G'. If $\varphi(u) = \{1, \ldots, t-1\}$, then $\{u, x, y\}$ is a triangle with at least $s + t - 1$ edges, a contradiction to $\tau(G) < \chi'(G) = s + t - 1$. Hence there is a colour $\alpha \in \bar{\varphi}(u)$. Choose two edges $e_1 \in E_1$ and $e_2 \in E_2'$. Colour the $s - 1$ edges of $E' \setminus \{e_1\}$ with colours $t, t+1, \ldots, t+s-2$ so that e_2 is coloured with t. If $\alpha \in \bar{\varphi}(x)$, then we can colour the edge e_1 with α. Otherwise, there is an edge $e \in E(G) \setminus E'$ such that e is incident with x and $\varphi(e) = \alpha$. Since all edges joining x with y are in E', the edge e is not incident with y and we can recolour e with t and then colour e_1 with α. In both cases we obtain a $(t+s-2)$-edge colouring of G, a contradiction to $s + t - 1 = \chi'(G)$.

Case 2: $s = 2$. Since $s \leq \Delta(G)$, it follows from (c) that G contains a triangle $T = \{x, y, z\}$.

For $u \in \{y, z\}$, there is an edge e_u in G joining u and x. The pair (E', x) with $E' = \{e_y, e_z\}$ is an s-star of G and, therefore, $\chi'(G - E') \leq t - 1$. Let $G' = G - E'$, and let $\varphi : E(G') \longrightarrow \{1, \ldots, t-1\}$ be any $(t-1)$-edge-colouring of G'.

Since T contains at most $\tau(G) \leq \chi'(G) - 1 = t$ edges and two of these edges are not coloured, some colour $\alpha \in \{1, \ldots, t-1\}$ is not present on edges of T. By (b), $\alpha \in \varphi(x)$. Hence the following two subcases finish the proof of the theorem.

Case 2.1: $\alpha \in \bar{\varphi}(y) \cup \bar{\varphi}(z)$. By the symmetry between y and z, we can suppose that $\alpha \in \bar{\varphi}(y)$. By (a) and (b), there is a colour $\beta \in \bar{\varphi}(x)$ and an edge e' of colour β joining y and z. Uncolour e' and colour e_z with β. This results in a $(t-1)$-edge-colouring φ' of $G - E''$, where $E'' = \{e_y, e'\}$. Then $\alpha \in \bar{\varphi}'(y)$ and no edge joining x and z has colour α. Since (E'', y) is an s-star of G, this is a contradiction to (b).

Case 2.2: $\alpha \in \varphi(x) \cap \varphi(y) \cap \varphi(z)$. This means that for every $u \in T$, there is an edge $e^u \in E(G')$ of colour α joining u and some vertex $v_u \notin T$. Let $\beta \in \bar{\varphi}(x)$ and P be the component containing x of the subgraph $H_{\alpha,\beta}$ induced by the set of edges $\{e \in E(G') \,|\, \varphi(e) \in \{\alpha, \beta\}\}$. Obviously, P is a path starting at x. By (b), there is an edge e' of colour β joining y and z and we eventually consider two cases.

Subcase A: Edge e' does not belong to P. If we interchange the colours α and β on P, then we obtain a new $(t-1)$-edge-colouring φ' of G'. Then φ' is a $(t-1)$-edge-colouring of G' with $\alpha \in \bar{\varphi}'(x)$ and $\varphi'(e^y) = \varphi'(e^z) = \alpha$. In particular, no edge of $G' = G - E'$ joining y and z has colour α, a contradiction to (b).

Subcase B: Edge e' belongs to P. In this case, e^y and e^z also belong to P. By symmetry, we may assume that the subpath P' of P joining y with x does not contain z. Uncolour e' and colour $e_y \in E'$ with β. This results in a $(t-1)$-edge-colouring φ' of $G - \{e_z, e'\}$ for which Subcase A with z in place of x and e_y in place of e' holds. Since Subcase A is settled, this finishes the whole proof. ∎

References

[1] W. G. Brown and H. A. Jung, On odd circuits in chromatic graphs, *Acta Math. Acad. Sci. Hungar.* **20** (1999), 129–134.

[2] T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley Interscience, New York, 1995.

[3] N. N. Mozhan, On doubly critical graphs with chromatic number five, Technical Report 14, Omsk Institute of Technology, 1986 (in Russian).

[4] C. E. Shannon, A theorem on coloring the lines of a network, *J Math. Phys.* **28** (1949), 148–151.

[5] M. Stiebitz, K_5 is the only double-critical 5-chromatic graph, *Discrete Math.* **64** (1987), 91–93.

[6] M. Stiebitz, On k-critical n-chromatic graphs. In: Colloquia Mathematica Soc. János Bolyai **52**, Combinatorics, Eger (Hungary), 1987, 509–514.