# Rainbow $H$-factors of complete $s$-uniform $r$-partite hypergraphs *

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**Abstract**

We say a $s$-uniform $r$-partite hypergraph is complete, if it has a vertex partition $\{V_1, V_2, \ldots, V_r\}$ of $r$ classes and its hyperedge set consists of all the $s$-subsets of its vertex set which have at most one vertex in each vertex class. We denote the complete $s$-uniform $r$-partite hypergraph with $k$ vertices in each vertex class by $\mathcal{T}_{s,r}(k)$. In this paper we prove that if $h$, $r$ and $s$ are positive integers with $2 \leq s \leq r \leq h$ then there exists a constant $k = k(h, r, s)$ so that if $H$ is an $s$-uniform hypergraph with $h$ vertices and chromatic number $\chi(H) = r$ then any proper edge coloring of $\mathcal{T}_{s,r}(k)$ has a rainbow $H$-factor.

**Keywords:** $H$-factors, Rainbow, uniform hypergraphs.

## 1 Introduction

A hypergraph is a pair $(V, E)$ where $V$ is a set of elements, called vertices, and $E$ is a set of non-empty subsets of $V$ called hyperedges or edges. A hypergraph $H$ is called

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$s$-uniform or an $s$-hypergraph if every edge has cardinality $s$. A graph is just a 2-uniform hypergraph. We say a hypergraph is $r$-partite if it has a vertex partition $\{V_1, V_2, \ldots, V_r\}$ of $r$ classes such that each hyperedge has at most one vertex in each vertex class, and a $s$-uniform $r$-partite hypergraph is complete, if it has a vertex partition $\{V_1, V_2, \ldots, V_r\}$ of $r$ classes and its hyperedge set consists of all the $s$-subsets of its vertex set which have at most one vertex in each vertex class. We denote the complete $s$-uniform $r$-partite hypergraph with $k$ vertices in each vertex class by $\mathcal{T}_{s,r}(k)$.

If $H$ is a hypergraph with $h$ vertices and $G$ is hypergraph with $hn$ vertices, we say that $G$ has an $H$-factor if it contains $n$ vertex disjoint copies of $H$. For example, a $K_2$-factor of a graph is simply a perfect matching. We say an edge coloring of a hypergraph is proper if any two edges sharing a vertex receive distinct colors. We say a subhypergraph of an edge-colored hypergraph is rainbow if all of its edges have distinct colors, and a rainbow $H$-factor is an $H$-factor whose components are rainbow $H$-subhypergraphs.

Many graph theoretic parameters have corresponding rainbow variants. Erdős and Rado[4] were among the first to consider the problems of this type. For graphs, Jamison, Jiang and Ling[3], and Chen, Schelp and Wei[2] considered Ramsey type variants where an arbitrary number of colors can be used; Alon et. al.[1] studied the function $f(H)$ which is the minimum integer $n$ such that any proper edge coloring of $K_n$ has a rainbow copy of $H$; and Keevash et. al.[5] considered the rainbow Turán number $ex^*(n; H)$ which is the largest integer $m$ such that there exists a properly edge-colored graph with $n$ vertices and $m$ edges but containing no rainbow copy of $H$. Recently, Yuster[6] proved that for every fixed graph $H$ with $h$ vertices and chromatic number $\chi(H)$, there exists a constant $K = K(H)$ such that every proper edge coloring of a graph with $hn$ vertices and with minimum degree at least $hn(1 - 1/\chi(H)) + K$ has a rainbow $H$-factor.

For hypergraphs, El-Zanati et al[7] discussed the existence of a rainbow 1-factor in 1-factorizations of $r$-uniform hypergraph; in[8], Bollobás et al considered the edge colorings with local restriction of the complete $r$-uniform hypergraphs. In this paper, we discuss the rainbow $H$-factor in hypergraphs and extend the main result in [6] to uniform hypergraphs. The main idea of our proof also comes from [6], although the details are more complex. The main result in this paper is:

**Theorem 1** *If $h$, $r$ and $s$ are positive integers with $2 \le s \le r \le h$ then there exists a constant $k = k(h, r, s)$ so that if $H$ is an $s$-uniform hypergraph with $h$ vertices and chromatic number $\chi(H) = r$ then any proper edge coloring of $\mathcal{T}_{s,r}(k)$ has a rainbow $H$-factor.*

## 2 Proof of Theorem 1

Let $H$ be a $s$-uniform hypergraph with $h$ vertices and $\chi(H) = r$. It is not difficult to check that $\mathcal{T}_{s,r}(h)$ has an $H$-factor for $\mathcal{T}_{s,r}(h)$ and $H$ have the same chromatic number. So it suffices to show that there exists $k = k(h, r, s)$ such that any proper edge-colored $\mathcal{T}_{s,r}(k)$ has a rainbow $\mathcal{T}_{s,r}(h)$-factor. We shall prove a slightly stronger statement. For $0 < p \le h$, Let $\mathcal{T}_{s,r}(h,\ p)$ be the complete $s$-uniform $r$-partite hypergraph with $h$ vertices in each vertex class, except the last vertex class which has only $p$ vertices. Define $\mathcal{T}_{s,r}(h;0) = \mathcal{T}_{s,r-1}(h;h)$. We prove that there exists $k = k(h,r,s,p)$ such that any proper edge-colored $\mathcal{T}_{s,r}(kh;kp)$ has a rainbow $\mathcal{T}_{s,r}(h;p)$- factor.

Let $h$ be fixed, we prove the result by induction on $r$, and for each $r$, by induction on $p \geq 1$. The base case $r = s$ and $p = 1$ is trivial since every subhypergraph of a proper edge-colored hypergraph $\mathcal{T}_{s,s}(h;1)$ is rainbow. Given $r \geq s$, assuming the result holds for $r$ and $p-1 \geq 1$, we prove it for $r$ and $p$ (if $p = 1$ then $p - 1 = 0$ so we use the induction on $\mathcal{T}_{s,r-1}(h;h)$). Let $k = k(h,r,s,p-1)$ and let $t$ be sufficiently large ($t$ will be chosen later). Consider a proper edge-coloring of $\mathcal{T} = \mathcal{T}_{s,r}(kth;ktp)$. We let $c(x_1, x_2, \ldots, x_s)$ denote the color of the edge $\{x_1, x_2, \ldots, x_s\}$. Denote the first $r-1$ vertex classes of $\mathcal{T}$ by $V_1, \ldots, V_{r-1}$ and the last vertex class by $U_r$. Let $V_r$ be an arbitrary subset of size $k(p-1)t$ and $W = U_r \setminus V_r$ the remaining set with $|W| = kt$. For $i = 1, \ldots, r$, we randomly partition $V_i$ into $t$ subsets $V_i(1), \ldots, V_i(t)$, each of the same size. Each of the $r$ random partitions is performed independently, and each partition is equally likely. Let $S(j)$ be the subhypergraph of $\mathcal{T}$ induced by $V_1(j) \cup V_2(j) \cup \cdots \cup V_r(j)$, for $j = 1, \ldots, t$. Notice that $S(j)$ is a properly edge-colored $\mathcal{T}_{s,r}(kh;k(p-1))$ and hence, by the induction hypothesis $S(j)$ has a rainbow $\mathcal{T}_{s,r}(h;p-1)$-factor. Let $B = (X \cup W; F)$ be a bipartite graph where $X = \{S(j) : j = 1, \ldots, t\}$ and there exists an edge $(S(j), w) \in F$ if for all $1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq r$ and for all $x_{i_k} \in V_{i_k}(j)$ $(k = 1, 2, \ldots, s-1)$, the color $c(x_{i_1}, x_{i_2}, \ldots, x_{i_{s-1}}, w)$ does not appear at all in $S(j)$.

If we can show that, with positive probability, $B$ has a 1-to-$k$ assignment in which each $S(j) \in X$ is assigned to precisely $k$ elements of $W$ and each $w \in W$ is assigned to a unique $S(j)$ then we can show that $\mathcal{T}$ has a rainbow $\mathcal{T}_{s,r}(h;p)$-factor. Indeed, consider $S(j)$ and the unique set $X_j$ of $k$ elements of $W$ that are matched to $S(j)$. Since $S(j)$ has a rainbow $\mathcal{T}_{s,r}(h;p-1)$-factor, we can arbitrarily assign a unique element of $X_j$ to each element of this factor and obtain a $\mathcal{T}_{s,r}(h;p)$ which is also rainbow because all the edges of this $\mathcal{T}_{s,r}(h;p)$ incident with the assigned vertex have colors that do not appear at all in other edges of this $\mathcal{T}_{s,r}(h;p)$. Now we use the 1-to-$k$ extension of Hall's Theorem to prove that $B$ has the required 1-to-$k$ assignment. Namely, we will show that, with positive probability, $|N(Y)| \geq k|Y|$ for each $Y \subseteq X$. (Hall's Theorem is simply the case $k = 1$.) To guarantee this condition, it suffices to prove that, with positive probability, each vertex of $X$ has degree greater than $(k-1/2)t$ in $B$ and each vertex of $W$ has degree greater than $t/2$ in $B$. Because, if $|Y| \leq t/2$, then $|N(Y)| \geq (k - \frac{1}{2})t \geq k|Y|$; if $|Y| > t/2$, then that each vertex of $W$ has degree greater than $t/2$ in $B$ implies that $N(Y) = W$, so $|N(Y)| \geq k|Y|$.

We first prove that each vertex of $X$ has degree greater than $(k-1/2)t$ in $B$. Consider $S(j) \in X$. Let $C(j)$ be the set of all colors appearing in $S(j)$. As $S(j)$ is a $\mathcal{T}_{s,r}(kh;k(p-1))$ we have that $|C(j)| < |E(\mathcal{T}_{s,r}(kh,kh))| = \binom{r}{s}(kh)^s$. For each vertex $x$ of $S(j)$, let $W_x \subset W$ be the set of vertices $w \in W$ such that there exists an edge in $\mathcal{T}$ incident to both $x$ and $w$ with color in $C(j)$. Obviously, $|W_x| \leq (s-1)|C(j)|$ since $\mathcal{T}$ is $s$-uniform and no color appears more than once in edges incident with $x$ for the coloring is proper. Let $W(j)$ be the union of all $W_x$ taken over all vertices of $S(j)$. Then, $|W(j)| < (khr)(s-1)\binom{r}{s}(kh)^s \leq \frac{1}{s}(khr)^{s+1}$. Because each $v \in W \setminus W(j)$ is a neighbor of $S(j)$ in $B$, thus, if we take

$t \geq (khr)^{s+1}$, we have that each $S(j)$ has more than $(k-1/2)t$ neighbors in $B$.

Now we prove the second part: each vertex of $W$ has degree greater than $t/2$ in $B$. Fix some $w \in W$ and let $d_B(w)$ denote the degree of $w$ in $B$. As $d_B(w)$ is a random variable, and since $|W| = kt$, it suffices to prove that $Pr\{d_B(w) \leq t/2\} < 1/kt$ which implies that $Pr\{\exists w : d_B(w) \leq t/2\} < 1$. To simplify notation we let $l_i$ be the size of the $i$'th vertex class of each $S(j)$. Thus $l_i = kh$ for $i = 1, \ldots, r-1$ and $l_r = k(p-1)$. Recall that the $i$'th vertex class of $S(j)$ is formed by taking the $j$'th block of a random partition of $V_i$ into $t$ blocks of equal size $l_i$. Alternatively, one can view the $i$'th vertex class of $S(j)$ as the elements $l_i(j-1)+1, \ldots, l_i j$ of a random permutation of $V_i$ for $i = 1, \ldots, r$. Therefore, Let $\pi_i$ be a random permutation of $V_i$. Thus, for $i = 1, \ldots, r$, $\pi_i(l) \in V_i$ for $l = 1, \ldots, l_i t$. We define the $a$'th vertex of $i$'th vertex class of $S(j)$ to be $\pi_i(l_i(j-1)+a)$ for $i = 1, \ldots, r$ and $a = 1, \ldots, l_i$.

We define the following events. For $2s-1$ vertex classes $V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}}$ with $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq r$ and $1 \leq \beta_1 < \beta_2 < \cdots < \beta_{s-1} \leq r-1$ for a block $S(j)$ where $1 \leq j \leq t$, and positive indices $a_{\alpha_i} \leq l_{\alpha_i}$, $b_{\beta_i} \leq l_{\beta_i}$, let $x_{j,\alpha_i}$ be the $a_{\alpha_i}$'th vertex of vertex class $V_{\alpha_i}$ in $S(j)$ $(1 \leq i \leq s)$, let $y_{j,\beta_k}$ be the $b_{\beta_k}$'th vertex of vertex class $V_{\beta_k}$ in $S(j)$ $(1 \leq k \leq s-1)$. Denote by $A(V_{\alpha_1}, \ldots, V_{\alpha_s}, V_{\beta_1}, \ldots, V_{\beta_{s-1}}, j, a_{\alpha_1}, \ldots, a_{\alpha_s}, b_{\beta_1}, \ldots, b_{\beta_{s-1}})$ the event that $c(x_{j,\alpha_i}, \ldots, x_{j,\alpha_s}) = c(y_{j,\beta_1}, \ldots, y_{j,\beta_{s-1}}, w)$. We now prove the following claim.

**Claim 1** *If $d_B(w) \leq t/2$ then there exist $V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots,$ $b_{\beta_{s-1}}$ and there exists $J \subset \{1,\ 2, \ldots, t\}$ with $|J| > t/(khr)^{2s-1}$ such that for each $j \in J$ the event $A(V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ j,\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})$ holds.*

**Proof of Claim 1.** If $d_B(w) \leq t/2$ then there exists $J' \subset \{1,\ 2, \ldots, t\}$ with $|J'| > t/2$ such that for each $j \in J'$ some event $A(\ldots,\ j,\ \ldots)$ holds. There are $\binom{r}{s}$ choices for $V_{\alpha_1}, \ldots, V_{\alpha_s}$, $\binom{r-1}{s-1}$ choices for $V_{\beta_1}, \ldots, V_{\beta_{s-1}}$, and at most $kh$ choices for each of $a_{\alpha_i},\ b_{\beta_i}$. Hence there exist $V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}}$ and some $J \subset J'$ with

$$|J| \geq \frac{|J'|}{\binom{r}{s}\binom{r-1}{s-1}(kh)^{2s-1}} > \frac{t}{(khr)^{2s-1}},$$

such that for each $j \in J$ the event

$$A(V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ j,\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})$$

holds. So we complete the proof of Claim 1. □

For each subset $J \subset \{1,\ 2, \ldots, t\}$ of cardinality $|J| = \lceil t/(khr)^{2s-1} \rceil$, let

$$\begin{aligned} &A(J, V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}}) \\ &\quad = \cap_{j \in J} A(V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ j,\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}}). \end{aligned}$$

**Claim 2** *If the probability of each of the events*

$$A(J,\ V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})$$

*is smaller than $k^{-2s}h^{-2s+1}r^{-2s+1}2^{-t}t^{-1}$ for each subset $J \subset \{1,\ 2, \ldots, t\}$ of cardinality $|J| = \lceil t/(khr)^{2s-1} \rceil$, then $Pr\{d_B(v) \leq t/2\} < 1/kt$.*

**Proof of Claim 2.** From Claim 1 and the fact that there are less than $2^t$ possible choices for $J$ and less than $(khr)^{2s-1}$ possible choices for $V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s}$, $b_{\beta_1}, \ldots, b_{\beta_{s-1}}$ where $a_{\alpha_i} \le l_{\alpha_i}$ $(1 \le i \le s)$ and $b_{\beta_i} \le l_{\beta_i}$ $(1 \le i \le s-1)$, we have

$$\begin{aligned} Pr\{d_B(v) \le t/2\} &\le \sum_J Pr\left\{A(J, V_{\alpha_1}, \ldots, V_{\alpha_s}, V_{\beta_1}, \ldots, V_{\beta_{s-1}}, a_{\alpha_1}, \ldots, a_{\alpha_s}, b_{\beta_1}, \ldots, b_{\beta_{s-1}})\right\} \\ &< 2^t (khr)^{2s-1} k^{-2s} h^{-2s+1} r^{-2s+1} 2^{-t} t^{-1} = 1/kt, \end{aligned}$$

where the sum is taken over all the events

$$A(J,\ V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})$$

with $J \subset \{1, 2, \ldots, t\}$ of cardinality $\lceil t/(khr)^{2s-1} \rceil$. $\square$

By Claim 2, in order to complete the proof of Theorem 1 it suffices to prove the following claim.

**Claim 3** *Let* $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_s \le r$, $1 \le \beta_1 < \beta_2 < \cdots < \beta_{s-1} \le r-1$, $a_{\alpha_i} \le l_{\alpha_i}$ $(1 \le i \le s)$ *and* $b_{\beta_i} \le l_{\beta_i}$ $(1 \le i \le s-1)$. *If* $J \subset \{1,\ 2, \ldots, t\}$ *of cardinality* $|J| = \lceil t/(khr)^{2s-1} \rceil$, *then*

$$Pr\left\{A(J,\ V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})\right\} < \frac{1}{k^{2s} h^{2s-1} r^{2s-1} 2^t t}.$$

**Proof of Claim 3.** For convenience, let

$$A = A(J,\ V_{\alpha_1}, \ldots, V_{\alpha_s},\ V_{\beta_1}, \ldots, V_{\beta_{s-1}},\ a_{\alpha_1}, \ldots, a_{\alpha_s},\ b_{\beta_1}, \ldots, b_{\beta_{s-1}})$$

and $\Delta = \lceil t/(khr)^{2s-1} \rceil$. We may assume, without loss of generality, that $J = \{1, \ldots, \Delta\}$. For $j \in J$, let $x_{j,\alpha_i}$ be the $a_{\alpha_i}$'th vertex of vertex class $V_{\alpha_i}$ in $S(j)$, let $y_{j,\alpha_i}$ be the $b_{\beta_i}$'th vertex of vertex class $V_{\beta_i}$ in $S(j)$. Suppose that we are given the identity of the $(2s-1)(j-1)+s-1$ vertices

$$x_{1,\alpha_1}, \ldots, x_{1,\alpha_s},\ y_{1,\alpha_1}, \ldots, y_{1,\beta_{s-1}}, \ldots, x_{j-1,\alpha_1}, \ldots, x_{j-1,\alpha_s},\ y_{j-1,\alpha_1}, \ldots, y_{j-1,\beta_{s-1}}$$

and $y_{j,\alpha_1}, \ldots, y_{j,\beta_{s-1}}$ (we assume here that all vertices are distinct otherwise $Pr\{A\} = 0$ for our edge coloring is proper). If we can show that given this information, the probability that $c(x_{j,\alpha_1}, \ldots, x_{j,\alpha_s}) = c(y_{j,\alpha_1}, \ldots, y_{j,\beta_{s-1}},\ w)$ is less than $q$ where $q$ only depends on $t,\ h,\ r,\ s,\ p$, then, by the product formula of conditional probabilities we have $Pr\{A\} < q^{\Delta}$. Thus, assume that we are given the identity of the $(2s-1)(j-1)+s-1$ vertices

$$x_{1,\alpha_1}, \ldots, x_{1,\alpha_s},\ y_{1,\alpha_1}, \ldots, y_{1,\beta_{s-1}}, \ldots, x_{j-1,\alpha_1}, \ldots, x_{j-1,\alpha_s},\ y_{j-1,\alpha_1}, \ldots, y_{j-1,\beta_{s-1}}$$

and $y_{j,\alpha_1}, \ldots, y_{j,\beta_{s-1}}$. In particular, we know the color $c(y_{j,\alpha_1}, \ldots, y_{j,\beta_{s-1}},\ v) = c$. Now we evaluate the probability that $c(x_{j,\alpha_1}, \ldots, x_{j,\alpha_s}) = c$. For $1 \le i \le s$, let

$$\begin{aligned} V'_{j,\alpha_i} = V_{\alpha_i} \setminus \{&x_{1,\alpha_1}, \ldots, x_{1,\alpha_s}, y_{1,\alpha_1}, \ldots, y_{1,\beta_{s-1}}, \ldots, x_{j-1,\alpha_1}, \ldots, x_{j-1,\alpha_s}, \\ &y_{j-1,\alpha_1}, \ldots, y_{j-1,\beta_{s-1}},\ y_{j,\alpha_1}, \ldots, y_{j,\beta_{s-1}}\}. \end{aligned}$$

Each vertex of $V'_{j,\alpha_i}$ has an equal chance of being $x_{j,\alpha_i}$. Thus, each edge of $V'_{j,\alpha_1} \times V'_{j,\alpha_1} \times \cdots \times V'_{j,\alpha_s}$ has an equal chance of being the edge $\{x_{j,\alpha_1}, \ldots, x_{j,\alpha_s}\}$. Obviously, $|V'_{j,\alpha_i}| \geq tkh - 2\Delta$. Since our coloring is proper, the color $c$ appears at most $tkh$ times in $V'_{j,\alpha_1} \times V'_{j,\alpha_1} \times \cdots \times V'_{j,\alpha_s}$. Hence,

$$\begin{aligned} Pr\{c(x_{j,\alpha_1}, \ldots, x_{j,\alpha_s}) = c\} &\leq \frac{tkh}{|V'_{j,\alpha_1}||V'_{j,\alpha_2}| \cdots |V'_{j,\alpha_s}|} \\ &\leq \frac{tkh}{(tkh-2\Delta)^s} < \frac{tkh}{(tkh - tkh/2)^2} = \frac{tkh}{(tkh/2)^s}. \end{aligned}$$

It is not difficult to check that

$$\left(\frac{tkh}{(tkh/2)^s}\right)^{\frac{t}{(khr)^{2s-1}}} < \frac{1}{k^{2s}h^{2s-1}r^{2s-1}2^t t}$$

holds for sufficiently large $t$, an integer-valued function on $k,\ h,\ r,\ s$, by taking log both sides. It implies that for sufficiently large $t$, an integer-valued function on $k,\ h,\ r,\ s$, we have

$$Pr\{A\} < \left(\frac{tkh}{(tkh/2)^s}\right)^{\Delta} \leq \left(\frac{tkh}{(tkh/2)^s}\right)^{\frac{t}{(khr)^{2s-1}}} < \frac{1}{k^{2s}h^{2s-1}r^{2s-1}2^t t}.$$

This completes the proof of Claim 3.
So we have completed the induction step and the proof of Theorem 1. □

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