# The number of 0-1-2 increasing trees as two different evaluations of the Tutte polynomial of a complete graph

C. Merino*

Instituto de Matemáticas,
Universidad Nacional Autónoma de México,
Circuito Exterior, C.U. Coyoacán 04510, México D.F.
merino@matem.unam.mx

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**Abstract**

If $T_n(x,y)$ is the Tutte polynomial of the complete graph $K_n$, we have the equality $T_{n+1}(1,0) = T_n(2,0)$. This has an almost trivial proof with the right combinatorial interpretation of $T_n(1,0)$ and $T_n(2,0)$. We present an algebraic proof of a result with the same flavour as the latter: $T_{n+2}(1,-1) = T_n(2,-1)$, where $T_n(1,-1)$ has the combinatorial interpretation of being the number of 0–1–2 increasing trees on $n$ vertices.

## 1 Introduction

Given a graph $G = (V,E)$, we define the *rank function* of $G$, $r : \mathcal{P}(E) \to \mathbb{Z}$ as $r(A) = |V| - k(A)$ for $A \subseteq E$, where $k(A)$ is the number of connected components in the graph $(V,A)$. The 2-variable graph polynomial $T(G;x,y)$, known as the *Tutte polynomial* of $G$, is defined as

$$T(G;x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{1}$$

The Tutte polynomial of $G$ has many interesting combinatorial interpretations when evaluated on different points $(x,y)$ and along several algebraic curves. One that is particularly interesting is along the line $x = 1$ which can be interpreted as the generating function of critical configuration of the sandpile model, see [8], or as the generating function of the $G$-parking functions, see [9]. When the graph $G$ is the complete graph on $n$ vertices, $K_n$, the latter is the classical generating function of parking functions or the inversion enumerator of labelled trees on $n$ vertices, see [10].

In the following section we prove the main theorem of the paper:

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**Theorem 1.** $T(K_n; 2, -1) = T(K_{n+2}; 1, -1)$.

The last section shows how this result is related to the number of 0-1-2 increasing trees on $n$ vertices.

# 2 $T(K_n; 2, -1)$ and $T(K_{n+2}; 1, -1)$

Let us assume that the vertices of $K_n$ are labelled $1, 2, \ldots, n$. For a spanning tree $A$ of $K_n$, an *inversion* in $A$ is a pair of vertices labelled $i$,$j$ such that $i > j$ and $i$ is on the unique path from 1 to $j$ in $A$. Let inv$A$ be the number of inversions in $A$. The *inversion enumerator* $J_n(y)$ is then defined as the generating function of spanning trees arranged by number of inversions, that is,

$$J_n(y) = \sum_A y^{\mathrm{inv}A} ,$$

where the sum is taken over all spanning trees of $K_n$. Now, from [10], we obtain the exponential generating function of the inversion enumerators,

$$\sum_{n\geq 0} J_{n+1}(y)(y-1)^n \frac{t^n}{n!} = \frac{\sum_{n\geq 0} y^{\binom{n+1}{2}} \frac{t^n}{n!}}{\sum_{n\geq 0} y^{\binom{n}{2}} \frac{t^n}{n!}}. \tag{2}$$

Note that our notation differs from [10], as Stanley uses $I_n(y)$ for $J_{n+1}(y)$.

Let $T_n(x, y)$ be the Tutte polynomial of $K_n$. Welsh in [11] gives the following exponential generating function for $T_n(x, y)$

$$1 + (x-1) \sum_{n\geq 1} (y-1)^n T_n(x, y) \frac{t^n}{n!} = \left( \sum_{n\geq 0} y^{\binom{n}{2}} \frac{t^n}{n!} \right)^{(x-1)(y-1)} \tag{3}$$

With these two general results it is easy to prove the following:

**Theorem 2.** *For $n \geq 0$, $J_{n+2}(-1) = T_n(2, -1)$.*

*Proof.* By taking $y = -1$ in Equation (2) we get

$$\sum_{n\geq 0} J_{n+1}(-1)(-2)^n \frac{t^n}{n!} = \frac{\sum_{n\geq 0} (-1)^{\binom{n+1}{2}} \frac{t^n}{n!}}{\sum_{n\geq 0} (-1)^{\binom{n}{2}} \frac{t^n}{n!}} = \frac{F(t)}{H(t)}.$$

Clearly, $F(t) = H'(t)$, where $H'(t)$ is the derivative of $H(t)$. Then, by integrating both sides of the previous expression and multiplying through by -2 we arrive at the equality

$$\sum_{n\geq 1} J_n(-1)(-2)^n \frac{t^n}{n!} = (-2) \ln |H(t)|.$$

The function $H(t)$ is the exponential generating function of the sequence 1, 1, -1, -1, 1, 1, -1, -1,. . ., so $H(t) = \cos(t) + \sin(t)$. Substituting this value on the above identity we obtain

$$\sum_{n\geq 1} J_n(-1)(-2)^n \frac{t^n}{n!} = (-2) \ln|\cos(t) + \sin(t)|. \tag{4}$$

Now, by differentiating twice both sides of equation (4) we conclude that

$$\sum_{n\geq 0} J_{n+2}(-1)(-2)^n \frac{t^n}{n!} = \frac{1}{(\cos(t) + \sin(t))^2}. \tag{5}$$

Taking $x = 2$ and $y = -1$ in Equation (3), we get the following identities

$$1 + \sum_{n\geq 1} (-2)^n T_n(2, -1) \frac{t^n}{n!} = \left( \sum_{n\geq 0} (-1)^{\binom{n}{2}} \frac{t^n}{n!} \right)^{-2} = \frac{1}{(\cos(t) + \sin(t))^2}. \tag{6}$$

Therefore, from Equations (5) and (6),

$$1 + \sum_{n\geq 1} T_n(2, -1) \frac{(-2)^n t^n}{n!} = \sum_{n\geq 0} J_{n+2}(-1) \frac{(-2)^n t^n}{n!}.$$

As $T_0(2, -1) = 1$, we obtain the result by equating the corresponding coefficients. □

It is known that $T_n(1, y) = J_n(y)$, see [7]. Thus, Theorem 1 follows by the previous result.

A permutation $\sigma \in S_n$ is an *up-down permutation* if $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$. Let $a_n$ be the number of up-down permutation in $S_n$ for $n \geq 1$ and set $a_0 = 1$. The sequence $a_n$ has a nice exponential generating function, namely

$$\sum_{n\geq 0} a_n \frac{t^n}{n!} = \tan(t) + \sec(t)\ .$$

The result is originally from [1] but a proof may also be found in [7]. The fact that the value $J_{n+1}(-1)$ equals $a_n$ is mentioned in [6] but a proof of this together with other evaluations of $J_n(x)$ is given in [7]. As a corollary we obtain

**Corollary 3.** *For $n \geq 0$, $T_n(2, -1) = a_{n+1}$ and*

$$\sum_{n\geq 0} T_n(2, -1) \frac{t^n}{n!} = \sec(t)(\tan(t) + \sec(t)).$$

## 3 The Tutte polynomial and increasing trees

A spanning tree in $K_n$ with root at 1 is said to be *increasing* whenever its vertices increase along the paths away from the root. A *0–1–2 increasing tree* is an increasing tree where all the vertices have at most 2 edges going out. A remarkable result stated in [4] and proved in [5] (see also a bijective proof in [3]) is that $a_n$ equals the number of 0–1–2 increasing trees on $n$ vertices. By using Corollary 3 we get

**Corollary 4.** $T_n(2,-1)$ *equals the number of 0–1–2 increasing trees on* $n+1$ *vertices.*

Thus, the number of 0–1–2 increasing trees on $n$ vertices corresponds two different evaluations of the Tutte polynomial of a complete graph, that is $T_{n-1}(2,-1)$ and $T_{n+1}(1,-1)$.

A similar situation occur for the number of permutations on $n$ letters. The quantity $T(G;2,0)$ equals the number of acyclic orientations of $G$ while $T(G;1,0)$ equals the number of acyclic orientations of $G$ with a unique predefined source, see [2]. If we use this combinatorial interpretation with $K_n$, clearly we get that $T_{n+1}(1,0) = T_n(2,0)$. In fact, it is easy to find the exact values, $T_n(2,0) = n!$ and $T_n(1,0) = n-1!$. That is, the number of permutations on $n$ letters occurs as two different evaluations of the Tutte polynomial of a complete graph, $T_n(2,0)$ and $T_{n+1}(1,0)$.

Increasing spanning trees correspond to spanning trees with no inversions. Thus, $J_n(0) = T_n(1,0)$ equals the number of increasing trees in $K_n$. By deleting the vertex 1 in $K_{n+1}$ we get a bijection between increasing trees in $K_{n+1}$ and increasing spanning forests in $K_n$. Here a forest is increasing if it is increasing in each component. Therefore, we get the interpretation of $T_n(2,0)$ as the number of increasing spanning forests in $K_n$.

Using the same technique we get a bijection between 0–1–2 increasing trees on $n+1$ vertices and 0–1–2 increasing forests on $n$ vertices with at most 2 components. Thus we get

**Corollary 5.** $T_n(2,-1)$ *equals the number of 0–1–2 increasing forests on* $n$ *vertices with at most 2 components.*

There are several combinatorial interpretations for evaluations of $T(G;x,y)$ when $x,y \geq 0$, and even when $x,y \leq 0$ probably because of the relationship of the Tutte polynomial with the partition function of the Potts model of statistical mechanics. But the situation is quite different when $y < 0 < x$ or $x < 0 < y$. I would like to think that Corollary 5 is just the tip of the iceberg and that more combinatorial interpretations for $T(G;x,y)$ in these regions exist.

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