On the number of orthogonal systems in vector spaces over finite fields

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Abstract

Iosevich and Senger (2008) showed that if a subset of the d-dimensional vector space over a finite field is large enough, then it contains many k-tuples of mutually orthogonal vectors. In this note, we provide a graph theoretic proof of this result.

1 Introduction

A classical set of problems in combinatorial geometry deals with the question of whether a sufficiently large subset of $\mathbb{R}^d$, $\mathbb{Z}^d$ or $\mathbb{F}_q^d$ contains a given geometric configuration. In a recent paper [3], Iosevich and Senger showed that a sufficiently large subset of $\mathbb{F}_q^d$, the d-dimensional vector space over the finite field with q elements, contains many k-tuple of mutually orthogonal vectors. Using geometric and character sum machinery, they proved the following result (see [3] for the motivation of this result).

Theorem 1.1 *([3]) Let $E \subset \mathbb{F}_q^d$, such that*

$$|E| \geqslant Cq^{d\frac{k-1}{k}+\frac{k-1}{2}+\frac{1}{k}} \tag{1.1}$$

with a sufficiently large constant $C > 0$, where $0 < \binom{k}{2} < d$. Let λ_k be the number of k-tuples of k mutually orthogonal vectors in E. Then

$$\lambda_k = (1 + o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}. \tag{1.2}$$

In this note, we provide a different proof to this result using graph theoretic methods. The main result of this note is the following.

Theorem 1.2 *Let $E \subset \mathbb{F}_q^d$, such that*

$$|E| \gg q^{\frac{d}{2}+k-1}, \tag{1.3}$$

where $d > 2(k-1)$. Then the number of k-tuples of k mutually orthogonal vectors in E is

$$(1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}. \tag{1.4}$$

Note that Theorem 1.1 only works in the range $d > \binom{k}{2}$ (as larger tuples of mutually orthogonal vectors are out of range of the methods uses) while Theorem 1.2 works in a wider range $d > 2(k-1)$. Moreover, Theorem 1.2 is stronger than Theorem 1.1 in the same range.

1.1 Sharpness of results

It is also interesting to note that the exponent $\frac{d}{2}+1$ cannot be improved in the case $k = 2$. In [3], Iosevich and Senger constructed a set $E \subset \mathbb{F}_q^d$ such that $|E| \geq cq^{\frac{d+1}{2}+1}$, for some $c > 0$, but no pair of its vectors are orthogonal (see Lemma 3.2 in [3]). Their basic idea is to construct $E = E_1 \oplus E_2$ where $E_1 \subset \mathbb{F}_q^2$ and $E_2 \subset \mathbb{F}_q^{d-2}$, such that $|E_1| \approx q^{1/2}$ and $|E_2| \approx q^{\frac{d-1}{2}}$ with the sum set of their respective dot product sets does not contain 0. We hope to demonstrate in the future that the exponent $\frac{d}{2}+k-1$ cannot, in general, be improved, for any $k > 2$.

2 Proof of Theorem 1.2

We call a graph $G = (V, E)$ (n, d, λ)-graph if G is a d-regular graph on n vertices with the absolute values of each of its eigenvalues but the largest one is at most λ. It is well-known that if $\lambda \ll d$ then an (n, d, λ)-graph behaves similarly as a random graph $G_{n,d/n}$. Let H be a fixed graph of order s with r edges and with automorphism group $\mathrm{Aut}(H)$. Using the second moment method, it is not difficult to show that for every constant p the random graph $G(n, p)$ contains

$$(1+o(1))p^r(1-p)^{\binom{s}{2}-r}\frac{n^s}{|\mathrm{Aut}(H)|} \tag{2.1}$$

induced copies of H. Alon extended this result to (n, d, λ)-graphs. He proved that every large subset of the set of vertices of an (n, d, λ)-graph contains the "correct" number of copies of any fixed small subgraph (Theorem 4.10 in [2]).

Theorem 2.1 *([2]) Let H be a fixed graph with r edges, s vertices and maximum degree Δ, and let $G = (V, E)$ be an (n, d, λ)-graph, where, say, $d \leqslant 0.9n$. Let $m < n$ satisfies $m \gg \lambda\left(\frac{n}{d}\right)^{\Delta}$. Then, for every subset $U \subset V$ of cardinality m, the number of (not necessarily induced) copies of H in U is*

$$(1+o(1))\frac{m^s}{|\mathrm{Aut}(H)|}\left(\frac{d}{n}\right)^r. \tag{2.2}$$

Note that the above theorem, proved for simple graphs in [2], remains true if we allow loops (i.e. edges that connects a vertex to itself) in the graph G. There is no different between the proof in [2] for simple graph and the proof for graph with loops.

We recall a well-known construction of Alon and Krivelevich [1]. Let $PG(q,d)$ denote the projective geometry of dimension $d-1$ over finite field $\mathbb{F}_q$. The vertices of $PG(q,d)$ correspond to the equivalence classes of the set of all non-zero vectors $x = (x_1, \ldots, x_d)$ over $\mathbb{F}_q$, where two vectors are equivalent if one is a multiple of the other by an element of the field. Let $G_P(q,d)$ denote the graph whose vertices are the points of $PG(q,d)$ and two (not necessarily distinct) vertices x and y are adjacent if and only if $x_1y_1 + \ldots + x_dy_d = 0$. This construction is well known. In the case $d = 2$, this graph is called the Erdős-Rényi graph. It is easy to see that the number of vertices of $G_P(q,d)$ is $n_{q,d} = (q^d - 1)/(q-1)$ and that it is $d_{q,d}$-regular for $d_{q,d} = (q^{d-1} - 1)/(q-1)$. The eigenvalues of G are easy to compute ([1]). Let A be the adjacency matrix of G. Then, by properties of $PG(q,d)$, $A^2 = AA^T = \mu J + (d_{q,d} - \mu)I$, where $\mu = (q^{d-2} - 1)/(q-1)$, J is the all one matrix and I is the identity matrix, both of size $n_{q,d} \times n_{q,d}$. Thus the largest eigenvalue of A is $d_{q,d}$ and the absolute value of all other eigenvalues is $\sqrt{d_{q,d} - \mu} = q^{(d-2)/2}$.

Now we are ready to give a proof of Theorem 1.2. Let $G(q,d)$ denote the graph whose vertices are the points of $\mathbb{F}_q^d - (0, \ldots, 0)$ and two (not necessarily distinct) vertices x and y are adjacent if and only if they are orthogonal, i.e. $x_1y_1 + \ldots + x_dy_d = 0$. Then $G(q,d)$ is just the product of $q-1$ copies of $G_P(q,d)$. Therefore, it is easy to see that the number of vertices of G is $N_{q,d} = (q-1)n_{q,d} = q^d - 1$ and that it is $D_{q,d}$-regular for $D_{q,d} = (q-1)d_{q,d} = q^{d-1} - 1$. The eigenvalues of $G(q,d)$ are also easy to compute. Let V be the adjacency matrix of $G(q,d)$. Then by the properties of $PG(q,d)$,

$$V^2 = VV^T = \rho J_{N_{q,d}} + (D_{q,d} - \rho) \bigoplus_{n_{q,d}} J_{q-1}, \tag{2.3}$$

where $\rho = (q-1)\mu = q^{d-2} - 1$, $J_{N_{q,d}}$ is the all one matrix of size $N_{q,d} \times N_{q,d}$ and J_{q-1} is the all one matrix of size $(q-1) \times (q-1)$. Thus, all eigenvalues of V^2 are all eigenvalues of $(q-1)\rho J_{n_{q,d}} + (q-1)(D_{q,d} - \rho) I_{n_{q,d}}$ and zeros (with $J_{n_{q,d}}$ is the all one matrix and $I_{n_{q,d}}$ is the identity matrix, both of size $n_{q,d} \times n_{q,d}$). Therefore, the largest eigenvalue of V is $D_{q,d}$ and the absolute values of all other eigenvalues are either $\sqrt{(q-1)(D_{q,d} - \rho)} = (q-1)q^{(d-2)/2}$ or 0. This implies that $G(q,d)$ is a $(q^d - 1, q^{d-1} - 1, (q-1)q^{(d-2)/2})$-graph.

Let K_k be a complete graph with k vertices then K_k has $\binom{k}{2}$ edges and the degree of each vertex is $k-1$. Let $E \subset \mathbb{F}_q^d$, such that

$$|E| \gg q^{\frac{d}{2} + k - 1}, \tag{2.4}$$

where $d \geqslant 2k-1$. We consider E as a subset of the vertex set of $G(q,d)$ then the number of k-tuples of k mutually orthogonal vectors in E is the number of copies of K_k in E. Set $E_1 = E - \{0, \ldots, 0\}$ then we have $|E| - 1 \leq |E_1| \leq |E|$. We have

$$|E_1| \geq |E| - 1 \gg q^{\frac{d}{2} + k - 1} \geq (q-1)q^{(d-2)/2} \left(\frac{q^d - 1}{q^{d-1} - 1} \right)^{k-1}. \tag{2.5}$$

From Theorem 2.1 and (2.5), the number of copies of K_k in E_1 is

$$(1+o(1))\frac{|E_1|^k}{k!}\left(\frac{q^{d-1}-1}{q^d-1}\right)^{\binom{k}{2}} = (1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}. \tag{2.6}$$

Let K_{k-1} be a complete graph with $k-1$ vertices then K_{k-1} has $\binom{k-1}{2}$ edges and the degree of each vertex is $k-2$.

We have $(q-1)q^{(d-2)/2}\left(\frac{q^d-1}{q^{d-1}-1}\right)^{k-1} > (q-1)q^{(d-2)/2}\left(\frac{q^d-1}{q^{d-1}-1}\right)^{k-2}$. Thus, from Theorem 2.1 and (2.5), the number of copies of K_{k-1} in E_1 is

$$\begin{aligned}(1+o(1))\frac{|E_1|^{k-1}}{(k-1)!}\left(\frac{q^{d-1}-1}{q^d-1}\right)^{\binom{k-1}{2}} &= (1+o(1))\frac{|E|^{k-1}}{(k-1)!}q^{-\binom{k-1}{2}} &(2.7)\\ &\ll (1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}, &(2.8)\end{aligned}$$

as $|E| \gg q^{\frac{d}{2}+k-1} \gg q^{k-1}$. From (2.6) and (2.8), the number of copies of K_k in E is

$$(1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}}.$$

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This implies that the number of the number of k-tuples of k mutually orthogonal vectors in E is also

$$(1+o(1))\frac{|E|^k}{k!}q^{-\binom{k}{2}},$$

completing the proof of Theorem 1.2.

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