On the non–existence of certain hyperovals in dual André planes of order 2^{2h}

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Abstract

No regular hyperoval of the Desarguesian affine plane $AG(2, 2^{2h})$, with $h > 1$, is inherited by a dual André plane of order 2^{2h} and dimension 2 over its kernel.

1 Introduction

The general question on existence of ovals in finite non–Desarguesian planes is still open and appears to be difficult. It has been shown by computer search that there exist some planes of order 16 without ovals; see [11]. On the other hand, ovals have been constructed in several finite planes; one of the most fruitful approaches in this search has been that of inherited oval, due to Korchmáros [5, 6].

Korchmáros' idea relies on the fact that any two planes π_1 and π_2 of the same order have the same number of points and lines; thus their point sets, as well as some lines, may be identified. If Ω is an oval of π_1, it might happen that Ω, regarded as a point set, turns also out to be an oval of π_2, although π_1 and π_2 differ in some (in general several) point–line incidences; in this case Ω is called an *inherited oval* of π_2 from π_1; see also [2, Page 728].

In practice, it is usually convenient to take π_1 to be the Desarguesian affine plane $AG(2, q)$ of order a prime power q. The case in which π_2 is the Hall plane $H(q^2)$ of order q^2 was investigated in [5], and inherited ovals were found. For q odd, this also proves the

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existence of inherited ovals in the dual plane of $H(q^2)$, which is a Moulton plane $M(q^2)$ of the same order.

Moulton planes have been originally introduced in [10], by altering some of the lines of a Desarguesian plane constructed over the real field, while keeping the original point set fixed. In particular, each line of the Moulton plane turns out to be either a line of the original plane or the union of two half–lines of different slope with one point in common.

This construction, when considering planes of finite order q^2, may be carried out as follows. Let $||\cdot||$ denote the norm function

$$||\cdot||: \begin{cases} \mathrm{GF}(q^2) \to \mathrm{GF}(q) \\ x \mapsto x^{q+1} \end{cases}$$

Take a proper subset U of $\mathrm{GF}(q)^\star$ and consider the following operation defined over the set $\mathrm{GF}(q^2)$

$$a \odot b = \begin{cases} ab & \text{if } ||b|| \notin U \\ a^q b & \text{if } ||b|| \in U. \end{cases}$$

The set $(\mathrm{GF}(q^2), +, \odot)$ is a pre-quasifield which is a quasifield for $U \neq \{1\}$. Every pre-quasifield coordinatizes a translation plane; see [4, Section 5.6]. In our case this translation plane is an affine André plane $A(q^2)$ of order q^2 and dimension 2 over its kernel; see [8]. In the case in which U consists of a single element of $GF(q^2)$ the translation plane is the affine Hall plane of order q^2 and its dual plane is the affine Moulton plane of order q^2. For details on these planes see [3, 8].

Write $\mathcal{M}_U(q^2) = (\mathcal{P}, \mathcal{L})$ for the incidence structure whose the point–set $\mathcal{P}$ is the same as that of $AG(2, q^2)$, and whose lines in $\mathcal{L}$ are either of the form

$$[c] = \{P(x, y) : x = c, y \in \mathrm{GF}(q^2)\}$$

or

$$[m, n] = \{P(x, y) : y = m \odot x + n\}.$$

The affine plane $\mathcal{M}_U(q^2)$ is the dual of an affine André plane $A(q^2)$ of order q^2. Completing $\mathcal{M}_U(q^2)$ with its points at infinity in the usual way gives a projective plane $\overline{\mathcal{M}_U(q^2)}$ called *the projective closure of* $\mathcal{M}_U(q^2)$.

Write $\Phi = \{P(x, y) : ||x|| \notin U\}$ and $\Psi = \{P(x, y) : ||x|| \in U\}$. Clearly, $\mathcal{P} = \Phi \cup \Psi$.

If an arc $\mathcal{A}$ of $PG(2, q^2)$ is in turn an arc in $\overline{\mathcal{M}_U(q^2)}$ then, $\mathcal{A}$ is *an inherited arc* of $\overline{\mathcal{M}_U(q^2)}$.

Any hyperoval of the Desarguesian projective plane $PG(2, q^2)$ obtained from a conic by adding its nucleus is called *regular*. Let consider the set Ω of the affine points in $AG(2, q^2)$ of a regular hyperoval. If $\Omega \subseteq \Phi$, that is for each point $P(x, y) \in \Omega$ the norm of x is an element of $\mathrm{GF}(q) \setminus U$, then Ω is clearly an inherited hyperoval of $\mathcal{M}_U(q^2)$.

In [1, Theorem 1.1], it is proven that for $q > 5$ an odd prime power, any arc of the Moulton plane $\mathcal{M}_t(q^2)$ with $t \in GF(q)$, obtained as $\mathcal{C}^\star = \mathcal{C} \cap \Phi$, where $\mathcal{C}$ is an ellipse in $AG(2, q^2)$ is complete.

In this paper the case where q is even and $|U| < \frac{q}{4} - 1$ is addressed. We prove the following.

Theorem 1. *Suppose Ω to be the set of the affine points of a regular hyperoval of the projective closure $PG(2, 2^{2h})$ of $AG(2, 2^{2h})$, with $h > 1$. Then, $\Omega^* = \Omega \cap \Phi$ is a complete arc in the projective closure of $\mathcal{M}_U(2^{2h})$.*

Theorem 2. *The arc consisting of the affine points of a regular hyperoval of $PG(2, 2^{2h})$ with $h > 1$ is not an inherited arc in the projective closure of $\mathcal{M}_U(2^{2h})$.*

We shall also see that any oval arising from a regular hyperoval of $AG(2, 2^{2h})$ by deleting a point cannot be inherited by $\mathcal{M}_U(2^{2h})$. The hypothesis on Ω being a regular hyperoval cannot be dropped; see [11] for examples of hyperovals in the Moulton plane of order 16.

2 Proof of Theorem 1

We begin by showing the following lemma, which is a slight generalisation of Lemma 2.1. in [1].

Lemma 3. *Let q be any prime power. A pencil of affine lines $\mathcal{L}(P)$ of $\mathcal{M}_U(q^2)$ with centre $P(x_0, y_0)$, either consists of lines of a Baer subplane $\mathcal{B}$ of $PG(2, q^2)$, or is a pencil in $AG(2, q^2)$ with the same centre, according as $||x_0|| \in U$ or not. In particular, in the former case, the $q^2 + 1$ lines in $\mathcal{L}(P)$ plus the q vertical lines $X = c$ with $||c|| = x_0^{q+1}$ and $c \neq x_0$ are the lines of $\mathcal{B}$.*

Proof. The pencil $\mathcal{L}(P)$ consists of the lines

$$r_m : y = m \odot x - m \odot x_0 + y_0,$$

with $m \in GF(q^2)$, plus the vertical line $\ell : x = x_0$. First suppose $||x_0|| \in U$. In this case $m \odot x_0 = m^q x_0$ and the line r_m of $\mathcal{L}(P)$ corresponds to the point $(m, m^q x_0 - y_0)$ in the dual of $\mathcal{M}_U(q^2)$, which is an André plane.

As m varies over $GF(q^2)$ we get q^2 affine points of the Baer subplane $\mathcal{B}'$ in $PG(2, q^2)$ represented by $y = x^q x_0 - y_0$. The points at infinity of $\mathcal{B}'$ are those points (c) such that $c^{q+1} = ||x_0||$. As the dual of a Baer subplane is a Baer subplane, it follows that the lines in $\mathcal{L}(P)$ are the lines of a Baer subplane $\mathcal{B}$ in $PG(2, q^2)$. More precisely, the lines in $\mathcal{L}(P)$ plus the q vertical lines $x = c$, $||c|| = ||x_0||$, with $c \neq x_0$, are the lines of $\mathcal{B}$.

In the case in which $||x_0|| \notin U$ the line $r_m : y = m \odot x - m x_0 + y_0$ in $\mathcal{L}(P)$ corresponds to the point

$$(m, m x_0 - y_0)$$

in the dual of $\mathcal{M}_U(q^2)$. As m varies over $GF(q^2)$ we get q^2 affine points in $AG(2, q^2)$ on the line $y = x_0 x - y_0$. Finally, the dual of infinite point of $y = x_0 x - y_0$ is the vertical line through $P(x_0, y_0)$. The result follows. $\square$

Let Ω denote a regular hyperoval in $AG(2, q^2)$, $q = 2^h$, $h > 1$. It will be shown that for any point $P(x_0, y_0)$ with $||x_0|| \in U$ there is at least a 2–secant to $\Omega^\star = \Omega \cap \Phi$ in $\mathcal{M}_U(q^2)$ through P.

Assume $\mathcal{B}$ to be the Baer subplane in $PG(2,q^2)$ containing the lines of the pencil $\mathcal{L}(P)$ in $\mathcal{M}_U(q^2)$ and the q vertical lines $X=c$ with $||c||=x_0^{q+1}$, $c\neq x_0$. Write Δ for the set of all points of Ω not covered by a vertical line of $\mathcal{B}$ and also let $n=|\Delta|$ and $m=q^2+2-n$. The vertical lines of $\mathcal{B}$ cover at most $2(q+1)$ points of Ω; thus, $q^2-2q\leq n\leq q^2+2$. We shall show that there is at least a line in $\mathcal{B}$ meeting Δ in two points.

Let $T\in\Delta$; since $T\notin\mathcal{B}$, there is a unique line ℓ_T of $\mathcal{B}$ through T. Every point $Q\in\Omega\setminus\Delta$ lies on at most $q+1-(m-1)=q-m+2$ lines ℓ_T with $T\in\Delta$. Suppose by contradiction that for every $T\in\Delta$,

$$\ell_T\cap\Omega=\{T,Q\},\text{with } Q\in\Omega\setminus\Delta.$$

The total number of lines ℓ_T obtained as Q varies in $\Omega\setminus\Delta$ does not exceed $m(q-m+2)$. So, $n=q^2-m+2\leq m(q-m+2)$. As m is a non–negative integer, this is possible only for $q=2$.

Since ℓ_T is not a vertical line, it turns out to be a chord of Ω^* in $\mathcal{M}_U(q^2)$ passing through $P(x_0,y_0)$. This implies that no point $P(x_0,y_0)\in\Psi$ may be aggregated to $\Omega^\star$ in order to obtain an arc.

This holds true in the case $P(x_0,y_0)\in\Phi$. In $AG(2,q^2)$ there pass $(q^2+2)/2$ secants to Ω through a point $P(x_0,y_0)\notin\Omega$ and, hence, $N=(q^2+2)/2-s$ secants to Ω^*, where $s\leq 2(q+1)|U|$. So by the hypothesis $|U|<q/4-1$, we obtain $N>0$; this implies that no point $P(x_0,y_0)\in\Phi$ may be aggregated to $\Omega^\star$ in order to obtain a larger arc. The same argument works also when P is assumed to be a point at infinity. Theorem 1 is thus proved.

3 Proof of Theorem 2

We shall use the notion of *conic blocking set*; see [7]. A conic blocking set $\mathcal{B}$ is a set of lines in a Desarguesian projective plane met by all conics; a conic blocking set $\mathcal{B}$ is *irreducible* if for any line of $\mathcal{B}$ there is a conic intersecting $\mathcal{B}$ in just that line.

Lemma 4 (Theorem 4.4,[7]). *The line–set*

$$\mathcal{B}=\{y=mx : m\in\mathrm{GF}(q)\}\cup\{x=0\}$$

is an irreducible conic blocking set in $\mathrm{PG}(2,q^2)$*, where* $q=2^h$*,* $h>1$*.*

Lemma 5. *Let Ω be a regular hyperoval of $PG(2,q^2)$, with $q=2^h$, $h>1$. Then, there are at least two points $P(x,y)$ in Ω such that $||x||\in U$.*

Proof. To prove the lemma we show that the set $\Psi'=\Psi\cup Y_\infty$, is a conic blocking set. We observe that the conic blocking set of Lemma 4 is actually a degenerate Hermitian curve of $\mathrm{PG}(2,q^2)$ with equation $x^qy-xy^q=0$. Since all degenerate Hermitian curves are projectively equivalent, this implies that any such a curve is a conic blocking set. On the other hand, Ψ' may be regarded as the union of degenerate Hermitian curves of equation $x^{q+1}=cz^{q+1}$, as c varies in U. Thus, Ψ' is also a conic blocking set. Suppose now $\Omega = \mathcal{C} \cup N$, where $\mathcal{C}$ is a conic of nucleus N. Take $P \in \Psi' \cap \mathcal{C}$. If $P = Y_\infty$ then at most one of the vertical lines $X = c$, with $||c|| \in U$, is tangent to $\mathcal{C}$; hence there are at least q points $P'(x, y) \in \Omega$ with $||x|| \in U$ and thus $|\Psi \cap \Omega| \geq q$.

Next, assume that $P = P(x, y) \in \Psi$. If the line $[x]$ is secant to $\mathcal{C}$ the assertion immediately follows. If the line $[x]$ is tangent to $\mathcal{C}$ then the nucleus N lies on $[x]$. Now, either N is an affine point in Ψ or $N = Y_\infty$. In the former case we have $|\Psi \cap \Omega| \geq 2$; in the latter, the lines $X = c$ with $||c|| \in U$ are all tangent to $\mathcal{C}$; hence, there are at least other $q + 1$ points $P'(x, y) \in \Omega$ such that $||x|| \in U$. $\square$

Now, let Ω be a regular hyperoval in $AG(2, q^2)$, with $q = 2^h$ and $h > 1$. From Lemma 5 we deduce that $|\Omega^* \cap \Phi| \leq q^2$; furthermore, Theorem 2 guaranties that Ω^* is a complete arc in the projective closure of $\mathcal{M}_U(q^2)$, whence Theorem 2 follows.

Remark 1. The largest arc of $\mathcal{M}_U(q^2)$ contained in a regular hyperoval of $AG(2, q^2)$, with $q = 2^h$, has at most q^2 points; in particular any oval which arises from a hyperoval of $AG(2, q^2)$ by deleting a point cannot be an oval of $\mathcal{M}_U(q^2)$. For an actual example of a q^2–arc of $\mathcal{M}_U(q^2)$ coming from a regular hyperoval of $AG(2, q^2)$ see [9]. This also shows that the result of [5] cannot be extended to even q.

References

[1] V. Abatangelo, B. Larato, *Canonically Inherited Arcs in Moulton Planes of Odd Order*, to appear on Innovations in Incidence Geometry.

[2] C.J. Colbourn, J.H. Dinitz, *Handbook of Combinatorial Designs* (2nd ed.), Chapman & Hall (2007).

[3] P. Dembowski, *Finite Geometries*, Springer (1968).

[4] N.L. Johnson, V. Jha, M. Biliotti, *Handbook of finite translation planes*. Pure and Applied Mathematics, **289** Chapman & Hall/CRC, Boca Raton, FL, (2007).

[5] G. Korchmáros, *Ovali nei piani di Moulton d'ordine dispari*, Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, 395–398. Atti dei Convegni Lincei, No. **17**, Accad. Naz. Lincei, Rome (1976).

[6] G. Korchmáros, *Ovali nei piani di Hall di ordine dispari*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **56** (1976) no. 3, 315–317.

[7] L. D. Holder, *Conic blocking sets in Desarguesian projective planes*, J. Geom. **80** (2004), no. 1-2, 95–105.

[8] D.Hughes, F. Piper, *Projective planes*, Springer (1973).

[9] G. Menichetti, *k–archi completi in piani di Moulton d'ordine q^m*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **60** (1976), no. 6, 775–781.

[10] F.R. Moulton, *A simple non–desarguesian plane geometry*, Trans. Am. Math. Soc. **3** (1902), 192–195.

[11] T. Penttila, G.F. Royle, M.K. Simpson, *Hyperovals in the known projective planes of order* 16, J. Combin. Des. **4** (1996), no. 1, 59–65.