Intersecting and cross-intersecting families of labeled sets

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Abstract

A family $\mathcal{A}$ of sets is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect. Families $\mathcal{A}_1, ..., \mathcal{A}_p$ are said to be *cross-intersecting* if, for any $i, j \in \{1, ..., p\}$ such that $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$.

For $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n$, $2 \leq k_1 \leq ... \leq k_n$, let $\mathcal{L}_{\mathbf{k}}$ be the family of *labeled n-sets* given by $\mathcal{L}_{\mathbf{k}} := \{\{(1, l_1), ..., (n, l_n)\} \colon l_i \in \{1, ..., k_i\}, i = 1, ..., n\}$. We point out a relationship between intersecting families and cross-intersecting families of labeled sets, and we show that, if $\mathcal{A}_1, ..., \mathcal{A}_p$ are cross-intersecting sub-families of $\mathcal{L}_{\mathbf{k}}$, then

$$\sum_{j=1}^{p} |\mathcal{A}_j| \leq \begin{cases} k_1 k_2 ... k_n & \text{if } p \leq k_1; \\ p k_2 ... k_n & \text{if } p \geq k_1. \end{cases}$$

We also determine the cases of equality. We then obtain a more general inequality, a special case of which is a sharp bound for cross-intersecting families of *permutations*.

1 Old and new intersection results for labeled sets

We start with some standard notation for sets. $\mathbb{N}$ is the set of positive integers $\{1, 2, ...\}$. For $m, n \in \mathbb{N}$, $m < n$, the set $\{m, m+1, ..., n\}$ is denoted by $[m, n]$, and if $m = 1$ then we also write $[n]$. The *power set* $\{A \colon A \subseteq X\}$ of a set X is denoted by 2^X, and a *uniform* sub-family $\{Y \subseteq X \colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

We denote the union of all sets in a family $\mathcal{F}$ by $U(\mathcal{F})$. For $u \in U(\mathcal{F})$, the family of sets in $\mathcal{F}$ that contain u is called a *star of $\mathcal{F}$* with *centre u*.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect. Note that a star of a family is trivially intersecting.

The classical Erdős-Ko-Rado (EKR) Theorem [6] says that, for $r \leq n/2$, an intersecting sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-1}{r-1}$, i.e. the size of a star of $\binom{[n]}{r}$. By the

Hilton-Milner Theorem [9], if $r < n/2$ then $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$. Many results were inspired by the EKR Theorem; see [5].

Families $\mathcal{A}_1, ..., \mathcal{A}_p$ are said to be *cross-intersecting* if, for any $i, j \in [p]$ such that $i \neq j$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$.

Hilton [8] determined the following nice EKR-type result for cross-intersecting sub-families of $\binom{[n]}{r}$ $(r \leq n/2)$.

Theorem 1.1 (Hilton [8]) *Let $r \leq n/2$ and $p \geq 2$. Let $\mathcal{A}_1, ..., \mathcal{A}_p$ be cross-intersecting sub-families of $\binom{[n]}{r}$. Then*

$$\sum_{j=1}^{p} |\mathcal{A}_j| \leq \begin{cases} \binom{n}{r} & \text{if } p \leq \frac{n}{r}; \\ p\binom{n-1}{r-1} & \text{if } p \geq \frac{n}{r}. \end{cases}$$

Unless $p = 2 = n/r$, the bound is attained if and only if one of the following holds:
(i) $p < n/r$ and, for some $q \in [p]$, $\mathcal{A}_q = \binom{[n]}{r}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [p]\backslash\{q\}$;
(ii) $p > n/r$ and $|\mathcal{A}_1| = ... = |\mathcal{A}_p| = \binom{n-1}{r-1}$;
(iii) $p = n/r$ and $\mathcal{A}_1, ..., \mathcal{A}_p$ are as in (i) or (ii).

The EKR Theorem follows from this result: set $p > n/r$ and $\mathcal{A}_1 = ... = \mathcal{A}_p$. In [2] it is shown that in case (ii) we must have $\mathcal{A}_1 = ... = \mathcal{A}_p = \{A \in \binom{[n]}{r} \colon i \in A\}$ for some $i \in [n]$.

For $\mathbf{k} = (k_1, ..., k_n)$, $k_1, ..., k_n \in \mathbb{N}\backslash\{1\}$, we define the family $\mathcal{L}_{\mathbf{k}}$ of *labeled n-sets* by

$$\mathcal{L}_{\mathbf{k}} := \{\{(1, l_1), ..., (n, l_n)\} \colon l_i \in [k_i],\ i = 1, ..., n\}.$$

An equivalent formulation for $\mathcal{L}_{\mathbf{k}}$ is $[k_1] \times [k_2] \times ... \times [k_n]$, but it is more convenient to work with n-sets than work with n-tuples (the alternative formulation demands that we change the setting of families of sets to one of sets of n-tuples).

In this note we are concerned with the sizes of intersecting and cross-intersecting families of labeled n-sets. Note that, if we allow $k_i = 1$ for some $i \in [n]$, then the problem becomes trivial because we get that all sets contain the point $(i, 1)$.

The obvious EKR-type problem for labeled sets was treated by Berge [1].

Theorem 1.2 (Berge [1]) *Let $m \in [n]$ such that $k_m = \min\{k_i \colon i \in [n]\}$. If $\mathcal{A}$ is an intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$, then the size of $\mathcal{A}$ is at most $|\mathcal{L}_{\mathbf{k}}|/k_m = k_1k_2...k_n/k_m$, i.e. the size of a star of $\mathcal{L}_{\mathbf{k}}$ with centre $(m, 1)$.*

We shall reproduce the remarkably short proof of this result.

For an integer q, let $\theta^q_{\mathbf{k}} : \mathcal{L}_{\mathbf{k}} \to \mathcal{L}_{\mathbf{k}}$ be the *translation operation* defined by

$$\theta^q_{\mathbf{k}}(A) := \{(a, b + q \bmod k_a) \colon (a, b) \in A\},$$

and define $\Theta^q_{\mathbf{k}} : 2^{\mathcal{L}_{\mathbf{k}}} \to 2^{\mathcal{L}_{\mathbf{k}}}$ by

$$\Theta^q_{\mathbf{k}}(\mathcal{F}) := \{\theta^q_{\mathbf{k}}(A) \colon A \in \mathcal{F}\}.$$

Now let m and $\mathcal{A}$ be as in Theorem 1.2. For any $A \in \mathcal{A}$ and $q \in [k_m - 1]$, we have $\theta^q_{\mathbf{k}}(A) \cap A = \emptyset$ and hence $\theta^q_{\mathbf{k}}(A) \notin \mathcal{A}$. Therefore $\mathcal{A}, \Theta^1_{\mathbf{k}}(\mathcal{A}), ..., \Theta^{k_m-1}_{\mathbf{k}}(\mathcal{A})$ are k_m disjoint sub-families of $\mathcal{L}_{\mathbf{k}}$. So $k_m|\mathcal{A}| \leq |\mathcal{L}_{\mathbf{k}}| = k_1k_2...k_n$ and hence Theorem 1.2.

Livingston [10] determined which families $\mathcal{A}$ attain the bound in Theorem 1.2 for the case when the k_i's are all the same.

Theorem 1.3 (Livingston [10]) *If $3 \leq k_1 = k_2 = ... = k_n$ and $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$, then $\mathcal{A}$ is a star of $\mathcal{L}_{\mathbf{k}}$.*

Using the shifting technique (see [7]) in an inductive argument, we can extend Theorem 1.3 to the following result.

Theorem 1.4 *Let $m \in [n]$ such that $k_m = \min\{k_i \colon i \in [n]\}$. Suppose $k_m \geq 3$ and $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$. Then $\mathcal{A}$ is a star of $\mathcal{L}_{\mathbf{k}}$ with centre in $C := \{(i, l) \colon i \in [n],\, k_i = k_m,\, l \in [k_m]\}$.*

Proof. We use induction on $\sum_{i=1}^n k_i$. The case $k_i = k_m$ for $i = 1, ..., n$ is Theorem 1.3, so we assume there exists $h \in [n]$ such that $k_h > k_m$. We may assume $m = 1$ and $h = n$. Let $\delta \colon \mathcal{A} \to \mathcal{L}_{\mathbf{k}}$ be the shift operation defined by

$$\delta(A) := \begin{cases} (A \backslash \{(n, k_n)\}) \cup \{(n, 1)\} & \text{if } (n, k_n) \in A; \\ A & \text{otherwise,} \end{cases}$$

and let $\mathcal{B} := \{\delta(A) \colon A \in \mathcal{A},\, \delta(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta(A) \in \mathcal{A}\}$. Clearly $|\mathcal{B}| = |\mathcal{A}|$.

We now show that, since $\mathcal{A}$ is intersecting, $\mathcal{B}$ is intersecting and, moreover, no two sets in $\mathcal{B}$ intersect only on (n, k_n). Let B_1 and B_2 be two arbitrary sets in $\mathcal{B}$. If neither set contains (n, k_n) then either they intersect on $(n, 1)$ or at least one of them does not contain $(n, 1)$ either and hence must intersect the other. Now suppose without loss of generality $(n, k_n) \in B_1$. Then $B_1, \delta(B_1) \in \mathcal{A}$. If $(n, k_n) \in B_2$ then B_2 is also in $\mathcal{A}$ and hence $\emptyset \neq \delta(B_1) \cap B_2 = (B_1 \cap B_2) \backslash \{(n, k_n)\}$. Suppose $(n, k_n) \notin B_2$. If $B_2 \in \mathcal{A}$ then $(n, k_n) \notin B_1 \cap B_2 \neq \emptyset$ is obvious. If $B_2 \notin \mathcal{A}$ then $(n, 1) \in B_2$, the set $A_2 := (B_2 \backslash \{(n, 1)\}) \cup \{(n, k_n)\}$ is in $\mathcal{A}$, and hence we have $(n, k_n) \notin B_1 \cap B_2 = \delta(B_1) \cap A_2 \neq \emptyset$.

Defining $\mathcal{B}_1 := \{B \in \mathcal{B} \colon (n, k_n) \notin B\}$, $\mathcal{B}_2 := \{B \in \mathcal{B} \colon (n, k_n) \in B\}$ and $\mathcal{B}_2' := \{B \backslash \{(n, k_n)\} \colon B \in \mathcal{B}_2\}$, it follows that $\mathcal{B}_1 \cup \mathcal{B}_2'$ is intersecting. Now $\mathcal{B}_1 \subset \mathcal{L}_{\mathbf{k}_1}$ and $\mathcal{B}_2' \subset \mathcal{L}_{\mathbf{k}_2}$, where $\mathbf{k}_1 = (k_1, ..., k_{n-1}, k_n - 1)$ and $\mathbf{k}_2 = (k_1, ..., k_{n-1})$. Let $\mathcal{S} := \{S \in \mathcal{L}_{\mathbf{k}} \colon (1, 1) \in S\}$, and define $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_2'$ similarly to $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_2'$ respectively. By the inductive hypothesis, $|\mathcal{B}_1| \leq |\mathcal{S}_1|$ and $|\mathcal{B}_2'| \leq |\mathcal{S}_2'|$. So $|\mathcal{A}| \leq |\mathcal{S}|$ as $|\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2'|$ and $|\mathcal{S}| = |\mathcal{S}_1| + |\mathcal{S}_2'|$. Since $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{L}_{\mathbf{k}}$, we actually have $|\mathcal{A}| = |\mathcal{S}|$, which implies $|\mathcal{B}_1| = |\mathcal{S}_1|$ and $|\mathcal{B}_2'| = |\mathcal{S}_2'|$. By noting that $C \subseteq U(\mathcal{L}_{\mathbf{k}_2})$ (since $k_n > k_1$) and applying the inductive hypothesis, we get $\mathcal{B}_2' = \{B \in \mathcal{L}_{\mathbf{k}_2} \colon (a, b) \in B\}$ for some $(a, b) \in C$. Thus, for any set in $\{A \in \mathcal{L}_{\mathbf{k}_1} \colon (a, b) \notin A\}$, there clearly exists a set in $\mathcal{B}_2'$ that does not intersect with it. Since $\mathcal{B}_1 \cup \mathcal{B}_2'$ is intersecting, it follows that all sets in $\mathcal{B}_1 \cup \mathcal{B}_2'$ contain (a, b). So $\mathcal{B} \subseteq \mathcal{C} := \{A \in \mathcal{L}_{\mathbf{k}} \colon (a, b) \in A\}$. Since $(a, b) \notin \{(n, 1), (n, k_n)\}$, we deduce that $\mathcal{A} \subseteq \mathcal{C}$. By maximality of $\mathcal{A}$, $\mathcal{A} = \mathcal{C}$. Hence result. $\square$

We will now use Theorems 1.2 and 1.4 to obtain the following 'labeled sets' analogue of Theorem 1.1.

Theorem 1.5 *Let $m \in [n]$ such that $k_m = \min\{k_i \colon i \in [n]\}$, and let $C := \{(i,l) \colon i \in [n],\ k_i = k_m,\ l \in [k_m]\}$. Let $\mathcal{A}_1, ..., \mathcal{A}_p$ be cross-intersecting sub-families of $\mathcal{L}_{\mathbf{k}}$. Then*

$$\sum_{j=1}^{p} |\mathcal{A}_j| \leq \begin{cases} |\mathcal{L}_{\mathbf{k}}| = k_1 k_2 ... k_n & \text{if } p \leq k_m; \\ p|\{A \in \mathcal{L}_{\mathbf{k}} \colon (m,1) \in A\}| = p\frac{k_1 k_2 ... k_n}{k_m} & \text{if } p \geq k_m. \end{cases}$$

If $k_m \geq 3$, then the bound is attained if and only if one of the following holds:
(i) $p < k_m$ and, for some $q \in [p]$, $\mathcal{A}_q = \mathcal{L}_{\mathbf{k}}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [p] \backslash \{q\}$;
(ii) $p > k_m$ and $\mathcal{A}_1 = ... = \mathcal{A}_p = \{A \in \mathcal{L}_{\mathbf{k}} \colon (a,b) \in A\}$ for some $(a,b) \in C$;
(iii) $p = k_m$ and $\mathcal{A}_1, ..., \mathcal{A}_p$ are as in (i) or (ii).

Proof. Let $k_{n+1} := p$ and $\mathbf{k}' := (k_1, ..., k_n, k_{n+1})$. For each $j \in [k_{n+1}]$, let

$$\mathcal{A}'_j := \{A \cup \{(n+1, j)\} \colon A \in \mathcal{A}_j\}.$$

Now let $\mathcal{A}$ be the the disjoint union $\mathcal{A}'_1 \cup \mathcal{A}'_2 \cup ... \cup \mathcal{A}'_{k_{n+1}}$, which is a sub-family of $\mathcal{L}_{\mathbf{k}'}$. Take two arbitrary sets A_1 and A_2 in $\mathcal{A}$. So $A_1 \in \mathcal{A}'_{j_1}$ and $A_2 \in \mathcal{A}'_{j_2}$ for some $j_1, j_2 \in [k_{n+1}]$. Also $B_1 := A_1 \backslash \{(n+1, j_1)\}$ is in $\mathcal{A}_{j_1}$ and $B_2 := A_2 \backslash \{(n+1, j_2)\}$ is in $\mathcal{A}_{j_2}$. If $j_1 = j_2$ then A_1 and A_2 intersect on $(n+1, j_1)$. If $j_1 \neq j_2$ then $\mathcal{A}_{j_1}$ and $\mathcal{A}_{j_2}$ are cross-intersecting, and this gives us $B_1 \cap B_2 \neq \emptyset$ and hence $A_1 \cap A_2 \neq \emptyset$. So $\mathcal{A}$ is an intersecting sub-family of $\mathcal{L}_{\mathbf{k}'}$. Since $|\mathcal{A}| = \sum_{j=1}^{k_{n+1}} |\mathcal{A}_j|$, it follows by Theorem 1.2 (with $\mathbf{k}'$ instead of $\mathbf{k}$) that

$$\sum_{j=1}^{k_{n+1}} |\mathcal{A}_j| \leq \frac{|\mathcal{L}_{\mathbf{k}'}|}{\min\{k_i \colon i \in [n+1]\}} = \frac{k_1 k_2 ... k_{n+1}}{\min\{k_m, k_{n+1}\}},$$

which immediately yields the desired upper bound.

Now suppose $k_m \geq 3$. It is straightforward that the upper bound is attained in each of the cases (i), (ii) and (iii). We now assume that the upper bound is attained and prove the converse. We obviously divide the problem into the following cases.

Case 1: $k_{n+1} < k_m$. So we have $|\mathcal{A}| = \frac{|\mathcal{L}_{\mathbf{k}'}|}{k_{n+1}}$.

Suppose first that $k_{n+1} \geq 3$. Since $k_{n+1} < k_m \leq k_i$ for all $i \in [n]$, it follows by Theorem 1.4 that $\mathcal{A} = \{A \in \mathcal{L}_{\mathbf{k}'} \colon (n+1, q) \in A\}$ for some $q \in [k_{n+1}]$. By construction of $\mathcal{A}$, we get $\mathcal{A}_q = \mathcal{L}_{\mathbf{k}}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [k_{n+1}] \backslash \{q\}$.

Now suppose $k_{n+1} = 2$. Let $k'_{n+1} := k_m$, and let $\mathbf{k}'' := (k_1, ..., k_n, k'_{n+1})$. Since $k'_{n+1} \geq 3$ and $|\mathcal{A}| = \frac{|\mathcal{L}_{\mathbf{k}'}|}{k_{n+1}} = \frac{|\mathcal{L}_{\mathbf{k}''}|}{k'_{n+1}}$, Theorem 1.2 tells us that $\mathcal{A}$ is a largest intersecting sub-family of $\mathcal{L}_{\mathbf{k}''}$. Thus, by Theorem 1.4, $\mathcal{A} = \{A \in \mathcal{L}_{\mathbf{k}''} \colon (a,b) \in A\}$ for some $(a,b) \in U(\mathcal{L}_{\mathbf{k}''})$. Since $\mathcal{A} \subseteq \mathcal{L}_{\mathbf{k}'}$, we actually have $(a,b) \in U(\mathcal{L}_{\mathbf{k}'})$. Suppose $a \neq n+1$; then $|\mathcal{A}| = \frac{|\mathcal{L}_{\mathbf{k}'}|}{k_a}$ and hence, since $|\mathcal{A}| = \frac{|\mathcal{L}_{\mathbf{k}'}|}{k_{n+1}}$, we get $k_a = k_{n+1}$, which contradicts $k_a \geq k'_{n+1} \geq 3 > k_{n+1}$. So $a = n+1$. By construction of $\mathcal{A}$, we get $\mathcal{A}_b = \mathcal{L}_{\mathbf{k}}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [k_{n+1}] \backslash \{b\}$.

Case 2: $k_{n+1} > k_m$. So $k_m = \min\{k_1, ..., k_n, k_{n+1}\}$. By Theorem 1.4, we must have $\mathcal{A} = \{A \in \mathcal{L}_{\mathbf{k}'} \colon (a,b) \in A\}$ for some $(a,b) \in D := \{(i,l) \colon i \in [n+1],\ k_i = k_m,\ l \in [k_m]\}$. Since $k_{n+1} > k_m$, $a \neq n+1$ and $D = C$. Thus, by construction of $\mathcal{A}$, for each $j \in [k_{n+1}]$ we have $\mathcal{A}'_j = \{A \cup \{(n+1, j)\} \colon (a,b) \in A \in \mathcal{L}_{\mathbf{k}}\}$ and hence $\mathcal{A}_j = \{A \in \mathcal{L}_{\mathbf{k}} \colon (a,b) \in A\}$.

Case 3: $k_{n+1} = k_m$. By Theorem 1.4, $\mathcal{A} = \{A \in \mathcal{L}_{\mathbf{k}'} \colon (a,b) \in A\}$ for some $(a,b) \in C \cup (\{n+1\} \times [k_{n+1}])$. Therefore, if $(a,b) \in \{n+1\} \times [k_{n+1}]$ then $\mathcal{A}_1, ..., \mathcal{A}_{k_{n+1}}$ are as in Case 1, and if $(a,b) \in C$ then $\mathcal{A}_1, ..., \mathcal{A}_{k_{n+1}}$ are as in Case 2. □

2 Translation invariant families of labeled sets

In this section, we generalize the sharp bounds in the above results.

Definition 2.1 *We say that $\mathcal{F}$ is translation invariant with respect to $\mathcal{L}_{\mathbf{k}}$ if $\mathcal{F} \subseteq \mathcal{L}_{\mathbf{k}}$ and $\Theta^1_{\mathbf{k}}(\mathcal{F}) = \mathcal{F}$.*

When it is clear from the context that we are considering $\mathcal{F} \subseteq \mathcal{L}_{\mathbf{k}}$, then, if $\Theta^1_{\mathbf{k}}(\mathcal{F}) = \mathcal{F}$, we simply say that $\mathcal{F}$ is *translation invariant.*

For any $r, s \in \mathbb{N}$ with $r \leq s$, let

$$\mathcal{P}_{r,s} := \{\{(1,l_1), ..., (r,l_r)\} \colon l_1, ..., l_r \text{ are } \textit{distinct} \text{ elements of } [s]\}.$$

$\mathcal{P}_{r,s}$ is an example of a translation invariant sub-family of $\mathcal{L}_{\mathbf{k}}$ with the $n = r$ entries of $\mathbf{k}$ being all s. The special family $\mathcal{P}_{n,n}$ describes *permutations of* $[n]$.

One can find various other examples of translation invariant families. For example, if $k_1 = ... = k_m \leq k_{m+1} = ... = k_n$, then clearly $\{\{(1,l_1), ..., (m,l_1), (m+1,l_2), ..., (n,l_2)\} \colon l_1 \in [k_1],\ l_2 \in [k_{m+1}]\}$ is a translation invariant sub-family of $\mathcal{L}_{\mathbf{k}}$.

The following is a straightforward result.

Proposition 2.2 *Let $\mathcal{F} \subseteq \mathcal{L}_{\mathbf{k}}$. $\mathcal{F}$ is translation invariant if and only if $\Theta^q_{\mathbf{k}}(\mathcal{F}) = \mathcal{F}$ for any $q \in \mathbb{N}$.*

Theorem 2.3 *Let $\mathcal{F}$ be translation invariant with respect to $\mathcal{L}_{\mathbf{k}}$. Let $m \in [n]$ such that $k_m = \min\{k_i \colon i \in [n]\}$. Then the size of an intersecting sub-family of $\mathcal{F}$ is at most $|\mathcal{F}|/k_m$, which is the size of a star of $\mathcal{F}$ with centre $(m,1)$.*

Proof. For each $j \in [k_m]$, let $\mathcal{F}_j := \{F \in \mathcal{F} \colon (m,j) \in F\}$ (i.e. the star of $\mathcal{F}$ with centre (m,j)). So the families $\mathcal{F}_1, ..., \mathcal{F}_{k_m}$ partition $\mathcal{F}$. Clearly, for any $q \in \mathbb{N}$, $\theta^q_{\mathbf{k}}$ is an injective function. Consider $j_1, j_2 \in [k_m]$, $j_1 < j_2$. $\theta^{j_2 - j_1}_{\mathbf{k}}$ maps any set $F_{j_1} \in \mathcal{F}_{j_1}$ to a set $F_{j_2} \in \{A \in \mathcal{L}_{\mathbf{k}} \colon (m, j_2) \in A\}$, and $F_{j_2} \in \mathcal{F}_{j_2}$ by Proposition 2.2; so $|\mathcal{F}_{j_1}| \leq |\mathcal{F}_{j_2}|$. Similarly, by considering the mapping $\theta^{k_m - j_2 + j_1}_{\mathbf{k}}$, we obtain $|\mathcal{F}_{j_2}| \leq |\mathcal{F}_{j_1}|$. So $|\mathcal{F}_{j_1}| = |\mathcal{F}_{j_2}|$. This implies $|\mathcal{F}_1| = ... = |\mathcal{F}_{k_m}|$ and hence $|\mathcal{F}| = k_m|\mathcal{F}_1|$.

Now let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{F}$. Let $A \in \mathcal{A}$ and $q \in [k_m - 1]$. Since $\theta^q_{\mathbf{k}}(A) \cap A = \emptyset$, $\theta^q_{\mathbf{k}}(A) \notin \mathcal{A}$. By Proposition 2.2, $\theta^q_{\mathbf{k}}(A) \in \mathcal{F}$. Thus $\mathcal{A}, \Theta^1_{\mathbf{k}}(\mathcal{A}), ..., \Theta^{k_m-1}_{\mathbf{k}}(\mathcal{A})$ are k_m disjoint sub-families of $\mathcal{F}$. So we have $k_m|\mathcal{A}| \leq |\mathcal{F}| = k_m|\mathcal{F}_1|$. Hence result. □

Remark. Deza and Frankl [4] proved the above result for the special case $\mathcal{F} = \mathcal{P}_{n,n}$. Cameron and Ku [3] proved that the extremal intersecting sub-families of $\mathcal{P}_{n,n}$ are the stars of $\mathcal{P}_{n,n}$. However, there are translation invariant families $\mathcal{F}$ whose largest intersecting sub-families are not all stars of $\mathcal{F}$. For example, suppose $n \geq 3$ and all the n entries of $\mathbf{k}$ are n. Let $\mathcal{E} := \{E \in \mathcal{P}_{n,n} \colon |E \cap \{(1,1),(2,2),(3,3)\}| \geq 2\}$ and take $\mathcal{F}$ to be the translation invariant sub-family of $\mathcal{L}_{\mathbf{k}}$ given by $\mathcal{E} \cup \Theta^1_{\mathbf{k}}(\mathcal{E}) \cup ... \cup \Theta^{n-1}_{\mathbf{k}}(\mathcal{E})$. Clearly $\mathcal{E}$ is an intersecting sub-family of $\mathcal{F}$ that is not a star of $\mathcal{F}$, and $|\mathcal{E}| = |\mathcal{F}|/n$. By Theorem 2.3, $\mathcal{E}$ is a largest intersecting sub-family of $\mathcal{F}$.

Theorem 2.3 enables us to generalize the first part of Theorem 1.5.

Theorem 2.4 *Let $\mathcal{F}$ and m be as in Theorem 2.3. Let $\mathcal{A}_1, ..., \mathcal{A}_p$ be cross-intersecting sub-families of $\mathcal{F}$. Then*

$$\sum_{j=1}^{p} |\mathcal{A}_j| \leq \begin{cases} |\mathcal{F}| & \text{if } p \leq k_m; \\ p\frac{|\mathcal{F}|}{k_m} = p|\{F \in \mathcal{F} \colon (m,1) \in F\}| & \text{if } p \geq k_m. \end{cases}$$

Proof. Similarly to the proof of Theorem 1.5, we set $k_{n+1} := p$, $\mathbf{k}' := (k_1, ..., k_n, k_{n+1})$ and $\mathcal{A} := \bigcup_{j=1}^{p} \{A \cup \{(n+1, j)\} \colon A \in \mathcal{A}_j\}$, and we have that $\mathcal{A}$ is an intersecting family of size $\sum_{j=1}^{p} |\mathcal{A}_j|$. Let $\mathcal{F}' := \{F \cup \{(n+1,j)\} \colon F \in \mathcal{F}, j \in [p]\}$. So $\mathcal{A} \subset \mathcal{F}' \subseteq \mathcal{L}_{\mathbf{k}'}$ and $|\mathcal{F}'| = p|\mathcal{F}|$. Now clearly $\mathcal{F}'$ is translation invariant. Thus the result follows from Theorem 2.3 (with $\mathbf{k}'$ instead of $\mathbf{k}$). $\square$

For the particularly interesting case when $\mathcal{F}$ in the above theorem is $\mathcal{P}_{r,s}$ (which has size $\frac{s!}{(s-r)!}$), we suggest the following conjecture about the extremal structures.

Conjecture 2.5 *If $\mathcal{A}_1, ..., \mathcal{A}_p$ are cross-intersecting sub-families of $\mathcal{P}_{r,s}$ and $\sum_{j=1}^{p} |\mathcal{A}_j|$ is a maximum, then one of the following holds:*
(i) $p < s$ and, for some $q \in [p]$, $\mathcal{A}_q = \mathcal{P}_{r,s}$ and $\mathcal{A}_j = \emptyset$ for all $j \in [p] \backslash \{q\}$;
(ii) $p > s$ and $\mathcal{A}_1 = ... = \mathcal{A}_p = \{A \in \mathcal{P}_{r,s} \colon (a,b) \in A\}$ for some $(a,b) \in [r] \times [s]$;
(iii) $p = s$ and $\mathcal{A}_1, ..., \mathcal{A}_p$ are as in (i) or (ii).

If this conjecture is true, then the largest intersecting sub-families of $\mathcal{P}_{r,s}$ are the stars of $\mathcal{P}_{r,s}$. To see this, consider $\mathcal{A}_1 = ... = \mathcal{A}_p$ (so the families $\mathcal{A}_j$ must be intersecting) and $p > s$ in the above conjecture.

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