Subsums of a Zero-sum Free Subset of an Abelian Group

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Abstract

Let G be an additive finite abelian group and $S \subset G$ a subset. Let $\mathsf{f}(S)$ denote the number of nonzero group elements which can be expressed as a sum of a nonempty subset of S. It is proved that if $|S| = 6$ and there are no subsets of S with sum zero, then $\mathsf{f}(S) \geq 19$. Obviously, this lower bound is best possible, and thus this result gives a positive answer to an open problem proposed by R.B. Eggleton and P. Erdős in 1972. As a consequence, we prove that any zero-sum free sequence S over a cyclic group G of length $|S| \geq \frac{6|G|+28}{19}$ contains some element with multiplicity at least $\frac{6|S|-|G|+1}{17}$.

1 Introduction and Main Results

Let G be an additive abelian group and $S \subset G$ a subset. We denote by $\mathsf{f}(G, S) = \mathsf{f}(S)$ the number of nonzero group elements which can be expressed as a sum of a nonempty subset of S. For a positive integer $k \in \mathbb{N}$ let $\mathsf{F}(k)$ denote the minimum of all $\mathsf{f}(A, T)$, where the minimum is taken over all finite abelian groups A and all zero-sum free subsets $T \subset A$ with $|T| = k$. This invariant $\mathsf{F}(k)$ was first studied by R.B. Eggleton and P. Erdős in 1972 (see [4]). For every $k \in \mathbb{N}$ they obtained a subset S in a cyclic group G with $|S| = k$ such that

$$\mathsf{F}(k) \leq \mathsf{f}(G, S) = \left\lfloor \frac{1}{2}k^2 \right\rfloor + 1 \tag{1.1}$$

(a detailed proof may be found in [8, Section 5.3]), and J.E. Olson ([10]) proved that

$$\mathsf{F}(k) \geq \frac{1}{9}k^2\,.$$

Moreover, Eggleton and Erdős determined $\mathsf{F}(k)$ for all $k \le 5$, and they stated the following conjecture (which holds true for $k \le 5$):

Conjecture 1.1. *For every $k \in \mathbb{N}$ there is a cyclic group G and a zero-sum free subset $S \subset G$ with $|S| = k$ such that $\mathsf{F}(k) = \mathsf{f}(G, S)$.*

Eggleton and Erdős conjectured that $\mathsf{F}(6) = 19$, and it will be a main aim of the present paper to verify this equality. Recently G. Bhowmik et. al. gave an example showing that $\mathsf{F}(7) \le 24$ (see [1]).

Apart from being of interest in their own rights, the invariants $\mathsf{F}(k)$, $k \in \mathbb{N}$, are useful tools in the investigation of various other problems in combinatorial and additive number theory. At the end of Section 8 we outline the connection to Olson's constant $\mathsf{Ol}(G)$. A further application deals with the study of the structure of long zero-sum free sequences. This is a topic going back to J.D. Bovey, P. Erdős and I. Niven ([2]) which found a lot of interest in recent years (see contributions by Gao, Geroldinger, Hamidoune, Savchev, Chen and others [5, 9, 11, 12], and [7, Section 7] for a recent survey). We will use the crucial new result, that $\mathsf{F}(6) = 19$, for further progress on this topic. For convenience we now state our main results (the necessary terminology will be fixed in Section 2).

Theorem 1.2. $\mathsf{F}(6) = 19$.

Theorem 1.3. *Let G be a cyclic group of order $n \ge 3$. If S is a zero-sum free sequence over G of length*

$$|S| \ge \frac{6n + 28}{19},$$

then S contains an element $g \in G$ with multiplicity

$$\mathsf{v}_g(S) \ge \frac{6|S| - n + 1}{17}.$$

In Section 2 we fix our notation and gather the tools needed in the sequel. In Section 3 we present the main idea for the proof of Theorem 1.2, formulate some auxiliary results (Theorem 3.2, Lemmas 3.3 and Lemma 3.4) and show that they easily imply Theorem 1.2. The Sections 4 to 7 are devoted to the proofs of these auxiliary results. In Section 8 we prove Theorem 1.3

Throughout this paper, let G denote an additive finite abelian group.

2 Preliminaries

We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$, and we define $\sup \emptyset = \max \emptyset = \min \emptyset = 0$.

We follow the conventions of [6] for the notation concerning sequences over an abelian group. Let $\mathcal{F}(G)$ denote the multiplicative, free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. An element $S \in \mathcal{F}(G)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

where all $\mathsf{v}_g(S) \in \mathbb{N}_0$ are uniquely determined and called the *multiplicity* of g in S. We say that S *contains* g if $\mathsf{v}_g(S) > 0$. A sequence $T \in \mathcal{F}(G)$ is called a *subsequence* of S if $T \mid S$ in $\mathcal{F}(G)$ (equivalently, $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$). Given any group homomorphism $\varphi \colon G \to G'$, we extend φ to a homomorphism of sequences, $\varphi \colon \mathcal{F}(G) \to \mathcal{F}(G')$, by letting $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. For a sequence S as above we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S\,,$$

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|] \qquad \text{the } \textit{maximum of the multiplicities} \text{ of } S\,,$$

$$\operatorname{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \qquad \text{the } \textit{support} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S\,,$$

$$\Sigma(S) = \Big\{\sum_{i \in I} g_i \Big| \emptyset \ne I \subset [1, l]\Big\} \quad \text{the } \textit{set of subsums} \text{ of } S\,,$$

and

$$\mathsf{f}(G, S) = \mathsf{f}(S) = |\Sigma(S) \setminus \{0\}| \quad \text{the } \textit{number of nonzero subsums} \text{ of } S\,.$$

We say that S is

- *zero-sum free* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- *squarefree* if $\mathsf{v}_g(S) \le 1$ for all $g \in G$.

The unit element $1 \in \mathcal{F}(G)$ is called the *trivial* sequence, and every other sequence is called *nontrivial*. Clearly, S is trivial if and only if S has length $|S| = 0$. In this paper we will deal with subsets of G and with sequences over G. For simplicity and consistency of notation, we will address sets as squarefree sequences throughout this manuscript. For $k \in \mathbb{N}$ we define

$$\begin{aligned}\mathsf{F}(G, k) = \min\big\{|\Sigma(S)| \;\big|\; & S \in \mathcal{F}(G) \text{ is a zero-sum free and} \\ & \text{squarefree sequence of length } |S| = k\big\}\,,\end{aligned}$$

and we denote by $\mathsf{F}(k)$ the minimum of all $\mathsf{F}(A, k)$ where A runs over all finite abelian groups A having a squarefree and zero-sum free sequence of length k. We gather some results on these invariants, which will be needed in the sequel.

Lemma 2.1. *[8, Theorem 5.3.1] It $t \in \mathbb{N}$ and $S = S_1 \cdot \ldots \cdot S_t \in \mathcal{F}(G)$ is zero-sum free, then*

$$\mathsf{f}(S) \geq \mathsf{f}(S_1) + \ldots + \mathsf{f}(S_t)\,.$$

Lemma 2.2.

1. $\mathsf{F}(1) = 1$, $\mathsf{F}(2) = 3$, $\mathsf{F}(3) = 5$ *and* $\mathsf{F}(4) = 8$.

2. *If $S \in \mathcal{F}(G)$ is squarefree, zero-sum free of length $|S| = 3$ and contains no elements of order 2, then $\mathsf{f}(S) \geq 6$.*

3. $\mathsf{F}(k) \geq \frac{1}{9}k^2$ *for all $k \in \mathbb{N}$.*

Proof. 1. See [8, Corollary 5.3.4.1].
2. See [8, Proposition 5.3.2.2].
3. See [10]. ∎

Lemma 2.3. *Let $S = S_1S_2 \in \mathcal{F}(G)$, $H = \langle \mathrm{supp}(S_1) \rangle$ and let $\varphi \colon G \to G/H$ denote the canonical epimorphism. Then we have*

$$\mathsf{f}(S) \geq \big(1 + \mathsf{f}(\varphi(S_2))\big)\mathsf{f}(S_1) + \mathsf{f}\big(\varphi(S_2)\big)\,.$$

Proof. W set $A = \sum(S_1) \cup \{0\}$ and $h = \big|\varphi\big(\Sigma(S_2) \cup \{0\}\big)\,\big|$. Then

$$|A| = 1 + \mathsf{f}(S_1) \quad \text{and} \quad h = 1 + \mathsf{f}\big(\varphi(S_2)\big)\,.$$

Suppose that

$$\varphi\left(\{0\} \cup \sum(S_2)\right) = \{\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_{h-1})\},$$

where $a_0 = 0$ and $a_i \in \sum(S_2)$ for all $i \in [1, h-1]$. Since $A \subset H = \langle \mathrm{supp}(S_1) \rangle$, it follows that

$$A \setminus \{0\}, a_1 + A, \ldots, a_{h-1} + A$$

are pairwise disjoint subsets of $\sum(S)$, and therefore

$$\begin{aligned} \mathsf{f}(S) &\geq |A \setminus \{0\}| + |a_1 + A| + \ldots + |a_{h-1} + A| \\ &= h\big(\mathsf{f}(S_1) + 1\big) - 1\,. \end{aligned}$$

∎

Lemma 2.4. *Let $S \in \mathcal{F}(G)$ be zero-sum free.*

1. *If $T \in \mathcal{F}\big(\mathrm{supp}(S)\big)$ and $U \in \mathcal{F}(G)$ such that $U \mid T$ and $TU^{-1} \mid S$, then $\sigma(U) \neq \sigma(T)$.*

2. *If $T_1, T_2 \in \mathcal{F}(G)$ are squarefree with $|T_1| = |T_2|$ and $|\gcd(T_1, T_2)| = |T_1| - 1$, then $\sigma(T_1) \neq \sigma(T_2)$.*

Proof. 1. Since S is zero-sum free and $TU^{-1} \mid S$, we have $\sigma(TU^{-1}) \neq 0$. Since $T = (TU^{-1})U$, we get $\sigma(T) = \sigma(TU^{-1}) + \sigma(U)$ and hence $\sigma(U) \neq \sigma(T)$.
2. Obvious. ∎

3 Proof of Theorem 1.2

Let $S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ be a squarefree, zero-sum free sequence of length $|S| = k \in \mathbb{N}$, and let $\mathcal{A}$ be the set of all nontrivial subsequences of S. We partition $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1 \uplus \ldots \uplus \mathcal{A}_r \,,$$

where two subsequences T, T' of S are in the same class $\mathcal{A}_\nu$, for some $\nu \in [1, r]$, if $\sigma(T) = \sigma(T')$. Thus we have $r = \mathsf{f}(S) = |\Sigma(S)|$. For a subset $\mathcal{B} \subset \mathcal{A}$ we set

$$\overline{\mathcal{B}} = \{ST^{-1} \mid T \in \mathcal{B}\} \,.$$

Then, for every $\nu \in [1, r]$, we clearly have $\overline{\mathcal{A}_\nu} \in \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, and $\overline{\mathcal{A}_\nu}$ will be called the *dual class* of $\mathcal{A}_\nu$. For a nontrivial subsequence T of S we denote by $[T]$ the class of T. The following easy observation will be useful.

Lemma 3.1. *Let all notations be as above, and let $i \in [1, r]$. Then the following statements hold*:

1. *For a subset $\mathcal{B} \subset \mathcal{A}$, we have $\mathcal{B} \in \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$ if and only if $\overline{\mathcal{B}} \in \{\mathcal{A}_1, \ldots, \mathcal{A}_r\}$, and $|\mathcal{B}| = |\overline{\mathcal{B}}|$.*
2. *$\mathcal{A}_i$ is the dual class of itself if and only if $\sigma(T) = \sigma(ST^{-1})$ for some $T \in \mathcal{A}_i$.*
3. *If $\mathcal{A}_i$ contains subsequences T and T' with $|T| = 1$ and $|T'| = k - 1$, then $S = TT'$ and $\mathcal{A}_i = \{T, T'\}$.*
4. *If $\mathcal{A}_i$ is the dual class of itself and $\mathcal{A}_i$ contains a subsequence of length 1, then $|\mathcal{A}_i| = 2$.*
5. *If $\mathcal{A}_i$ is the dual class of itself, then $|\mathcal{A}_i|$ is even.*
6. *$[S] = \{S\}$.*

In order to prove Theorem 1.2, we need the following three results.

Theorem 3.2. *Let $S \in \mathcal{F}(G)$ be a squarefree, zero-sum free sequence of length $|S| = k \in [4, 7]$. If S contains some element of order 2, then*

$$\mathsf{f}(S) \geq \left\lfloor \frac{k^2}{2} \right\rfloor + 1 \,.$$

Lemma 3.3. *Let $S \in \mathcal{F}(G)$ be a squarefree, zero-sum free sequence of length $|S| = 6$ which contains no elements of order 2. Then $|[x_k]| \le 4$ for all $k \in [1,6]$. Moreover, if $|[x_i]| = |[x_j]| = 4$ for some $i, j \in [1,6]$ with $i < j$, then*

$$\mathsf{f}(S) \ge 19\,.$$

Lemma 3.4. *Let $S \in \mathcal{F}(G)$ be a squarefree, zero-sum free sequence of length $|S| = 6$ which contains no elements of order 2, and let $\mathcal{A}_1, \ldots, \mathcal{A}_r$ be defined as above. Then $|\mathcal{A}_i| \le 5$ for all $i \in [1,r]$, and if $|\mathcal{A}_i| = 5$ for some $i \in [1,r]$, then*

$$\mathsf{f}(S) \ge 19\,.$$

Proof of Theorem 1.2, based on 3.2, 3.3 and 3.4

By [8, Corollary 5.3.4.2] it follows that $\mathsf{F}(6) \le 19$, and hence it suffices to verify the reverse inequality. Let $S = x_1 \cdot \ldots \cdot x_6 \in \mathcal{F}(G)$ be a squarefree zero-sum free sequence. We need to show

$$\mathsf{f}(S) \ge 19\,.$$

If S contains an element of order 2, then Theorem 3.2 implies that $\mathsf{f}(S) \ge 19$. So we may assume that S contains no elements of order 2. By Lemma 3.3 and Lemma 3.4, we may assume there exists at most one $i \in [1,r]$ such that $|[x_i]| = 4$ and that $|\mathcal{A}_j| \le 4$ for all $j \in [1,r]$.

We set

$$L = \sum_{i=1}^{r} |\mathcal{A}_i| = 2^6 - 1 = 63\,.$$

Assume that $S \in \mathcal{A}_r$. Then $\mathcal{A}_r = \{S\}$ and thus $\mathcal{A}_r$ contributes 1 to the sum L. Next let t be the number of those $i \in [1,6]$ with $[x_i] = \overline{[x_i]}$, say $x_1, \ldots, x_t$ have this property. If $i \in [1,t]$, then Lemma 3.1 implies that $[x_i] = \{x_i, x_i^{-1}S\}$ and hence $|[x_i]| = 2$. Thus we get $|[x_1]| + \ldots + |[x_t]| = 2t$. Since S is squarefree, $i, j \in [1,6]$ with $i \ne j$ implies that $\overline{[x_i]} \ne [x_j]$. Excluding the above self-dual classes, the remaining $[x_i]$ and $\overline{[x_i]}$ contribute at most $4 \times 2 + 3 \times 2(6 - t - 1) = 38 - 6t$ to the sum L, that is

$$\sum_{i=t+1}^{6} \left(|[x_i]| + |\overline{[x_i]}|\right) \le 38 - 6t\,.$$

Finally, by excluding $\mathcal{A}_r$, all $[x_i]$ and their dual class $\overline{[x_i]}$, we have $r-1-t-2(6-t)$ classes left. These remaining classes contribute at most $4 \times (r - 1 - t - 2(6-t)) = 4r - 52 + 4t$ to L. Adding up these numbers, we obtain

$$1 + 2t + (38 - 6t) + (4r - 52 + 4t) \ge L = 63.$$

This gives that $4r \ge 76$ and therefore $\mathsf{f}(S) = r \ge 19$ as desired. ∎

The proofs of Theorem 3.2 and of the Lemmas 3.3 and 3.4 will be given in Sections 4 to 7. Throughout these sections, let

$$S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$$

be a squarefree, zero-sum free sequence of length $|S| = k \in \mathbb{N}$, and let $\mathcal{A}_1, \ldots, \mathcal{A}_r$ be as introduced in the beginning of this section.

4 Proof of Theorem 3.2

Without loss of generality we may assume that $\mathrm{ord}(x_1) = 2$. We set $S = S_1 S_2$, where $S_1 = x_1$ and $S_2 = x_2 \cdot \ldots \cdot x_k$. Then $\mathsf{f}(S_1) = 1$. Let $H = \langle x_1 \rangle = \{0, x_1\}$ and $\varphi \colon G \to G/H$ the canonical epimorphism. Then $\varphi(S_2) = \varphi(x_2) \cdot \ldots \cdot \varphi(x_k)$.

First, we assert that $\varphi(S_2)$ is zero-sum free. Assume to the contrary that there is a nontrivial subsequence U of S_2 such that $\sigma(\varphi(U)) = \varphi(\sigma(U)) = 0$. Then $\sigma(U) \in H$. Since S is zero-sum free, $\sigma(U) \neq 0$, so $\sigma(U) = x_1$. Then $\sigma(S_1 U) = \sigma(S_1) + \sigma(U) = x_1 + x_1 = 0$, a contradiction. Thus $\varphi(S_2)$ is zero-sum free.

Next, we show that $\mathrm{h}(\varphi(S_2)) \leq 2$. Assume to the contrary that $\varphi(x_{i_1}) = \varphi(x_{i_2}) = \varphi(x_{i_3})$ for some pairwise distinct $i_1, i_2, i_3 \in [1, k]$. Then $\varphi(x_{i_1} - x_{i_2}) = \varphi(x_{i_1} - x_{i_3}) = 0$, so $x_{i_1} - x_{i_2}, x_{i_1} - x_{i_3} \in H$. Since S is squarefree, it follows that $x_{i_1} - x_{i_2} \neq 0$ and $x_{i_1} - x_{i_3} \neq 0$. Thus $x_{i_1} - x_{i_2} = x_{i_1} - x_{i_3} = x_1$, and so $x_{i_2} = x_{i_3}$, a contradiction. This proves that $\mathrm{h}(\varphi(S_2)) \leq 2$.

We distinguish four cases as follows.

Case 1: $k = 4$. Since $\mathrm{h}(\varphi(S_2)) \leq 2$, $\varphi(S_2)$ allows a product decomposition $\varphi(S_2) = U_1 U_2$ into squarefree sequences $U_1, U_2 \in \mathcal{F}(G/H)$ with $|U_1| = 2$ and $|U_2| = 1$. It follows from Lemma 2.2 and Lemma 2.1 that

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 3 + 1 = 4.$$

By Lemma 2.3, we have

$$\mathsf{f}(S) \geq (1 + \mathsf{f}(\varphi(S_2)))\mathsf{f}(S_1) + \mathsf{f}(\varphi(S_2)) \geq (1 + 4) \times 1 + 4 = 9,$$

and we are done.

Case 2: $k = 5$. Since $\mathrm{h}(\varphi(S_2)) \leq 2$, $\varphi(S_2)$ allows a product decomposition $\varphi(S_2) = U_1 U_2$ into squarefree sequences $U_1, U_2 \in \mathcal{F}(G/H)$ with $|U_1| = |U_2| = 2$. By Lemma 2.2 and Lemma 2.1, we have

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 3 + 3 = 6.$$

By Lemma 2.3, we have

$$\mathsf{f}(S) \geq (1 + \mathsf{f}(\varphi(S_2)))\mathsf{f}(S_1) + \mathsf{f}(\varphi(S_2)) \geq (1 + 6) \times 1 + 6 = 13,$$

and we are done.

Case 3: $k = 6$. By Lemma 2.3, we have $\mathsf{f}(S) \geq (1 + \mathsf{f}(\varphi(S_2)))\mathsf{f}(S_1) + \mathsf{f}(\varphi(S_2))$. If we can show that $\mathsf{f}(\varphi(S_2)) \geq 9$, then $\mathsf{f}(S) \geq 19$ as desired. Since $\mathrm{h}(\varphi(S_2)) \leq 2$, we have $|\mathrm{supp}(\varphi(S_2))| \geq 3$.

If $|\mathrm{supp}(\varphi(S_2))| \geq 4$, $\varphi(S_2)$ allows a product decomposition $\varphi(S_2) = U_1 U_2$ into squarefree sequences $U_1, U_2 \in \mathcal{F}(G/H)$ with $|U_1| = 4$ and $|U_2| = 1$. By Lemma 2.2 and Lemma 2.1,

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 8 + 1 = 9$$

and we are done.

Next, suppose $|\mathrm{supp}(\varphi(S_2))| = 3$ and $\varphi(S_2) = a^2 b^2 c$. Since $\varphi(S_2)$ is zero-sum free, we must have $\mathrm{ord}(a) \neq 2$ and $\mathrm{ord}(b) \neq 2$. If $\mathrm{ord}(c) \neq 2$, then we set $U_1 = a \cdot b \cdot c$ and $U_2 = a \cdot b$. By Lemma 2.1 and Lemma 2.2,

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 6 + 3 = 9,$$

and we are done. So we may assume that $\mathrm{ord}(c) = 2$. Then

$$a, a + b, 2a + b, 2a + 2b, c, a + c, a + b + c, 2a + b + c, 2a + 2b + c$$

are pairwise distinct, whence $\mathsf{f}(\varphi(S_2)) \geq 9$ and we are done.

Case 4: $k = 7$. If $\mathsf{f}(\varphi(S_2)) \geq 12$, then by Lemma 2.3, $\mathsf{f}(S) \geq (1 + \mathsf{f}(\varphi(S_2)))\mathsf{f}(S_1) + \mathsf{f}(\varphi(S_2)) \geq (1 + 12) \times 1 + 12 = 25$ as desired. It suffices to show $\mathsf{f}(\varphi(S_2)) \geq 12$. Since $\mathrm{h}(\varphi(S_2)) \leq 2$, we have $|\mathrm{supp}(\varphi(S_2))| \geq 3$.

If $\varphi(S_2)$ contains no elements of order 2, $\varphi(S_2)$ allows a product decomposition $\varphi(S_2) = U_1 U_2$ into squarefree sequences $U_1, U_2 \in \mathcal{F}(G/H)$ with $|U_1| = |U_2| = 3$. By Lemma 2.1 and Lemma 2.2,

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 6 + 6 = 12$$

and we are done.

If $\varphi(S_2)$ contains an element of order 2. Then $|\mathrm{supp}(\varphi(S_2))| \geq 4$. Since $\mathrm{h}(\varphi(S_2)) \leq 2$, $\varphi(S_2)$ allows a product decomposition $\varphi(S_2) = U_1 U_2$ into squarefree sequences $U_1, U_2 \in \mathcal{F}(G/H)$ such that $|U_1| = 4, |U_2| = 2$, and U_1 contains some element of order 2. It follows from Case 1 that $\mathsf{f}(U_1) \geq 9$. By Lemma 2.2 and Lemma 2.1,

$$\mathsf{f}(\varphi(S_2)) \geq \mathsf{f}(U_1) + \mathsf{f}(U_2) \geq 9 + 3 = 12$$

and we are done. ∎

5 On The Maximal Size of Classes

The following result provides an upper bound for $|\mathcal{A}_1|, \ldots, |\mathcal{A}_r|$, under the assumption that S contains no elements of order 2.

Lemma 5.1. *Suppose that S contains no elements of order 2. Then the following hold.*

1. *If $k \le 4$, then $|\mathcal{A}_i| \le 2$ for every $i \in [1, r]$.*

2. *If $k = 5$, then $|\mathcal{A}_i| \le 3$ for every $i \in [1, r]$.*

3. *If $k = 6$, then $|[x_i]| = |\overline{[x_i]}| \le 4$ for every $i \in [1, 6]$, and $|\mathcal{A}_i| \le 5$ for every $i \in [1, r]$.*

Proof. Take an arbitrary $i \in [1, r]$, and let

$$\mathcal{A}_i = \{S_1, \ldots, S_l\}$$

where $S_1, \ldots, S_l$ are subsequences of S and $1 \le |S_1| \le |S_2| \le \cdots \le |S_l|$. Then $|\mathcal{A}_i| = l$.

Case 1: $k \le 4$. The result follows from Lemma 2.4.

Case 2: $k = 5$.

If $\mathcal{A}_i = [x_j]$ for some $j \in [1, 5]$, then we may assume that $S_1 = x_j$. By Lemma 2.4, we have

$$S_\nu \,|\, x_j^{-1} x_1 \cdot \ldots \cdot x_5 \quad \text{for every} \quad \nu \in [2, l]\,.$$

Let $\mathcal{B} = \overline{\{S_2, \ldots, S_l\}}$. Then by Case 1 we have $|\mathcal{B}| \le 2$ and thus $l \le 3$. Therefore, $|[x_j]| = |\overline{[x_j]}| \le 3$ for every $j \in [1, 5]$.

Next we assume that $\mathcal{A}_i$ contains neither a sequence of length 1 nor a sequence of length 4. So $2 \le |S_1| \le \cdots \le |S_l| \le 3$. Assume to the contrary that $l \ge 4$. If $|S_1| = |S_2| = |S_3| = 2$, then there exist $m, n \in [1, 3]$ such that $|\gcd(S_m, S_n)| = 1$, a contradiction. So $|S_3| = 3$. If $|S_{l-2}| = |S_{l-1}| = |S_l| = 3$, then there exist $m, n \in \{l-2, l-1, l\}$ such that $|\gcd(S_m, S_n)| = 2$, a contradiction again. So $|S_{l-2}| = 2$. This forces that $l = 4$ and $|S_1| = |S_2| = 2, |S_3| = |S_4| = 3$. Now, let $S_1 = x_1 \cdot x_2, S_2 = x_3 \cdot x_4$. By Lemma 2.4, $x_5 \,|\, S_3$ and $x_5 \,|\, S_4$. Without loss of generality, we may assume that $x_1 \cdot x_3 \,|\, S_3$, so $x_2 \cdot x_4 \,|\, S_4$. Thus $\mathcal{A}_i = \{x_1 \cdot x_2,\ x_3 \cdot x_4,\ x_1 \cdot x_3 \cdot x_5,\ x_2 \cdot x_4 \cdot x_5\}$, and then $(x_1 + x_2) + (x_3 + x_4) = (x_1 + x_3 + x_5) + (x_2 + x_4 + x_5)$. Therefore, $0 = 2x_5$, a contradiction.

Case 3: $k = 6$. Assume that $\mathcal{A}_i = [x_j]$ for some $j \in [1, 6]$ and $S_1 = x_j$. As before, we have

$$S_\nu \,|\, x_j^{-1} x_1 \cdot \ldots \cdot x_6 \quad \text{for every} \quad \nu \in [2, l]\,.$$

Consider $\mathcal{B} = \overline{\{S_2, \ldots, S_l\}}$. By Case 2 we have $|\mathcal{B}| \le 3$ and thus $l \le 4$. Therefore, $|[x_j]| = |\overline{[x_j]}| \le 4$ for every $j \in [1, 6]$.

Next assume that $\mathcal{A}_i$ contains neither a sequence of length 1 nor of length 5, so $2 \le |S_1| \le |S_2| \le \cdots \le |S_l| \le 4$. We have to show that $l \le 5$. Assume to the contrary that $l \ge 6$. Define $T = S_1 \cdot \ldots \cdot S_l$.

For every $a \,|\, S$, we have that $|\{i \mid a \,|\, S_i\}| + |\{i \mid a \nmid S_i\}| = l \ge 6$. By Case 2, $|\{i \mid a \nmid S_i\}| \le 3$ and $|\{i \mid a \,|\, S_i\}| \le 3$. These force that $|\{i \mid a \,|\, S_i\}| = |\{i \mid a \nmid S_i\}| = 3$ and $l = 6$. Thus,

$$\mathsf{v}_a(T) = 3$$

for every $a \in S$. Hence, $|T| = 18$.

Let $r_t = |\{i \mid |S_i| = t\}|$ for every $t \in [2, 4]$. Then $2r_2 + 3r_3 + 4r_4 = |T| = 18$. Therefore, r_3 is even and hence $r_3 \in \{0, 2, 4, 6\}$. We distinguish two subcases according to whether $r_3 \ge 4$ or not.

Subcase 3.1: $r_3 \geq 4$. We may assume that $|S_2| = |S_3| = |S_4| = |S_5| = 3$. From $|T| = 18$ we infer that $|S_1| + |S_6| = 6$. If $|\gcd(S_1, S_6)| = 0$, then $S_1 = SS_6^{-1}$. By Lemma 3.1.2, $\mathcal{A}_i = \overline{\mathcal{A}_i}$. So, we may assume that $S_2 = SS_5^{-1}$. By Lemma 2.4.2, $|\gcd(S_3, S_2)| \leq 1$ and $|\gcd(S_3, S_5)| \leq 1$. Thus $|S_3| = |\gcd(S_3, S)| = |\gcd(S_3, S_2)| + |\gcd(S_3, S_5)| \leq 2$, a contradiction. Therefore, $|\gcd(S_1, S_6)| > 0$. Let $a \mid \gcd(S_1, S_6)$. Since $\mathsf{v}_a(T) = 3$ we may assume that $a \nmid S_i$ for every $i \in [2, 4]$. Therefore, S_2, S_3 and S_4 divide $a^{-1}S$ and we must have $|\gcd(S_n, S_m)| = 2$ for some distinct $m, n \in [2, 4]$, a contradiction to Lemma 2.4.2.

Subcase 3.2: $r_3 < 4$. Then, $r_3 \in \{0, 2\}$. From $|T| = 18$ we know that $r_2 \geq 2$ and $r_4 \geq 2$. We may assume that $|S_1| = |S_2| = 2$ and $|S_5| = |S_6| = 4$. Furthermore, we may assume that $S_1 = x_1 \cdot x_2, S_2 = x_3 \cdot x_4$. By Lemma 2.4 we infer that $x_5 \cdot x_6 \mid S_5$ and $x_5 \cdot x_6 \mid S_6$. So we may assume that $S_5 = x_1 \cdot x_3 \cdot x_5 \cdot x_6$ and $S_6 = x_2 \cdot x_4 \cdot x_5 \cdot x_6$. Again, by Lemma 2.4 we know that $|S_3| \neq 2$. It follows from $|T| = 18$ that $|S_3| = |S_4| = 3$. Since $\mathsf{v}_a(T) = 3$ for every $a \mid S$, we have $S_3 S_4 = S$, implying $\sigma(S_3) = \sigma(SS_3^{-1})$. By Lemma 3.1.2, $\mathcal{A}_i = \overline{\mathcal{A}_i}$. But $SS_1^{-1} = x_3 \cdot x_4 \cdot x_5 \cdot x_6 \notin \mathcal{A}_i$, a contradiction. This proves $l \leq 5$. ∎

6 Proof of $\mathsf{F}(5) = 13$

R.B. Eggleton and Erdős stated in [4] that they gave a proof of $\mathsf{F}(5) = 13$ in [3] as an appendix. Since we could not find this note, we include a proof of $\mathsf{F}(5) = 13$ here for completeness. Moreover, the ideas and methods in our proof will be used frequently in the sequel.

We denote by P_n the symmetric group on $[1, n]$. Note that it follows from [8, Corollary 5.3.4.2] that $\mathsf{F}(5) \leq 13$.

Lemma 6.1. *Let $T = (-2x) \cdot x \cdot (3x) \cdot (4x) \cdot (5x) \in \mathcal{F}(G)$ be a squarefree, zero-sum free sequence. Then $\mathsf{f}(T) \geq 13$.*

Proof. Obviously, $kx \in \Sigma(T)$ for all $k \in [1, 13]$. Since T is zero-sum free, $kx \neq 0$ holds for every $k \in [1, 13]$, and thus $ix \neq jx$ for any $i \neq j \in [1, 13]$. Therefore, $\mathsf{f}(T) \geq 13$. ∎

Lemma 6.2. *Let $S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ be as fixed at the end of Section 3, and suppose that $k = 5$. If $|[x_i]| = 3$ for some $i \in [1, 5]$, then $[x_i]$ is of one of the following forms:*

(1) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)},\ x_{\tau(4)} \cdot x_{\tau(5)}\}$.

(2) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)},\ x_{\tau(2)} \cdot x_{\tau(4)} \cdot x_{\tau(5)}\}$

for some $\tau \in P_5$.

Proof. Without loss of generality, we may assume that $i = 1$ and $[x_i] = \{x_1, S_2, S_3\}$ with $2 \leq |S_2| \leq |S_3|$. By Lemma 3.1, we know that $|S_3| \leq 3$. Note that

$$S_2 \mid x_2 \cdot \ldots \cdot x_5 \quad \text{and} \quad S_3 \mid x_2 \cdot \ldots \cdot x_5 \,.$$

By Lemma 2.4.2, we infer that $|S_2| = 2$. So, we may assume that $S_2 = x_2 \cdot x_3$. If $|S_3| = 2$, then $S_3 = x_4 \cdot x_5$. Therefore, $[x_1]$ is of form (1) and we are done. Otherwise, $|S_3| = 3$, by Lemma 2.4, we know that $S_3 = x_2 \cdot x_4 \cdot x_5$ or $S_3 = x_3 \cdot x_4 \cdot x_5$. Therefore, $[x_1]$ is of form (2). ∎

The following easy observation will also be useful.

Lemma 6.3. *Let $S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ be as fixed at the end of Section 3, and suppose that $k \geq 3$. Let a, b, c be distinct in $[1, k]$ such that $x_a = x_b + x_c$. Suppose that S contains no element of order 2. Then, $x_b - x_a \notin \operatorname{supp}(S)$.*

Proof. Assume to the contrary that $x_b - x_a = x_d$ for some $d \in [1, k]$. This together with $x_a = x_b + x_c$ gives that $x_c + x_d = 0$, a contradiction. ∎

Proof of $\mathsf{F}(5) = 13$.

Let $S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ be as fixed at the end of Section 3, and suppose that $k = 5$. We have to show

$$\mathsf{f}(S) \geq 13\,.$$

Assume to the contrary that $\mathsf{f}(S) < 13$ for some S. By Theorem 3.2, S contains no elements of order 2, and thus it follows from Lemma 5.1 that $|\mathcal{A}_i| \leq 3$ for all $i \in [1, r]$.

Recall that $\mathcal{A}_r = [S] = \{S\}$. We may assume that $|\mathcal{A}_1| \leq 2, \ldots, |\mathcal{A}_t| \leq 2$ and $|\mathcal{A}_{t+1}| = \ldots = |\mathcal{A}_{r-1}| = 3$. If $t \geq 4$, since $2t + 3(r-1-t) + 1 \geq 31$, then $r \geq (33+t)/3 \geq 37/3$. Therefore $r \geq 13$, a contradiction. Therefore, $t \leq 3$.

Now $|[x_i]| = |[x_j]| = 3$ for some $i, j \in [1, 5]$ with $i \neq j$. Without loss of generality, we may assume that $i = 1$. We distinguish two cases.

Case 1. $[x_1]$ is of form (1) in Lemma 6.2. We may assume that $[x_1] = \{x_1,\ x_2 \cdot x_3,\ x_4 \cdot x_5\}$. Without loss of generality, we may assume that $j = 2$. Let $[x_2] = \{x_2, S_2, S_3\}$ with $2 \leq |S_2| \leq |S_3|$. Since $x_1 = x_2 + x_3$, by Lemma 6.3 we know that $x_2 - x_1 \nmid S$. Thus $[x_2]$ is not of form (1). Therefore, by Lemma 6.2, $[x_2]$ is of form (2) and $|S_2| = 2$. Again by Lemma 6.3 we know that $x_1 \nmid S_2$. It follows from Lemma 2.4 that $x_2 \nmid S_2$. Since $x_1 = x_4 + x_5$ we have $S_2 \neq x_4 \cdot x_5$. Therefore, $S_2 = x_3 \cdot x_4$ or $S_2 = x_3 \cdot x_5$. So, we may assume that $S_2 = x_3 \cdot x_4$. Now by Lemma 6.2 we obtain that $S_3 = x_3 \cdot x_1 \cdot x_5$ or $S_3 = x_4 \cdot x_1 \cdot x_5$. Therefore, $x_3 + x_4 = x_3 + x_1 + x_5$ or $x_3 + x_4 = x_4 + x_1 + x_5$. Thus $x_4 - x_1 = x_5$ or $x_3 - x_1 = x_5$. This together with $x_1 = x_2 + x_3 = x_4 + x_5$ gives a contradiction to Lemma 6.3.

Case 2. $[x_1]$ is of form (2) in Lemma 6.2. We may assume that $[x_1] = \{x_1, x_2 \cdot x_3, x_2 \cdot x_4 \cdot x_5\}$. Now we have $x_3 = x_4 + x_5$. If $[x_j]$ is of form (1), then this reduces to Case 1. So we may assume that $[x_j]$ is of form (2). Let $[x_j] = \{x_j, S_2, S_3\}$ with $|S_2| = 2$ and $|S_3| = 3$. We distinguish subcases.

Subcase 2.1 $j = 2$. $[x_2] = \{x_2, S_2, S_3\}$. Note that $x_3 = x_4 + x_5$. By Lemma 6.3 and Lemma 2.4, we obtain that $S_2 = x_3 \cdot x_4$ or $S_2 = x_3 \cdot x_5$. Without loss of generality, we may assume that $S_2 = x_3 \cdot x_4$. Now by Lemma 6.2, we get $S_3 = x_3 \cdot x_1 \cdot x_5$ or $S_3 = x_4 \cdot x_1 \cdot x_5$. If $S_3 = x_4 \cdot x_1 \cdot x_5$, then $x_3 + x_4 = x_4 + x_1 + x_5$. Thus $x_4 + x_5 = x_3 = x_1 + x_5$, a contradiction. Therefore, $S_3 = x_3 \cdot x_1 \cdot x_5$. Now we have $x_1 = x_2 + x_3 = x_2 + x_4 + x_5$ and $x_2 = x_3 + x_4 = x_1 + x_3 + x_5$. Thus $x_1 = 5x_3, x_2 = 4x_3, x_4 = 3x_3, x_5 = -2x_3$. It follows from Lemma 6.1 that $\mathsf{f}(S) \geq 13$, a contradiction. Therefore, $|[x_2]| \leq 2$.

Subcase 2.2. $j = 4$. Now $[x_4] = \{x_4, S_2, S_3\}$. Since $x_3 = x_4 + x_5$, by Lemma 6.3 we have $x_3 \nmid S_2$. Therefore, $S_2 \mid x_1 \cdot x_2 \cdot x_5$. Hence, $S_2 = x_1 \cdot x_2$ or $S_2 = x_2 \cdot x_5$

or $S_2 = x_1 \cdot x_5$. If $S_2 = x_1 \cdot x_2$, by Lemma 2.4 we obtain that $S_3 = x_1 \cdot x_3 \cdot x_5$ or $S_3 = x_2 \cdot x_3 \cdot x_5$. Since $x_2 + x_3 + x_5 = x_1 + x_5 \neq x_1 + x_2$ we get $S_3 = x_1 \cdot x_3 \cdot x_5$. Now we have $x_1 = 4x_2, x_3 = 3x_2, x_4 = 5x_2, x_5 = -2x_2$ and thus it follows from Lemma 6.1 that $\mathsf{f}(S) \geq 13$, a contradiction. Therefore, $S_2 \neq x_1 \cdot x_2$. If $S_2 = x_2 \cdot x_5$, then by Lemma 2.4, we obtain that $S_3 = x_2 \cdot x_1 \cdot x_3$ or $S_3 = x_5 \cdot x_1 \cdot x_3$. Thus $x_2 + x_5 = x_2 + x_1 + x_3$ or $x_2 + x_5 = x_5 + x_1 + x_3$. So, $x_5 - x_3 = x_1$ or $x_2 - x_1 = x_3$, contradicting $x_3 = x_4 + x_5$ or $x_1 = x_2 + x_3$ (in view of Lemma 6.3). Hence, $S_2 = x_1 \cdot x_5$. As above, we obtain that $S_3 = x_1 \cdot x_2 \cdot x_3$ or $S_3 = x_5 \cdot x_2 \cdot x_3$. Since $x_1 + x_5 \neq x_1 + x_2 + x_3 = 2x_1$, we obtain that $S_3 = x_5 \cdot x_2 \cdot x_3$. Therefore,

$[x_4] = \{x_4,\ x_1 \cdot x_5,\ x_5 \cdot x_2 \cdot x_3\}$.

We assert that $|[x_5]| \leq 2$ in this subcase. Assume to the contrary that $|[x_5]| = 3$. As above, we may assume that $[x_5] = \{x_5,\ x_1 \cdot x_4,\ x_4 \cdot x_2 \cdot x_3\}$. Now we have $x_5 = x_1 + x_4$, a contradiction to $x_4 = x_1 + x_5$ (in view of Lemma 6.3). This proves the assertion.

Next, we show that $|[x_3]| \leq 2$ in this subcase. Assume to the contrary that $|[x_3]| = 3$. Then $[x_3] = \{x_3,\ x_4 \cdot x_5,\ T_3\}$ with $|T_3| = 3$.

By Lemma 2.4, $T_3 = x_4 \cdot x_1 \cdot x_2$ or $T_3 = x_5 \cdot x_1 \cdot x_2$. Since $x_5 + x_1 + x_2 \neq x_1 + 2x_5 = x_4 + x_5$, we have $T_3 \neq x_5 \cdot x_1 \cdot x_2$. Therefore, $T_3 = x_4 \cdot x_1 \cdot x_2$. Now we have $x_3 = x_4 + x_5 = x_4 + x_1 + x_2$. In view of $[x_1]$ and $[x_4]$, we derive that $x_1 = 3x_5, x_2 = -2x_5, x_3 = 5x_5, x_4 = 4x_5$ and thus $\mathsf{f}(S) \geq 13$ by Lemma 6.1, a contradiction. Therefore, we must have $|[x_3]| \leq 2$.

Since $x_3 = x_4 + x_5$, we have $[x_3] \neq \overline{[x_3]}$. Now $[x_2], [x_3], \overline{[x_3]}$ and $[x_5]$ are distinct and all have length not exceeding two, contradicting $t \leq 3$. Therefore, $j \neq 4$, or equivalently, $|[x_4]| \leq 2$.

Similarly, we conclude that $|[x_5]| \leq 2$.

Subcase 2.3. $j = 3$. Since $x_3 = x_4 + x_5$, we have $[x_3] = \{x_3,\ x_4 \cdot x_5,\ S_3\}$. By Lemma 2.4, $S_3 = x_4 \cdot x_1 \cdot x_2$ or $S_3 = x_5 \cdot x_1 \cdot x_2$. We may assume that $S_3 = x_4 \cdot x_1 \cdot x_2$. Then $x_4 + x_5 = x_4 + x_1 + x_2$, and thus $x_5 = x_1 + x_2$. Therefore, $[x_5], \overline{[x_5]}, [x_2]$ and $[x_4]$ are distinct, contradicting $t \leq 3$.

This completes the proof. ∎

7 On the number of maximal classes

Let $S = x_1 \cdot \ldots \cdot x_k \in \mathcal{F}(G)$ be as fixed at the end of Section 3, and suppose that $k = 6$. We shall prove Lemma 3.3 and Lemma 3.4 through a series of lemmas.

Lemma 7.1. *If S is of one of the following forms:*

(i) $S = (-7x) \cdot (-6x) \cdot (-5x) \cdot (-2x) \cdot x \cdot (3x)$;

(ii) $S = (-2x) \cdot x \cdot (3x) \cdot (4x) \cdot (5x) \cdot (7x)$;

(iii) $S = (-2x) \cdot x \cdot (3x) \cdot (4x) \cdot (5x) \cdot (6x)$;

(iv) $S = (-6x) \cdot (-5x) \cdot (-4x) \cdot (-3x) \cdot (-2x) \cdot x$;

(*v*) $S = x \cdot (2x) \cdot (3x) \cdot (4x) \cdot (5x) \cdot (6x)$,

then $\mathsf{f}(S) \geq 19$.

Proof. We give only the proof for the case when S is of form (i). The proofs for other cases are similar and are omitted.

Suppose that $S = (-7x) \cdot (-6x) \cdot (-5x) \cdot (-2x) \cdot x \cdot (3x)$. Clearly, $kx \in \Sigma(S)$ for any $k \in [-19, -1]$. Since S is zero-sum free, $kx \neq 0$ for any $k \in [-19, -1]$. Then $ix \neq jx$ for any $i, j \in [-19, -1]$, and therefore, $\mathsf{f}(S) \geq 19$ as desired. ∎

7.1. Classes of size 4 containing sequences of length 1

This subsection deals with classes of size 4 having a sequence of length 1, and it provides a proof for Lemma 3.3.

Lemma 7.2. *If* $|[x_i]| = 4$ *for some* $i \in [1, 6]$*, then there exists* $\tau \in P_6$ *such that* $[x_i]$ *is of one of the following forms:*

(b1) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)} \cdot x_{\tau(4)} \cdot x_{\tau(5)},\ x_{\tau(2)} \cdot x_{\tau(6)},\ x_{\tau(3)} \cdot x_{\tau(4)} \cdot x_{\tau(6)}\}$;

(b2) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)} \cdot x_{\tau(4)} \cdot x_{\tau(5)},\ x_{\tau(2)} \cdot x_{\tau(3)} \cdot x_{\tau(6)},\ x_{\tau(4)} \cdot x_{\tau(5)} \cdot x_{\tau(6)}\}$;

(b3) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)},\ x_{\tau(4)} \cdot x_{\tau(5)},\ x_{\tau(2)} \cdot x_{\tau(4)} \cdot x_{\tau(6)}\}$;

(b4) $\{x_{\tau(1)},\ x_{\tau(2)} \cdot x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(2)} \cdot x_{\tau(5)} \cdot x_{\tau(6)},\ x_{\tau(3)} \cdot x_{\tau(5)}\}$.

Proof. Let $[x_i] = \{S_1, S_2, S_3, S_4\}$ where S_1, S_2, S_3, S_4 are subsequences of S and $|S_1| \leq |S_2| \leq |S_3| \leq |S_4|$. Without loss of generality, we may assume that $S_1 = x_1$. By Lemma 2.4, we have

$$S_\nu \,|\, x_1^{-1}S = x_2 \cdot \ldots \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]$$

and $2 \leq |S_2| \leq |S_3| \leq |S_4| \leq 5$.

We first show that $3 \leq |S_4| \leq 4$. If $|S_4| = 5$, then $S_4 = x_2 \cdot \ldots \cdot x_6$. But $S_2 \,|\, x_2 \cdot \ldots \cdot x_6 = S_4$, a contradiction. If $|S_4| = 2$, then $|S_2| = |S_3| = 2$. By Lemma 2.4.2, S_2, S_3 and S_4 are pairwise disjoint. But

$$S_\nu \,|\, x_2 \cdot \ldots \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]\,,$$

a contradiction. Therefore, $3 \leq |S_4| \leq 4$.

We distinguish two cases.

Case 1: $|S_4| = 4$. Without loss of generality, we may assume that $S_4 = x_2 \cdot x_3 \cdot x_4 \cdot x_5$. Since

$$S_2 \,|\, x_2 \cdot \ldots \cdot x_6 \quad \text{and} \quad S_3 \,|\, x_2 \cdot \ldots \cdot x_6\,,$$

by Lemma 2.4, $x_6 \,|\, S_2$ and $x_6 \,|\, S_3$.

We claim that $|S_3| = 3$. If $|S_3| = 4$, since

$$S_3 \,|\, x_2 \cdot \ldots \cdot x_6 \quad \text{and} \quad S_4 \,|\, x_2 \cdot \ldots \cdot x_6\,,$$

then $|\gcd(S_3, S_4)| \geq 3$, a contradiction. If $|S_3| = 2$, then $|S_2| = 2$. Since $x_6 \mid S_3$ and $x_6 \mid S_2$, then $|\gcd(S_2, S_3)| = 1$, a contradiction again. So $|S_3| = 3$.

If $|S_2| = 2$, without loss of generality, we may assume that $S_2 = x_2 \cdot x_6$. Since $x_6 \mid S_3$, we have $x_2 \nmid S_3$. So

$$x_6 \mid S_3 \mid x_3 \cdot x_4 \cdot x_5 \cdot x_6.$$

Without loss of generality, we may assume that $S_3 = x_3 \cdot x_4 \cdot x_6$. Then $\mathcal{A}_i$ is of form $(b1)$.

If $|S_2| = 3$, without loss of generality, we may assume that $S_2 = x_2 \cdot x_3 \cdot x_6$. Since $x_6 \mid S_3$ and $|S_3| = 3$, by Lemma 2.4.2 we have $x_2, x_3 \nmid S_3$. Then $S_3 = x_4 \cdot x_5 \cdot x_6$, and $\mathcal{A}_i$ is of form $(b2)$.

Case 2: $|S_4| = 3$. Then $|S_2| \leq |S_3| \leq 3$.

If $|S_2| = 3$, then $|S_3| = 3$. Since

$$S_\nu \mid x_2 \cdot \ldots \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4],$$

there exist $m, n \in [2, 4]$ such that $|\gcd(S_m, S_n)| \geq 2$, a contradiction. So $|S_2| = 2$.

If $|S_3| = 2$, then $|S_2| = 2$ and $|\gcd(S_3, S_2)| = 0$. Without loss of generality, we may assume that $S_2 = x_2 \cdot x_3$ and $S_3 = x_4 \cdot x_5$. Since $S_4 \mid x_2 \cdot \ldots \cdot x_6$, by Lemma 2.4, we have $|\gcd(S_4, S_2)| = |\gcd(S_4, S_3)| = 1$. So $x_6 \mid S_4$. Without loss of generality, let $S_4 = x_2 \cdot x_4 \cdot x_6$. Then $\mathcal{A}_i$ is of form $(b3)$.

If $|S_3| = 3$, without loss of generality, let $S_3 = x_2 \cdot x_3 \cdot x_4$. Since $S_4 \mid x_2 \cdot \ldots \cdot x_6$ and $|S_4| = 3$, we have $|\gcd(S_4, S_3)| = 1$. Without loss of generality, let $S_4 = x_2 \cdot x_5 \cdot x_6$. By Lemma 2.4, we have $x_2 \nmid S_2$ and $|\gcd(S_2, S_3)| = |\gcd(S_2, S_4)| = 1$. Without loss of generality let $S_2 = x_3 \cdot x_5$. Then $\mathcal{A}_i$ is of form $(b4)$.

This completes the proof. ∎

Lemma 7.3. *If $x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_3 + x_6 = x_4 + x_5 + x_6$, then $\mathsf{f}(S) \geq 19$.*

Proof. Let

$a_1 = x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_3 + x_6 = x_4 + x_5 + x_6,$
$a_2 = x_2,$
$a_3 = x_4,$
$a_4 = x_6 = x_2 + x_3 = x_4 + x_5,$
$a_5 = x_1 + x_2 = x_2 + x_4 + x_5 + x_6,$
$a_6 = x_1 + x_4 = x_2 + x_3 + x_4 + x_6,$
$a_7 = x_2 + x_4,$
$a_8 = x_2 + x_6 = x_2 + x_4 + x_5,$
$a_9 = x_4 + x_6 = x_2 + x_3 + x_4,$
$a_{10} = x_1 + x_2 + x_4 = x_2 + x_3 + x_6 + x_2 + x_4,$
$a_{11} = x_1 + x_2 + x_6 = x_1 + x_2 + x_4 + x_5 = x_2 + x_6 + x_4 + x_5 + x_6,$
$a_{12} = x_1 + x_4 + x_6 = x_1 + x_2 + x_3 + x_4 = x_2 + x_3 + x_6 + x_4 + x_6,$
$a_{13} = x_2 + x_4 + x_6,$
$a_{14} = x_1 + x_2 + x_4 + x_6,$
$a_{15} = x_1+x_2+x_3+x_4+x_5 = x_1+x_2+x_3+x_6 = x_1+x_4+x_5+x_6 = x_2+x_3+x_6+x_4+x_5+x_6,$
$a_{16} = x_1 + x_2 + x_3 + x_4 + x_6,$

$a_{17} = x_1 + x_2 + x_4 + x_5 + x_6,$
$a_{18} = x_2 + x_3 + x_4 + x_5 + x_6 = x_1 + x_6 = x_1 + x_2 + x_3 = x_1 + x_4 + x_5,$
$a_{19} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6.$

A straightforward computation shows that

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}$$

are pairwise distinct, and we are done. ∎

Lemma 7.4. *If* $x_1 = x_2 + x_3 = x_4 + x_5 = x_2 + x_4 + x_6$*, then* $\mathsf{f}(S) \geq 19$.

Proof. Let

$a_1 = x_1 = x_2 + x_3 = x_4 + x_5 = x_2 + x_4 + x_6,$
$a_2 = x_2,$
$a_3 = x_3 = x_4 + x_6,$
$a_4 = x_4,$
$a_5 = x_5 = x_2 + x_6,$
$a_6 = x_1 + x_6 = x_2 + x_3 + x_6 = x_4 + x_5 + x_6 = x_3 + x_5,$
$a_7 = x_2 + x_3 + x_5 = x_1 + x_5 = x_2 + x_4 + x_5 + x_6 = x_1 + x_2 + x_6,$
$a_8 = x_3 + x_4 + x_5 = x_1 + x_3 = x_2 + x_3 + x_4 + x_6 = x_1 + x_4 + x_6,$
$a_9 = x_1 + x_2 + x_4 + x_6 = x_1 + x_2 + x_3 = x_1 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5,$
$a_{10} = x_1 + x_2 + x_3 + x_4 + x_6 = x_1 + x_3 + x_4 + x_5,$
$a_{11} = x_1 + x_2 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + x_5,$
$a_{12} = x_2 + x_3 + x_4 + x_5 + x_6 = x_1 + x_3 + x_5 = x_1 + x_2 + x_3 + x_6 = x_1 + x_4 + x_5 + x_6,$
$a_{13} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6,$
$a_{14} = x_1 + x_2 = x_2 + x_4 + x_5 = 2x_2 + x_3,$
$a_{15} = x_1 + x_4 = x_2 + x_3 + x_4 = 2x_4 + x_5,$
$a_{16} = x_2 + x_4,$
$a_{17} = x_1 + x_2 + x_3 + x_4 + x_5,$
$a_{18} = x_1 + x_2 + x_4,$
$a_{19} = x_1 + x_2 + x_3 + x_4,$
$a_{20} = x_1 + x_2 + x_4 + x_5,$
$a_{21} = x_1 + x_3 + x_4 + x_5 + x_6,$
$a_{22} = x_3 + x_4,$
$a_{23} = x_1 + x_3 + x_4 + x_6,$
$a_{24} = x_1 + x_2 + x_3 + x_5 + x_6,$
$a_{25} = x_1 + x_2 + x_5 + x_6.$

By Lemma 5.1, we have $a_i \notin \{a_1, a_{12}\}$ for every $i \in [1, 25] \setminus \{1, 12\}$. Since S contains no elements of order 2, by Lemma 2.4 we infer that $a_1, a_2, \ldots, a_{17}$ are pairwise distinct. Let

$$A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}\}.$$

By Lemma 2.4 and noting that S contains no elements of order 2, we obtain

$a_{18} \notin A \setminus \{a_3, a_5, a_6\},$

$a_{19} \notin A \setminus \{a_5, a_6\}$,
$a_{20} \notin A \setminus \{a_3, a_6\}$,
$a_{21} \notin A \setminus \{a_2, a_{14}, a_{16}\}$,
$a_{22} \notin A \setminus \{a_2, a_5, a_7, a_{11}, a_{14}\}$,
and
$a_{23} \notin A \setminus \{a_2, a_5, a_7, a_{11}, a_{14}, a_{16}\}$.

We distinguish four cases.

Case 1: $a_{18} = a_3$. That is $x_1 + x_2 + x_4 = x_3 = x_4 + x_6$. Then $x_6 = x_1 + x_2$. By Lemma 2.4, we infer that $a_{19} \notin A \setminus \{a_5\}$.

If $a_{19} = a_5$, that is $x_1 + x_2 + x_3 + x_4 = x_5 = x_2 + x_6 = x_2 + x_1 + x_2$, then $x_2 = x_3 + x_4$. Thus $x_1 = 4x_2, x_3 = 3x_2, x_4 = -2x_2, x_5 = 6x_2, x_6 = 5x_2$. By Lemma 7.1, $\mathsf{f}(S) \geq 19$.

Next, we may assume that $a_{19} \notin A$. By Lemma 2.4 and in view of $x_6 = x_1 + x_2$, we infer that $a_{21} \notin (A \setminus \{a_2\}) \cup \{a_{19}\}$. If $a_{21} \neq a_2$, then $A \cup \{a_{21}, a_{19}\}$ is a set of 19 distinct elements and we are done. So, we may assume that $a_{21} = a_2$, that is $x_1 + x_3 + x_4 + x_5 + x_6 = x_2$. Now, by Lemma 2.4, we obtain that $a_{23} \notin A \cup \{a_{19}\}$. Hence, $A \cup \{a_{23}, a_{19}\}$ is a set of 19 distinct elements.

Case 2: $a_{18} = a_5$. Then $x_6 = x_1 + x_4$. By interchanging x_2, x_3, a_{21} and a_{23} with x_4, x_5, a_{24} and a_{25} respectively, we can reduce this case to Case 1.

Case 3: $a_{18} = a_6$. Then $x_1 + x_2 + x_4 = x_1 + x_6 = x_2 + x_3 + x_6 = x_4 + x_5 + x_6 = x_3 + x_5$. Thus $x_6 = x_2 + x_4$. By Lemma 2.4 and noting that S contains no elements of order 2, we obtain that $A \cup \{a_{19}, a_{20}\}$ is a set of 19 distinct elements.

Case 4: $a_{18} \neq a_3, a_5, a_6$, that is $a_{18} \notin A$ and $x_6 \neq x_1 + x_2, x_1 + x_4, x_2 + x_4$. Let

$$B = A \cup \{a_{18}\}.$$

Since $x_6 \neq x_1 + x_4$ we infer that $a_{19} \neq a_6$. Note that $a_{19} \neq a_{18}$ we have $a_{19} \notin B \setminus \{a_5\}$. If $a_{19} \neq a_5$, then $B \cup \{a_{19}\}$ is a set of 19 distinct elements and we are done. Since $x_6 \neq x_1 + x_2$ we infer that, $a_{20} \neq a_6$ and $a_{20} \notin B \setminus \{a_3\}$. If $a_{20} \neq a_3$, then $B \cup \{a_{20}\}$ is a set of 19 distinct elements and we are done. So, we may assume that $a_{19} = a_5$ and $a_{20} = a_3$. Then, $x_6 = x_1 + x_3 + x_4 = x_1 + x_2 + x_5$. Therefore, $x_3 + x_4 \neq x_2$, i.e. $a_{22} \neq a_2$. By Lemma 2.4, and noting that $x_6 = x_1 + x_3 + x_4 = x_1 + x_2 + x_5$, we obtain that $a_{22} \notin \{a_5, a_7, a_{11}, a_{14}, a_{18}\}$. Therefore, $B \cup \{a_{22}\}$ is a set of 19 distinct elements. This completes the proof. ∎

Lemma 7.5. *If* $x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$ *and* $x_2 = x_3 + x_4 = x_4 + x_5 + x_6 = x_1 + x_3 + x_5 + x_6$*, then* $\mathsf{f}(S) \geq 19$.

Proof. Let
$a_1 = x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$,
$a_2 = x_2 = x_3 + x_4 = x_4 + x_5 + x_6 = x_1 + x_3 + x_5 + x_6$,
$a_3 = x_4 = x_1 + x_3 = x_2 + x_3 + x_6 = x_1 + x_5 + x_6$,
$a_4 = x_1 + x_6 = x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5 + x_6$,
$a_5 = x_2 + x_4 = x_1 + x_2 + x_3 = x_1 + x_2 + x_5 + x_6 = x_1 + x_3 + x_4 + x_5 + x_6$,
$a_6 = x_1 + x_4 + x_5 = x_2 + x_3 + x_4 = x_2 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + x_5 + x_6$,
$a_7 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$,

$a_8 = x_3 = x_5 + x_6$,
$a_9 = x_5$,
$a_{10} = x_6 = x_2 + x_5 = x_3 + x_4 + x_5$,
$a_{11} = x_1 + x_2 = x_1 + x_3 + x_4 = x_1 + x_4 + x_5 + x_6 = x_2 + x_3 + x_4 + x_6$,
$a_{12} = x_1 + x_5 = x_2 + x_3 = x_2 + x_5 + x_6 = x_3 + x_4 + x_5 + x_6$,
$a_{13} = x_4 + x_6 = x_1 + x_3 + x_6 = x_2 + x_4 + x_5 = x_1 + x_2 + x_3 + x_5$,
$a_{14} = x_1 + x_2 + x_6 = x_1 + x_3 + x_4 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5$,
$a_{15} = x_1 + x_2 + x_3 + x_4 = x_1 + x_2 + x_4 + x_5 + x_6$,
$a_{16} = x_4 + x_5 = x_1 + x_3 + x_5 = x_2 + x_3 + x_5 + x_6$,
$a_{17} = x_1 + x_4 = x_2 + x_4 + x_6 = x_1 + x_2 + x_3 + x_6$,
$a_{18} = x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5 = x_1 + x_2 + x_3 + 2x_6$,
$a_{19} = x_3 + x_6 = x_2 + x_3 + x_5$.

By using Lemma 2.4, we can check that $a_1, a_2, \ldots, a_{16}$ are pairwise distinct. Also, we have

$$a_{17} \neq a_1, \ldots, a_8, a_{10}, \ldots, a_{16};$$
$$a_{18} \neq a_1, \ldots, a_7, a_9, \ldots, a_{17};$$
$$a_{19} \neq a_1, \ldots, a_{14}, a_{16}, a_{17}, a_{18}.$$

If $a_{17} = a_9$, then $x_5 = x_1 + x_4 = x_1 + x_1 + x_5 + x_6$, so $0 = 2x_1 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$, a contradiction. Thus $x_5 \neq x_1 + x_4$. Therefore, $x_5 + x_6 \neq x_1 + x_4 + x_6$ and $x_2 + x_3 + x_5 \neq x_1 + x_2 + x_3 + x_4$. This implies that

$$a_{17} \neq a_9, a_{18} \neq a_8, a_{19} \neq a_{15}.$$

Therefore,

$$a_1, a_2, \ldots, a_{19}$$

are pairwise distinct, giving $\mathsf{f}(S) \geq 19$. ∎

Lemma 7.6. *If* $x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$ *and* $x_3 = x_5 + x_6 = x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5$, *then* $\mathsf{f}(S) \geq 19$.

Proof. Let
$a_1 = x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$,
$a_2 = x_1 + x_6 = x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5 + x_6$,
$a_3 = x_3 = x_5 + x_6 = x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5$,
$a_4 = x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_4$,
$a_5 = x_1 + x_2 + x_6 = x_1 + x_3 + x_4 + x_6 = x_3 + x_5 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5$,
$a_6 = x_3 + x_6 = x_2 + x_3 + x_5 = x_1 + x_2 + x_3 + x_4 = x_1 + x_2 + x_4 + x_5 + x_6$,
$a_7 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6 = x_1 + x_4 + x_5 + x_6 = x_3 + x_5$,
$a_8 = x_1 + x_5 = x_2 + x_3 = x_2 + x_5 + x_6 = x_1 + x_2 + x_4 + x_6 = x_3 + x_4 + x_5 + x_6$,
$a_9 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$,
$a_{10} = x_2 = x_3 + x_4 = x_4 + x_5 + x_6$,
$a_{11} = x_5 = x_1 + x_4 = x_2 + x_4 + x_6$,

$a_{12} = x_1+x_3 = x_2+x_3+x_6 = x_1+x_5+x_6 = x_1+2x_2+x_4+x_5+x_6 = x_1+x_3+x_4+2x_5$,

$a_{13} = x_1 + x_2 + x_3 = x_1 + x_2 + x_5 + x_6 = x_1 + x_3 + x_4 + x_5 + x_6 = 2x_1 + x_2 + x_4 + x_6 = x_1 + 2x_2 + x_3 + 2x_4 + x_5 + x_6$,

$a_{14} = x_1 + x_3 + x_5 = x_2 + x_3 + x_5 + x_6 = x_1 + x_2 + x_3 + x_4 + x_6 = x_1 + 2x_5 + x_6 = 2x_1 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + 2x_4 + 2x_5 + x_6$,

$a_{15} = x_1 + x_4 + x_5 = x_2 + x_3 + x_4 = x_2 + x_4 + x_5 + x_6$,

$a_{16} = x_2 + x_4$,

$a_{17} = x_4 + x_5$,

$a_{18} = x_4 + x_6 = x_2 + x_4 + x_5$,

$a_{19} = x_1+x_3+x_6 = x_1+x_2+x_3+x_5 = x_1+x_3+x_4+2x_5+x_6 = x_1+2x_2+x_3+x_4+x_6 = x_1 + x_2 + x_3 + 2x_4 + x_5 + 2x_6$.

By Lemma 5.1, we know that $|\mathcal{A}_i| \leq 5$ for all $i \in [1, r]$. Thus $a_j \neq a_i$ for every $i \in [1, 9]$ and every $j \in [1, 19] \setminus \{i\}$. Also, by Lemma 2.4, we have

$$a_{10}, a_{11}, \ldots, a_{19}$$

are pairwise distinct. Therefore, $a_1, a_2, \ldots, a_{19}$ are distinct, giving $\mathsf{f}(S) \geq 19$. ∎

Lemma 7.7. *If* $x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$ *and* $x_5 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6$*, then* $\mathsf{f}(S) \geq 19$.

Proof. Note that either $x_4 \neq x_1 + x_5 + x_6$ or $x_3 \neq x_1 + x_5 + x_6$. By the symmetry of x_3 and x_4 in $[x_1]$ and $[x_5]$, we may assume that $x_4 \neq x_1 + x_5 + x_6$. Let

$a_1 = x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$,

$a_2 = x_1 + x_6 = x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5 + x_6$,

$a_3 = x_5 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6$,

$a_4 = x_1 + x_5 = x_2 + x_5 + x_6 = x_3 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + x_4 + x_6$,

$a_5 = x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_3 + x_4$,

$a_6 = x_5 + x_6 = x_1 + x_2 + x_6 = x_1 + x_3 + x_4 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5$,

$a_7 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$,

$a_8 = x_2 = x_3 + x_4$,

$a_9 = x_1 + x_3 = x_2 + x_3 + x_6$,

$a_{10} = x_3 + x_5 = x_1 + x_2 + x_3$,

$a_{11} = x_3 + x_6 = x_2 + x_3 + x_5$,

$a_{12} = x_1 + x_3 + x_5 = x_2 + x_3 + x_5 + x_6 = x_1 + x_2 + 2x_3 + x_4 + x_6$,

$a_{13} = x_1 + x_3 + x_6 = x_1 + x_2 + x_3 + x_5 = x_2 + 2x_3 + x_4 + x_5 + x_6$,

$a_{14} = x_3 + x_5 + x_6 = x_1 + x_2 + x_3 + x_6 = x_1 + x_2 + 2x_3 + x_4 + x_5$,

$a_{15} = x_1 + x_3 + x_5 + x_6 = 2x_1 + x_2 + x_3 + x_6 = x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6$,

$a_{16} = x_1 + x_2 + x_3 + x_5 + x_6$,

$a_{17} = x_2 + x_3$,

$a_{18} = x_3$

$a_{19} = x_2 + x_3 + x_4$,

As before, by Lemma 5.1 we know that $a_j \neq a_i$ for every $i \in [1, 7]$ and every $j \in [1, 19] \setminus \{i\}$.

Since $x_4 \neq x_1 + x_5 + x_6$, using Lemma 2.4 we can verify that

$$a_8, \ldots, a_{19}$$

are pairwise distinct. Therefore $a_1, a_2, \ldots, a_{19}$ are pairwise distinct and we are done. ∎

Lemma 7.8. *If $x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$ and $x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_3$, then $\mathsf{f}(S) \geq 19$.*

Proof. Let

$a_1 = x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6$,
$a_2 = x_4$,
$a_3 = x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_3$,
$a_4 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6 = x_4 + x_5$,
$a_5 = x_1 + x_4 = x_2 + x_4 + x_6$,
$a_6 = x_1 + x_5 = x_2 + x_5 + x_6 = x_3 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + x_6$,
$a_7 = x_1 + x_6 = x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5 + x_6$,
$a_8 = x_4 + x_6 = x_2 + x_4 + x_5 = x_1 + x_2 + x_3 + x_4$,
$a_9 = x_1 + x_2 + x_6 = x_4 + x_5 + x_6 = x_1 + x_3 + x_4 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5$,
$a_{10} = x_1 + x_4 + x_5 = x_2 + x_4 + x_5 + x_6 = x_1 + x_2 + x_3 + x_4 + x_6$,
$a_{11} = x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5$,
$a_{12} = x_1 + x_2 + x_4 + x_6$,
$a_{13} = x_1 + x_2 + x_5 + x_6 = x_1 + x_3 + x_4 + x_5 + x_6$,
$a_{14} = x_1 + x_4 + x_5 + x_6$,
$a_{15} = x_1 + x_2 + x_4 + x_5 + x_6$,
$a_{16} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$,
$a_{17} = x_2 = x_3 + x_4$,
$a_{18} = x_2 + x_3 + x_4$,
$a_{19} = x_5 = x_1 + x_3 = x_2 + x_3 + x_6$.

Using Lemma 2.4, we can verify that

$$a_1, a_2, \ldots, a_{16}$$

are pairwise distinct, and we also have

$$\begin{aligned} &a_{17} \neq a_1, \ldots, a_{13}, a_{15}, a_{16}; \\ &a_{18} \neq a_1, \ldots, a_5, a_7, \ldots, a_{10}, a_{16}, a_{17}. \end{aligned}$$

If $a_{17} = a_{14}$, then $x_3 + x_4 = x_1 + x_4 + x_5 + x_6 = x_1 + x_4 + x_1 + x_3 + x_6$, so $0 = x_1 + x_1 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$, a contradiction. Thus $a_{17} \neq a_{14}$.

Since

$$\begin{aligned} &x_3 + x_4 + x_5 + x_6 = x_3 + x_4 + x_5 + x_2 + x_5 \neq x_2 + x_3 + x_4, \\ &x_1 + x_2 + x_4 + x_5 = x_1 + x_2 + x_4 + x_1 + x_3 \neq x_2 + x_3 + x_4, \\ &x_1 + x_2 + x_4 + x_6 = x_2 + x_3 + x_4 + x_5 + x_2 + x_4 + x_6 \neq x_2 + x_3 + x_4, \end{aligned}$$

$$
\begin{aligned}
&x_1+x_2+x_5+x_6 = x_2+x_3+x_4+x_5+x_2+x_1+x_3+x_6 \neq x_2+x_3+x_4,\\
&x_1+x_4+x_5+x_6 = x_2+x_3+x_4+x_5+x_4+x_1+x_3+x_6 \neq x_2+x_3+x_4,\\
&x_1+x_2+x_4+x_5+x_6 =\\
&\qquad\qquad\qquad x_2+x_3+x_4+x_5+x_2+x_4+x_1+x_3+x_6 \neq x_2+x_3+x_4,
\end{aligned}
$$

we have $a_{18} \neq a_6, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}$. Therefore

$$a_1, a_2, \ldots, a_{18}$$

are pairwise distinct.

By Lemma 2.4, we have $a_{19} \neq a_1, \ldots, a_{11}, a_{13}, \ldots, a_{18}$. If $a_{19} \neq a_{12}$, then $a_1, \ldots, a_{18}, a_{19}$ are distinct and we are done. So we may assume $a_{19} = a_{12}$. Thus $x_5 = x_1 + x_3 = x_2 + x_3 + x_6 = x_1 + x_2 + x_4 + x_6$. This implies that

$$x_1 = -5x_2, x_3 = -2x_2, x_4 = 3x_2, x_5 = -7x_2, x_6 = -6x_2.$$

By Lemma 7.1, we have $\mathsf{f}(S) \geq 19$. ∎

We are now ready to provide a proof of Lemma 3.3.

Proof of Lemma 3.3.

For every $k \in [1, 6]$, $|[x_k]| \leq 4$ follows from Lemma 5.1.

If $[x_i]$ or $[x_j]$ has form $(b2)$ or $(b3)$ described in Lemma 7.2, then by Lemma 7.3 or Lemma 7.4, $\mathsf{f}(S) \geq 19$. So we may assume that $[x_i]$ and $[x_j]$ have forms $(b1)$ or $(b4)$. Without loss of generality, we assume that $i = 1$. Let $[x_j] = \{S_1, S_2, S_3, S_4\}$ where S_1, S_2, S_3, S_4 are subsequences of S and $1 = |S_1| \leq |S_2| \leq |S_3| \leq |S_4|$. We distinguish cases.

Case 1: both $[x_1]$ and $[x_j]$ are of form $(b1)$. Then

$$|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 4.$$

and

$$[x_1] = \{x_1,\ x_2 \cdot x_3 \cdot x_4 \cdot x_5,\ x_2 \cdot x_6,\ x_3 \cdot x_4 \cdot x_6\}.$$

Thus,

$$
\begin{aligned}
&x_1 = x_2+x_3+x_4+x_5 = x_2+x_6 = x_3+x_4+x_6;\\
&x_2 = x_3+x_4;\\
&x_6 = x_2+x_5 = x_3+x_4+x_5.
\end{aligned}
$$

Subcase 1.1: $j = 2$. Then $S_1 = x_2$ and $S_2 = x_3 \cdot x_4$. By Lemma 7.2, $S_4 = x_1 \cdot x_3 \cdot x_5 \cdot x_6$ or $S_4 = x_1 \cdot x_4 \cdot x_5 \cdot x_6$. Without loss of generality, let $S_4 = x_1 \cdot x_3 \cdot x_5 \cdot x_6$. Also, by Lemma 7.2, $S_3 = x_1 \cdot x_4 \cdot x_5$, or $S_3 = x_1 \cdot x_4 \cdot x_6$, or $S_3 = x_4 \cdot x_5 \cdot x_6$. If $S_3 = x_1 \cdot x_4 \cdot x_5$, then $x_2 = x_1 + x_4 + x_5 = x_2 + x_4 + x_5 + x_6$, a contradiction. If $S_3 = x_1 \cdot x_4 \cdot x_6$, then $x_2 = x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5$, a contradiction again. So, $S_3 = x_4 \cdot x_5 \cdot x_6$. Then $x_2 = x_3 + x_4 = x_4 + x_5 + x_6 = x_1 + x_3 + x_5 + x_6$. Therefore, $\mathsf{f}(S) \geq 19$ by Lemma 7.5.

Subcase 1.2: $j = 3$. By Lemma 7.2, we have

$$S_\nu \,|\, x_1 \cdot x_2 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]\,.$$

Since $x_1 + x_2 + x_5 + x_6 = x_1 + x_3 + x_4 + x_5 + x_6 \neq x_3$, we have $S_4 \neq x_1 \cdot x_2 \cdot x_5 \cdot x_6$. Thus, $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_5$ or $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_6$ or $S_4 = x_1 \cdot x_4 \cdot x_5 \cdot x_6$ or $S_4 = x_2 \cdot x_4 \cdot x_5 \cdot x_6$.

(i) $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_5$. By Lemma 7.2, $x_6 \,|\, \gcd(S_2, S_3)$. Since

$$\begin{aligned}
&x_1 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_3,\\
&x_2 + x_6 = x_1 \neq x_3,\\
&x_4 + x_6 = x_2 + x_4 + x_5 \neq x_1 + x_2 + x_4 + x_5,
\end{aligned}$$

we have $S_2 \neq x_1 {\cdot} x_6$, $x_2 {\cdot} x_6$ or $x_4 {\cdot} x_6$. So $S_2 = x_5 {\cdot} x_6$. Note that $x_1 {+} x_2 {+} x_4 {+} x_5 = x_1 {+} x_4 {+} x_6$. We conclude that $S_3 = x_1 {\cdot} x_4 {\cdot} x_6$. Therefore, $x_3 = x_5 {+} x_6 = x_1 {+} x_4 {+} x_6 = x_1 {+} x_2 {+} x_4 {+} x_5$. Now, $\mathsf{f}(S) \geq 19$ by Lemma 7.6.

(ii) $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_6$. Then $x_5 \,|\, \gcd(S_2, S_3)$. Since

$$\begin{aligned}
&x_1 + x_5 = x_3 + x_4 + x_5 + x_6 \neq x_3,\\
&x_2 + x_5 = x_3 + x_4 + x_5 \neq x_3,
\end{aligned}$$

we have $S_2 \neq x_1 \cdot x_5$ or $S_2 \neq x_2 \cdot x_5$. If $S_2 = x_4 \cdot x_5$, then $x_4 + x_5 = x_1 + x_2 + x_4 + x_6 = x_1 + x_2 + x_4 + x_2 + x_5$, so $0 = x_1 + x_2 + x_2 = x_1 + x_2 + x_3 + x_4$, a contradiction. Thus $S_2 \neq x_4 \cdot x_5$, and then $S_2 = x_5 \cdot x_6$. By Lemma 7.2, $x_6 \nmid S_3$ and

$$x_5 \,|\, S_3 \,|\, x_1 \cdot x_2 \cdot x_4 \cdot x_5.$$

Therefore, $S_3 = x_1 \cdot x_2 \cdot x_5,\ x_1 \cdot x_4 \cdot x_5$ or $x_2 \cdot x_4 \cdot x_5$. But

$x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 \neq x_3$,

$x_1 + x_4 + x_5 = x_2 + x_4 + x_5 + x_6 \neq x_1 + x_2 + x_4 + x_6$,

and

$x_2 + x_4 + x_5 \neq x_2 + x_3 + x_4 + x_5 + x_2 + x_4 + x_6 = x_1 + x_2 + x_4 + x_6$,

a contradiction. Therefore, $S_4 \neq x_1 \cdot x_2 \cdot x_4 \cdot x_6$.

(iii) $S_4 = x_1 \cdot x_4 \cdot x_5 \cdot x_6$. Then $x_2 \,|\, \gcd(S_2, S_3)$. Since S contains no elements of order 2, we have $x_3 \neq x_2 + x_4$, so $S_2 \neq x_2 \cdot x_4$. Since

$$\begin{aligned}
&x_1 + x_2 = x_1 + x_3 + x_4 \neq x_3,\\
&x_2 + x_5 = x_3 + x_4 + x_5 \neq x_3,\\
&x_2 + x_6 = x_3 + x_4 + x_6 \neq x_3,
\end{aligned}$$

we have $S_2 \neq x_1 \cdot x_2$, or $S_2 \neq x_2 \cdot x_5$ or $S_2 \neq x_2 \cdot x_6$, a contradiction. Therefore $S_4 \neq x_1 \cdot x_4 \cdot x_5 \cdot x_6$.

(iv) $S_4 = x_2 \cdot x_4 \cdot x_5 \cdot x_6$. Then $x_1 \mid \gcd(S_2, S_3)$. Since

$$\begin{aligned}
&x_1 + x_2 = x_1 + x_3 + x_4 \neq x_3,\\
&x_1 + x_5 = x_3 + x_4 + x_5 + x_6 \neq x_3,\\
&x_1 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_3,
\end{aligned}$$

we have $S_2 \neq x_1 \cdot x_2$, or $S_2 \neq x_1 \cdot x_5$ or $S_2 \neq x_1 \cdot x_6$. Then $S_2 = x_1 \cdot x_4$, and thus $x_4 \nmid S_3$. So

$$x_1 \,|\, S_3 \,|\, x_1 \cdot x_2 \cdot x_5 \cdot x_6.$$

Since

$$\begin{aligned}
&x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5 \neq x_3,\\
&x_1 + x_2 + x_6 = x_1 + x_3 + x_4 + x_6 \neq x_3,\\
&x_1 + x_5 + x_6 = x_2 + x_3 + x_4 + x_5 + x_5 + x_6 \neq x_2 + x_4 + x_5 + x_6,
\end{aligned}$$

we have $S_3 \neq x_1 \cdot x_2 \cdot x_5$, $S_3 \neq x_1 \cdot x_2 \cdot x_6$ or $S_3 \neq x_1 \cdot x_5 \cdot x_6$, a contradiction. Therefore, $S_4 \neq x_2 \cdot x_4 \cdot x_5 \cdot x_6$.

Subcase 1.3: $j = 4$. By the symmetry of x_3 and x_4 in $[x_1]$, This reduces to subcase 1.2.

Subcase 1.4: $j = 5$. Then

$$S_\nu \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]\,.$$

Since $x_1 + x_3 + x_4 + x_6 = x_1 + x_2 + x_3 + x_4 + x_5 \neq x_5$, we have $S_4 \neq x_1 \cdot x_3 \cdot x_4 \cdot x_6$. Thus, $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_4$ or $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_6$ or $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_6$ or $S_4 = x_2 \cdot x_3 \cdot x_4 \cdot x_6$.

(i) $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_4$. Then $x_6 \mid \gcd(S_2, S_3)$. Since

$$\begin{aligned}
&x_1 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_5,\\
&x_2 + x_6 = x_2 + x_2 + x_5 \neq x_5,\\
&x_3 + x_6 = x_2 + x_3 + x_5 \neq x_5,\\
&x_4 + x_6 = x_2 + x_4 + x_5 \neq x_5,
\end{aligned}$$

we have $S_2 \neq x_1 \cdot x_6$, or $S_2 \neq x_2 \cdot x_6$ or $S_2 \neq x_3 \cdot x_6, x_4 \cdot x_6$, a contradiction. Therefore $S_4 \neq x_2 \cdot x_4 \cdot x_5 \cdot x_6$.

(ii) $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_6$. Then $x_4 \mid \gcd(S_2, S_3)$. Since

$$\begin{aligned}
&x_3 + x_4 = x_2 \neq x_5,\\
&x_4 + x_6 = x_2 + x_4 + x_5 \neq x_5,
\end{aligned}$$

we have $S_2 \neq x_3 \cdot x_4$ or $S_2 \neq x_4 \cdot x_6$. Then $S_2 = x_1 \cdot x_4$ or $S_2 = x_2 \cdot x_4$.

If $S_2 = x_2 \cdot x_4$, then $x_2 \nmid S_3$. So

$$x_4 \,|\, S_4 \,|\, x_1 \cdot x_3 \cdot x_4 \cdot x_6.$$

Since

$$x_1 + x_3 + x_4 = x_1 + x_2 \neq x_2 + x_4,$$
$$x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5 \neq x_2 + x_4,$$
$$x_3 + x_4 + x_6 = x_2 + x_3 + x_4 + x_5 \neq x_2 + x_4,$$

we have $S_3 \neq x_1 \cdot x_3 \cdot x_4$, $S_3 \neq x_1 \cdot x_4 \cdot x_6$ or $S_3 \neq x_3 \cdot x_4 \cdot x_6$, a contradiction. Therefore, $S_2 = x_1 \cdot x_4$. Note that $x_1 + x_4 = x_2 + x_4 + x_6$. We have $S_3 = x_2 \cdot x_4 \cdot x_6$, so $x_5 = x_1 + x_4 = x_2 + x_4 + x_6 = x_1 + x_2 + x_3 + x_6$. This implies that

$$x_1 = -5x_2, x_3 = 3x_2, x_4 = -2x_2, x_5 = -7x_2, x_6 = -6x_2.$$

By Lemma 7.1, we have $\mathsf{f}(S) \geq 19$.

(iii) $S_4 = x_1 \cdot x_2 \cdot x_4 \cdot x_6$. By the symmetry of x_3 and x_4 in $[x_1]$, This reduces to the case when $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_6$.

(iv) $S_4 = x_2 \cdot x_3 \cdot x_4 \cdot x_6$. Note that $x_2 + x_3 + x_4 + x_6 = x_1 + x_2 = x_1 + x_3 + x_4$, so $S_2 = x_1 \cdot x_2, S_3 = x_1 \cdot x_3 \cdot x_4$. Then $x_5 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6$. By Lemma 7.7, we have $\mathsf{f}(S) \geq 19$.

Subcase 1.5: $j = 6$. Then $S_1 = x_6, S_2 = x_2 \cdot x_5, S_3 = x_3 \cdot x_4 \cdot_5$. By Lemma 7.2, $S_4 = x_1 \cdot x_2 \cdot x_3 \cdot x_4$. Thus,

$$x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_3 + x_4$$

and

$$x_5 = x_1 + x_2 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6.$$

By Lemma 7.7, we have $\mathsf{f}(S) \geq 19$.

Case 2: $[x_1]$ is of form $(b1)$ and $[x_j]$ is of form $(b4)$. Then

$$|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 3.$$

By Lemma 7.2, we have

$$\begin{aligned} \mathrm{supp}(S_3S_4) &= \mathrm{supp}(SS_1^{-1}), \\ |\gcd(S_3, S_4)| &= 1, \\ |\gcd(S_2, S_3)| &\geq 1, \\ |\gcd(S_2, S_4)| &\geq 1, \\ |\gcd(S_2, S_3, S_4)| &= 0. \end{aligned}$$

Now

$$[x_1] = \{x_1,\ x_2 \cdot x_3 \cdot x_4 \cdot x_5,\ x_2 \cdot x_6,\ x_3 \cdot x_4 \cdot x_6\}$$

and

$$x_1 = x_2 + x_3 + x_4 + x_5 = x_2 + x_6 = x_3 + x_4 + x_6;$$
$$x_2 = x_3 + x_4;$$

$$x_6 = x_2 + x_5 = x_3 + x_4 + x_5.$$

Subcase 2.1: $j = 2$. Let $S_1 = x_2$ and $S_2 = x_3 \cdot x_4$. Then

$$S_3 \,|\, x_1 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{and} \quad S_4 \,|\, x_1 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6\,.$$

Without loss of generality, let $x_3 \,|\, S_3$. Then $x_4 \nmid S_3$, and thus $x_4 \,|\, S_4$ and $x_3 \nmid S_4$. So, $S_3 = x_1 \cdot x_3 \cdot x_5$ or $S_3 = x_1 \cdot x_3 \cdot x_6$ or $S_3 = x_3 \cdot x_5 \cdot x_6$.

Since

$$x_1 + x_3 + x_5 = x_3 + x_4 + x_6 + x_3 + x_5 \neq x_3 + x_4,$$
$$x_1 + x_3 + x_6 = x_2 + x_3 + x_4 + x_5 + x_3 + x_6 \neq x_3 + x_4,$$

we have $S_3 \neq x_1 \cdot x_3 \cdot x_5$ or $S_3 \neq x_1 \cdot x_3 \cdot x_6$. So $S_3 = x_3 \cdot x_5 \cdot x_6$, and then $S_4 = x_1 \cdot x_4 \cdot x_5$ or $S_4 = x_1 \cdot x_4 \cdot x_6$. But

$$x_1 + x_4 + x_5 = x_3 + x_4 + x_6 + x_4 + x_5 \neq x_3 + x_4,$$
$$x_1 + x_4 + x_6 = x_2 + x_3 + x_4 + x_5 + x_4 + x_6 \neq x_3 + x_4,$$

a contradiction.

Subcase 2.2: $j = 3$. Then

$$S_\nu \,|\, x_1 \cdot x_2 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]\,.$$

Since $x_2 = x_3 + x_4$, we have $x_3 \neq x_1 + x_2, x_2 + x_4, x_2 + x_5, x_2 + x_6$, so $x_2 \nmid S_2$. Then $S_2 \,|\, x_1 \cdot x_4 \cdot x_5 \cdot x_6$. Since

$$x_1 + x_5 = x_3 + x_4 + x_5 + x_6 \neq x_3,$$
$$x_1 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_3,$$

we have $S_2 \neq x_1 \cdot x_5$ or $S_2 \neq x_1 \cdot x_6$. Thus, $S_2 = x_1 \cdot x_4$ or $S_2 = x_4 \cdot x_5$ or $S_2 = x_4 \cdot x_6$ or $S_2 = x_5 \cdot x_6$.

Next, we show that if $x_2 \,|\, S_3$ (resp. S_4), then $x_4 \,|\, S_3$ (resp. S_4). Suppose on the contrary that $x_2 \,|\, S_3$, but $x_4 \nmid S_3$. Then $x_3 = \sigma(S_3) = \sigma(x_2^{-1} x_3 x_4 S_3)$, a contradiction. So if $x_2 \,|\, S_3$ (resp. S_4), then $x_4 \,|\, S_3$ (resp. S_4).

(i) $S_2 = x_1 \cdot x_4$. Note that $x_1 + x_4 = x_2 + x_4 + x_6$. So we may assume $S_3 = x_2 \cdot x_4 \cdot x_6$. Then $x_2 \nmid S_4$, otherwise $x_2 \,|\, S_4$ and $x_4 \,|\, S_4$, a contradiction. Since $\operatorname{supp}(S_3 S_4) = \operatorname{supp}(S S_1^{-1})$, then $x_1 \,|\, S_4$ and $x_4 \nmid S_4$. Then $S_4 = x_1 \cdot x_5 \cdot x_6$. So $x_3 = x_1 + x_4 = x_2 + x_4 + x_6 = x_1 + x_5 + x_6$. Thus

$$x_1 = -7x_4, x_2 = -5x_4, x_3 = -6x_4, x_5 = 3x_4, x_6 = -2x_4.$$

Therefore, $\mathsf{f}(S) \geq 19$ by Lemma 7.1.

(ii) $S_2 = x_4 \cdot x_5$. Now, let $x_4 \,|\, S_3$. Then $x_5 \nmid S_3$, $x_5 \,|\, S_4$ and $x_4 \nmid S_4$. Thus $x_2 \nmid S_4$, and therefore, $S_4 = x_1 \cdot x_5 \cdot x_6$. But $x_1 + x_5 + x_6 = x_2 + x_3 + x_4 + x_5 + x_5 + x_6 \neq x_4 + x_5$, a contradiction.

(iii) $S_2 = x_4 \cdot x_6$. Note that $x_4 + x_6 = x_2 + x_4 + x_5$, so we may assume that $S_3 = x_2 \cdot x_4 \cdot x_5$. Then $x_2 \nmid S_4$, $x_4 \nmid S_4$, and thus $S_4 = x_1 \cdot x_5 \cdot x_6$. But $x_1 + x_5 + x_6 = x_3 + x_4 + x_6 + x_5 + x_6 \neq x_4 + x_6$, a contradiction.

(iv) $S_2 = x_5 \cdot x_6$. Let $x_5 \,|\, S_3$. Then $x_6 \nmid S_3$, $x_6 \,|\, S_4$ and $x_5 \nmid S_4$. Thus $S_3 = x_1 \cdot x_4 \cdot x_5$ or $S_3 = x_2 \cdot x_4 \cdot x_5$ or $S_3 = x_1 \cdot x_2 \cdot x_5$. But

$$
\begin{aligned}
x_1 + x_4 + x_5 &= x_2 + x_6 + x_4 + x_5 \neq x_5 + x_6, \\
x_2 + x_4 + x_5 &= x_4 + x_6 \neq x_5 + x_6, \\
x_1 + x_2 + x_5 &= 2x_2 + x_6 + x_5 \neq x_5 + x_6,
\end{aligned}
$$

a contradiction.

Subcase 2.3: $j = 4$. By the symmetry of x_3 and x_4 in $[x_1]$, this reduces to subcase 2.2.

Subcase 2.4: $j = 5$. Then

$$S_\nu \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4] \,.$$

Since $x_6 = x_2 + x_5$, we have $x_5 \neq x_1 + x_6, x_2 + x_6, x_3 + x_6, x_4 + x_6$, so $x_6 \nmid S_2$. Thus $S_2 \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_4$. Since $x_3 + x_4 = x_2 \neq x_5$, we have $S_2 \neq x_3 \cdot x_4$. Then $S_2 = x_1 \cdot x_2$ or $S_2 = x_1 \cdot x_3$ or $S_2 = x_1 \cdot x_4$ or $S_2 = x_2 \cdot x_3$ or $S_2 = x_2 \cdot x_4$.

(i) $S_2 = x_1 \cdot x_2$. Note that $x_1 + x_2 = x_2 + x_3 + x_4 + x_6$, a contradiction. So $S_2 \neq x_1 \cdot x_2$.

(ii) $S_2 = x_1 \cdot x_3$. Note that $x_1 + x_3 = x_2 + x_3 + x_6$. We may assume $S_3 = x_2 \cdot x_3 \cdot x_6$. Since $\mathrm{supp}(S_3 S_4) = \mathrm{supp}(S S_1^{-1}) = \{x_1, x_2, x_3, x_4, x_6\}$, we have $x_1 \cdot x_4 \,|\, S_4$. If $x_3 \,|\, S_4$, then $S_2 \,|\, S_4$, a contradiction. So $x_3 \nmid S_4$. Since $x_1 + x_2 + x_4 = x_1 + x_3 + x_4 + x_4 \neq x_1 + x_3$, we have $S_4 \neq x_1 \cdot x_2 \cdot x_4$. So $S_4 = x_1 \cdot x_4 \cdot x_6$. However, $x_1 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5$, a contradiction. So $S_2 \neq x_1 \cdot x_3$.

By the symmetry of x_3 and x_4 in $[x_1]$, we may also assume that $S_2 \neq x_1 \cdot x_4$.

(iii) $S_2 = x_2 \cdot x_3$. Without loss of generality, let $x_2 \,|\, S_3$. Then $x_3 \nmid S_3$, $x_3 \,|\, S_4$ and $x_2 \nmid S_4$. Thus $S_3 = x_1 \cdot x_2 \cdot x_4$ or $S_3 = x_1 \cdot x_2 \cdot x_6$ or $S_3 = x_2 \cdot x_4 \cdot x_6$. Since $x_1 + x_2 + x_6 = x_2 + x_3 + x_4 + x_5 + x_2 + x_6 \neq x_2 + x_3$, we have $S_3 \neq x_1 \cdot x_2 \cdot x_6$.

If $S_3 = x_1 \cdot x_2 \cdot x_4$, then $x_3 \cdot x_6 \,|\, S_4$. Thus $S_4 = x_1 \cdot x_3 \cdot x_6$ or $S_4 = x_3 \cdot x_4 \cdot x_6$. But

$$
\begin{aligned}
x_1 + x_3 + x_6 &= x_2 + x_3 + x_4 + x_5 + x_3 + x_6 \neq x_2 + x_3, \\
x_3 + x_4 + x_6 &= x_1 \neq x_5,
\end{aligned}
$$

a contradiction. So $S_3 \neq x_1 \cdot x_2 \cdot x_4$.

If $S_3 = x_2 \cdot x_4 \cdot x_6$, then $x_1 \cdot x_3 \,|\, S_4$. Thus $S_4 = x_1 \cdot x_3 \cdot x_4$ or $S_4 = x_1 \cdot x_3 \cdot x_6$. But

$$
\begin{aligned}
x_1 + x_3 + x_6 &= x_2 + x_3 + x_4 + x_5 + x_3 + x_6 \neq x_2 + x_3, \\
x_1 + x_3 + x_4 &= x_1 + x_2 \neq x_2 + x_3,
\end{aligned}
$$

a contradiction, so $S_3 \neq x_2 \cdot x_4 \cdot x_6$. Thus $S_2 \neq x_2 \cdot x_3$. By the symmetry of x_3 and x_4 in $[x_1]$, we also conclude that $S_2 \neq x_2 \cdot x_4$, a contradiction again.

Subcase 2.5: $j = 6$. Let $S_1 = x_6, S_2 = x_2 \cdot x_5$ and $S_3 = x_3 \cdot x_4 \cdot x_5$. By Lemma 7.2, $S_4 = x_1 \cdot x_2 \cdot x_3$ or $S_4 = x_1 \cdot x_2 \cdot x_4$. By the symmetry of x_3 and x_4 in $[x_1]$, we may assume

$S_4 = x_1 \cdot x_2 \cdot x_3$. Thus $x_6 = x_2 + x_5 = x_3 + x_4 + x_5 = x_1 + x_2 + x_3$. By Lemma 7.8, we have $\mathsf{f}(S) \geq 19$.

Case 3: both $[x_1]$ and $[x_j]$ are of form $(b4)$. Then

$$|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 3.$$

As in Case 3, we have

$$\begin{aligned}\operatorname{supp}(S_3S_4) &= \operatorname{supp}(SS_1^{-1}),\\ |\gcd(S_3, S_4)| &= 1,\\ |\gcd(S_2, S_3)| &\geq 1,\\ |\gcd(S_2, S_4)| &\geq 1,\\ |\gcd(S_2, S_3, S_4)| &= 0.\end{aligned}$$

Now,

$$[x_1] = \{x_1,\ x_2 \cdot x_3 \cdot x_4,\ x_2 \cdot x_5 \cdot x_6,\ x_3 \cdot x_5\}$$

and

$$\begin{aligned}&x_1 = x_2 + x_3 + x_4 = x_2 + x_5 + x_6 = x_3 + x_5;\\ &x_3 = x_2 + x_6;\\ &x_5 = x_2 + x_4.\end{aligned}$$

Subcase 3.1: $j = 2$. Then

$$S_\nu \,|\, x_1 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4]\,.$$

Since $x_3 = x_2+x_6, x_5 = x_2+x_4$, we have $x_2 \neq x_1+x_3, x_3+x_4, x_3+x_5, x_3+x_6, x_1+x_5, x_4+x_5$, or $x_5 + x_6$, so $x_3, x_5 \nmid S_2$. Thus $S_2 = x_1 \cdot x_4$ or $S_2 = x_1 \cdot x_6$ or $S_2 = x_4 \cdot x_6$. But

$$\begin{aligned}x_1 + x_4 = x_2 + x_5 + x_6 + x_4 \neq x_2,\\ x_1 + x_6 = x_2 + x_3 + x_4 + x_6 \neq x_2,\end{aligned}$$

so $S_2 = x_4 \cdot x_6$.

Without loss of generality, let $x_4 \,|\, S_3$. Then $x_6 \nmid S_3$. So $S_3 = x_1 \cdot x_3 \cdot x_4$ or $S_3 = x_1 \cdot x_4 \cdot x_5$ or $S_3 = x_3 \cdot x_4 \cdot x_5$. But

$$\begin{aligned}x_1 + x_3 + x_4 = x_2 + x_5 + x_6 + x_3 + x_4 \neq x_2,\\ x_3 + x_4 + x_5 = x_2 + x_6 + x_4 + x_5 \neq x_2,\end{aligned}$$

so $S_3 = x_1 \cdot x_4 \cdot x_5$. Since $\operatorname{supp}(S_3S_4) = \operatorname{supp}(SS_1^{-1})$ and $|\gcd(S_3, S_4)| = 1$, we have $S_4 = x_1 \cdot x_3 \cdot x_6$ or $S_4 = x_3 \cdot x_5 \cdot x_6$. But $x_3 + x_5 + x_6 = x_3 + x_2 + x_4 + x_6 \neq x_2$, so $S_4 = x_1 \cdot x_3 \cdot x_6$. That gives $x_2 = x_4 + x_6 = x_1 + x_4 + x_5 = x_1 + x_3 + x_6$. Therefore,

$$x_5 = x_2 + x_4 = x_2 + x_1 + x_3 = x_4 + x_6 + x_1 + x_3.$$

This reduces to Case 2.

Subcase 3.2: $j = 3$. Let $S_1 = x_3$ and $S_2 = x_2 \cdot x_6$. Then

$$S_3 \,|\, x_1 \cdot x_2 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{and} \quad S_4 \,|\, x_1 \cdot x_2 \cdot x_4 \cdot x_5 \cdot x_6 \,.$$

If $x_5 \nmid S_3$ then $|x_5 S_3| = 4$ and $\sigma(x_5 S_3) = x_3 + x_5 = x_1$, a contradiction. Therefore $x_5 \,|\, S_3$. Similarly, $x_5 \in S_4$. Let $S_3' = x_5^{-1} S_3$ and $S_4' = x_5^{-1} S_4$. Then, $S_3' S_4' = x_1 \cdot x_2 \cdot x_4 \cdot x_6$ and $\gcd(S_3', S_4') = 1$. Since $S_2 = x_2 \cdot x_6$, we may assume that $x_2 \,|\, S_3'$ and $x_6 \,|\, S_4'$. Therefore, $S_3' = x_1 \cdot x_2$ and $S_4' = x_4 \cdot x_6$, or $S_3' = x_2 \cdot x_4$ and $S_4' = x_1 \cdot x_6$. Hence, $S_3 = x_1 \cdot x_2 \cdot x_5$ and $S_4 = x_4 \cdot x_5 \cdot x_6$, or $S_3 = x_2 \cdot x_4 \cdot x_5$ and $S_4 = x_1 \cdot x_5 \cdot x_6$. Thus, $x_2 + x_6 = x_4 + x_5 + x_6$ or $x_2 + x_6 = x_1 + x_5 + x_6$. But $x_1 + x_5 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_2 + x_6$ and $x_4 + x_5 + x_6 = 2x_4 + x_2 + x_6 \neq x_2 + x_6$, a contradiction.

Subcase 3.3: $j = 4$. Then

$$S_\nu \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_5 \cdot x_6 \quad \text{for every} \quad \nu \in [2, 4] \,.$$

Since $x_5 = x_2 + x_4$ and $x_3 = x_2 + x_6$, we have $x_4 \neq x_1 + x_5, x_2 + x_5, x_3 + x_5, x_5 + x_6$ or $x_2 + x_6$. So $S_2 = x_1 \cdot x_2$ or $x_1 \cdot x_3$ or $x_1 \cdot x_6$ or $x_2 \cdot x_3$ or $x_3 \cdot x_6$. Since $|S_3| = |S_4| = 3$ and

$$\begin{aligned} x_1 + x_3 &= x_2 + x_5 + x_6 + x_3, \\ x_1 + x_6 &= x_2 + x_3 + x_4 + x_6, \end{aligned}$$

$S_2 \neq x_1 \cdot x_3$ or $S_2 \neq x_1 \cdot x_6$.

(i) $S_2 = x_1 \cdot x_2$. Note that $x_1 + x_2 = x_3 + x_5 + x_2$, so we may assume that $S_3 = x_2 \cdot x_3 \cdot x_5$. Since $\gcd(S_2, S_3, S_4) = 1$ and $\operatorname{supp}(S_3 S_4) = \operatorname{supp}(S S_1^{-1}) = \{x_1, x_2, x_3, x_5, x_6\}$ and $|\gcd(S_3, S_4)| = 1$, we have $S_4 = x_1 \cdot x_3 \cdot x_6$ or $S_4 = x_1 \cdot x_5 \cdot x_6$. But

$$\begin{aligned} x_1 + x_3 + x_6 &= x_1 + x_2 + x_6 + x_6 \neq x_1 + x_2, \\ x_1 + x_5 + x_6 &= x_1 + x_2 + x_4 + x_6 \neq x_4, \end{aligned}$$

a contradiction. So $S_2 \neq x_1 \cdot x_2$.

(ii) $S_2 = x_2 \cdot x_3$. Without loss of generality, let $x_2 \,|\, S_3$. Then $x_3 \nmid S_3$. So $S_3 = x_1 \cdot x_2 \cdot x_5$ or $S_3 = x_1 \cdot x_2 \cdot x_6$ or $S_3 = x_2 \cdot x_5 \cdot x_6$. But

$$\begin{aligned} x_1 + x_2 + x_6 &= x_1 + x_3 \neq x_2 + x_3, \\ x_2 + x_5 + x_6 &= x_1 \neq x_4, \end{aligned}$$

so $S_3 = x_1 \cdot x_2 \cdot x_5$. Since $x_2 \nmid S_4$ and $x_3 \,|\, S_4$, we have $S_4 = x_1 \cdot x_3 \cdot x_5$ or $S_4 = x_1 \cdot x_3 \cdot x_6$ or $S_4 = x_3 \cdot x_5 \cdot x_6$. But

$$\begin{aligned} x_1 + x_3 + x_5 &= x_1 + x_3 + x_2 + x_4 \neq x_2 + x_3, \\ x_3 + x_5 + x_6 &= x_3 + x_2 + x_4 + x_6 \neq x_4, \end{aligned}$$

so $S_4 = x_1 \cdot x_3 \cdot x_6$. This gives that $x_4 = x_2 + x_3 = x_1 + x_2 + x_5 = x_1 + x_3 + x_6$. Then

$$x_3 = x_1 + x_5 = x_1 + x_2 + x_4 = x_3 + x_5 + x_2 + x_4 .$$

This reduces to Case 2.

(iii) $S_2 = x_3 \cdot x_6$. Without loss of generality, let $x_3 \mid S_3$, then $x_6 \mid S_4$ and $x_3 \nmid S_4$. So $S_4 = x_1 \cdot x_2 \cdot x_6$ or $S_4 = x_1 \cdot x_5 \cdot x_6$ or $S_4 = x_2 \cdot x_5 \cdot x_6$. But

$$\begin{aligned} &x_1 + x_2 + x_6 = x_3 + x_5 + x_2 + x_6 \neq x_3 + x_6, \\ &x_1 + x_5 + x_6 = x_2 + x_3 + x_4 + x_5 + x_6 \neq x_4, \\ &x_2 + x_5 + x_6 = x_1 \neq x_4, \end{aligned}$$

a contradiction.

Subcase 3.4: $j = 5$. By the symmetry of x_3, x_6 and x_5, x_4 in $[x_1]$, this reduces to subcase 3.2.

Subcase 3.5: $j = 6$. By the symmetry of x_3, x_6 and x_5, x_4 in $[x_1]$, this reduces to subcase 3.3.

This completes the proof. ∎

7.2. Classes of size 5

This subsection deals with classes of size 5, and it provides a proof of Lemma 3.4.

Lemma 7.9. *If $|\mathcal{A}_i| = 5$, then there exists $\tau \in P_6$ such that $\mathcal{A}_i$ or the dual class of $\mathcal{A}_i$ is of one of the following forms:*

(c1). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(1)} \cdot x_{\tau(3)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(2)} \cdot x_{\tau(4)},\ x_{\tau(5)} \cdot x_{\tau(1)} \cdot x_{\tau(4)}\}$;

(c2). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(1)} \cdot x_{\tau(3)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(2)},\ x_{\tau(5)} \cdot x_{\tau(1)} \cdot x_{\tau(4)}\}$;

(c3). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(1)} \cdot x_{\tau(3)},\ x_{\tau(5)} \cdot x_{\tau(1)} \cdot x_{\tau(4)},\ x_{\tau(6)} \cdot x_{\tau(2)} \cdot x_{\tau(4)}\}$;

(c4). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(5)} \cdot x_{\tau(6)} \cdot x_{\tau(1)},\ x_{\tau(5)} \cdot x_{\tau(2)} \cdot x_{\tau(3)},\ x_{\tau(6)} \cdot x_{\tau(2)} \cdot x_{\tau(4)}\}$;

(c5). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(1)} \cdot x_{\tau(3)} \cdot x_{\tau(6)} \cdot x_{\tau(5)},\ x_{\tau(1)} \cdot x_{\tau(4)} \cdot x_{\tau(3)},\ x_{\tau(2)} \cdot x_{\tau(6)} \cdot x_{\tau(3)},\ x_{\tau(5)} \cdot x_{\tau(4)} \cdot x_{\tau(6)}\}$;

(c6). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(1)} \cdot x_{\tau(3)} \cdot x_{\tau(6)} \cdot x_{\tau(5)},\ x_{\tau(1)} \cdot x_{\tau(4)} \cdot x_{\tau(3)},\ x_{\tau(2)} \cdot x_{\tau(6)} \cdot x_{\tau(3)},\ x_{\tau(5)} \cdot x_{\tau(2)} \cdot x_{\tau(4)}\}$;

(c7). $\{x_{\tau(1)} \cdot x_{\tau(2)},\ x_{\tau(1)} \cdot x_{\tau(3)} \cdot x_{\tau(4)},\ x_{\tau(1)} \cdot x_{\tau(5)} \cdot x_{\tau(6)},\ x_{\tau(2)} \cdot x_{\tau(3)} \cdot x_{\tau(5)},\ x_{\tau(2)} \cdot x_{\tau(4)} \cdot x_{\tau(6)}\}$.

Proof. Let

$$\mathcal{A}_i = \{S_1, \ldots, S_5\}$$

where $S_1, \ldots, S_5$ are subsequences of S and $1 \leq |S_1| \leq \ldots \leq |S_5|$.

Let $T = S_1 S_2 S_3 S_4 S_5$. As in the proof of Lemma 5.1, we have $\mathrm{supp}(T) = S$ and $2 \leq \mathsf{v}_a(T) \leq 3$ for every $a \in S$.

By Lemma 5.1 we have

$$2 \leq |S_1| \leq \cdots \leq |S_5| \leq 4.$$

By Lemma 2.4, we infer that $\mathcal{A}_i$ contains at most three sequences of length 2, and three sequences of length 4.

Next, we distinguish cases.

Case 1: $\mathcal{A}_i$ contains three sequences of lengths 2. Then $|S_1| = |S_2| = |S_3| = 2$. Let $S_1 = x_1 \cdot x_2, S_2 = x_3 \cdot x_4$ and $S_3 = x_5 \cdot x_6$. Then by Lemma 2.4, we have $|S_4| = |S_5| = 3$. Since $\mathsf{v}_a(T) \geq 2$ for every $a \,|\, S$, we have $S_4S_5 = S$. Thus $\sigma(S_4) = \sigma(S_5) = \sigma(S_4^{-1}S)$. Then $\mathcal{A}_i$ is the dual class of itself, but $|\mathcal{A}_i| = 5$, a contradiction.

Case 2: $\mathcal{A}_i$ contains two sequences of length 2. Than $|S_1| = |S_2| = 2$. Without loss of generality, let

$$S_1 = x_1 \cdot x_2, S_2 = x_3 \cdot x_4.$$

If $|S_j| \geq 3$ for some $j \in [3, 5]$, then $\gcd(S_j, x_5 \cdot x_6) \neq 1$. Furthermore, if $|S_j| = 4$, then $x_5 \cdot x_6 \,|\, S_j$ and $|\gcd(S_1, S_j)| = |\gcd(S_2, S_j)| = 1$. Also, we may assume that $\mathcal{A}_i$ contains at most two sequences of length 4. Otherwise, we may consider $\overline{\mathcal{A}_i}$ instead and it contains three sequences of length 2. This reduces to Case 1, and we are done.

Subcase 2.1: $\mathcal{A}_i$ contains two sequences of lengths 4. Then $|S_3| = 3$ and $|S_4| = |S_5| = 4$. Since $x_5 \cdot x_6 \,|\, S_4$ and $|\gcd(S_1, S_4)| = |\gcd(S_2, S_4)| = 1$, we may assume $S_4 = x_5 \cdot x_6 \cdot x_1 \cdot x_3$. Since $x_5 \cdot x_6 \,|\, S_5$ and $|\gcd(S_5, S_4)| \leq 2$, we have $S_5 = x_5 \cdot x_6 \cdot x_2 \cdot x_4$.

Without loss of generality, let $x_5 \,|\, S_3$. Then $x_6 \nmid S_3$. If $x_1, x_2 \nmid S_3$, then $S_3 = x_3 \cdot x_4 \cdot x_5$ and $S_2 \,|\, S_3$, a contradiction. If $x_1 \,|\, S_3$, then $x_2, x_3 \nmid S_3$. Therefore, $S_3 = x_5 \cdot x_1 \cdot x_4$ and $\mathcal{A}_i$ is of form $(c1)$. If $x_2 \,|\, S_3$, then similarly we have $S_3 = x_5 \cdot x_2 \cdot x_3$, and thus $\mathcal{A}_i$ is of form $(c1)$ again.

Subcase 2.2: $\mathcal{A}_i$ contains one sequence of length 4. Then $|S_5| = 4$ and $|S_3| = |S_4| = 3$. Since $x_5 \cdot x_6 \,|\, S_5$ and $|\gcd(S_1, S_5)| = |\gcd(S_2, S_5)| = 1$, we may assume $S_5 = x_5 \cdot x_6 \cdot x_1 \cdot x_3$. Note that $\gcd(S_3, x_5 \cdot x_6) \neq 1$ and $\gcd(S_4, x_5 \cdot x_6) \neq 1$. We may assume that $|\gcd(S_3, x_5 \cdot x_6)| \geq |\gcd(S_4, x_5 \cdot x_6)|$.

If $x_5 \cdot x_6 \,|\, S_3$, then $x_1, x_3 \nmid S_3$. Without loss of generality, let $x_2 \,|\, S_3$. Then $S_3 = x_5 \cdot x_6 \cdot x_2$. Next, we may assume $x_5 \,|\, S_4$. Then $x_2, x_6 \nmid S_4$. If $x_1 \nmid S_4$, then $S_4 = x_5 \cdot x_3 \cdot x_4$ and $S_2 \,|\, S_4$, a contradiction. So $x_1 \,|\, S_4$. By Lemma 2.4, $|\gcd(S_4, S_5)| \leq 2$, so $x_3 \nmid S_4$. Therefore $S_4 = x_5 \cdot x_1 \cdot x_4$. Then $\mathcal{A}_i$ is of form $(c2)$.

Now, suppose $\gcd(S_3, x_5 \cdot x_6) = x_5$, and then $|\gcd(S_4, x_5 \cdot x_6)| = 1$. Since $\mathsf{v}_{x_6}(T) \geq 2$, then $x_6 \,|\, S_4$ and thus $x_5 \nmid S_4$. Hence, $S_4 \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_6$. If $x_1 \,|\, S_3$, then $x_2, x_3 \nmid S_3$, so $S_3 = x_5 \cdot x_1 \cdot x_4$. Since $\mathsf{v}_{x_1}(T) \leq 3$, $x_1 \nmid S_4$. and thus $S_4 \,|\, x_2 \cdot x_3 \cdot x_4 \cdot x_6$. Note that $|\gcd(x_3 \cdot x_4, S_4)| \leq 1$. We have $S_4 = x_6 \cdot x_2 \cdot x_3$ or $S_4 = x_6 \cdot x_2 \cdot x_4$. If $S_4 = x_6 \cdot x_2 \cdot x_3$, then $S_4 = SS_3^{-1}$, so $\mathcal{A}_i$ is the dual class of itself. Since $|\mathcal{A}_i| = 5$, $\mathcal{A}_i$ is not self-dual, a contradiction. Thus $S_4 = x_6 \cdot x_2 \cdot x_4$ and then $\mathcal{A}_i$ is of form $(c3)$.

Next, assume that $x_1 \nmid S_3$. By the symmetry of x_1 and x_3 in $\{S_1, S_2, S_5\}$, we may also assume that $x_3 \nmid S_3$. By the symmetry of S_3 and S_4, we also have $x_1, x_3 \nmid S_4$. Then,

$$S_3 \,|\, x_2 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{and} \quad S_4 \,|\, x_2 \cdot x_4 \cdot x_5 \cdot x_6\,,$$

so $|\gcd(S_3, S_4)| \geq 2$, a contradiction.

Subcase 2.3: $\mathcal{A}_i$ contains no sequence of length 4. Then $|S_3| = |S_4| = |S_5| = 3$. Since $\gcd(S_j, x_5 \cdot x_6) \neq 1$ for every $j = 3, 4, 5$, we may assume $x_5 \,|\, \gcd(S_3, S_4)$. If $x_5 \,|\, S_5$, then

$$x_5^{-1} S_\nu \,|\, x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_6 \quad \text{for every} \quad \nu \in [3, 5]\,.$$

Since $|x_5^{-1}S_3| = |x_5^{-1}S_4| = |x_5^{-1}S_5| = 2$, there exist $m, n \in [3,5]$ such that $|\gcd(x_5^{-1}S_m, x_5^{-1}S_n)| \ge 1$, so $|\gcd(S_m, S_n)| \ge 2$, a contradiction. Thus $x_5 \nmid S_5$, and therefore $x_6 \mid S_5$. By the symmetry of S_3, S_4, we may assume $x_6 \mid S_3$ and $x_6 \nmid S_4$. This gives that $x_5 \cdot x_6 \mid S_3$. By the symmetry of x_1, x_2, x_3 and x_4 in $\{S_1, S_2\}$, we may assume $S_3 = x_5 \cdot x_6 \cdot x_1$. Since $x_5 \mid S_4$, we have $x_6, x_1 \nmid S_4$. so $S_4 = x_5 \cdot x_2 \cdot x_3$ or $S_4 = x_5 \cdot x_2 \cdot x_4$ or $S_4 = x_5 \cdot x_3 \cdot x_4$. But $S_2 \nmid S_4$, so $S_4 \ne x_5 \cdot x_3 \cdot x_4$. By the symmetry of x_3 and x_4 in $\{S_1, S_2, S_3\}$, we may assume that $S_4 = x_5 \cdot x_2 \cdot x_3$. Since $x_6 \mid S_5$, we have $x_1 \nmid S_5$, so $S_5 = x_6 \cdot x_2 \cdot x_3$ or $S_5 = x_6 \cdot x_2 \cdot x_4$ or $S_5 = x_6 \cdot x_3 \cdot x_4$. Note that $x_6 + x_2 + x_3 \ne x_5 + x_2 + x_3, x_6 + x_3 + x_4 \ne x_3 + x_4$, we must have $S_5 = x_6 \cdot x_2 \cdot x_4$. Hence, $\mathcal{A}_i$ is of form $(c4)$.

Case 3: $\mathcal{A}_i$ contains exactly one sequence of length 2. We may also assume $\mathcal{A}_i$ contains at most one sequence of length 4 (otherwise, we may consider $\overline{\mathcal{A}_i}$ instead and we are back to one of the above cases). Let $S_1 = x_1 \cdot x_2$.

Subcase 3.1: $\mathcal{A}_i$ contains exactly one sequence of length 4. Then $|S_5| = 4$. If $\gcd(S_5, S_1) = 1$, then $\mathcal{A}_i$ is the dual class of itself, giving a contradiction. So $\gcd(S_5, S_1) \ne 1$. Without loss of generality, we may assume that $S_5 = x_1 \cdot x_3 \cdot x_5 \cdot x_6$.

If $x_1 \nmid S_2S_3S_4$, then we have

$$S_\nu \mid x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{for every} \quad \nu \in [2,4]\,.$$

Since $|S_2| = |S_3| = |S_4| = 3$, there exist $m, n \in [2,4]$ such that $|\gcd(S_m, S_n)| \ge 2$, a contradiction. So $x_1 \mid S_2S_3S_4$. But $\mathsf{v}_a(T) \le 3$ for every $a \mid S$, so we have $\mathsf{v}_{x_1}(S_2S_3S_4) = 1$. Without loss of generality, let $x_1 \mid S_2$. Then $x_2 \nmid S_2$ and $x_1 \nmid S_3S_4$. If $x_4 \nmid S_2$, then $S_2 \mid x_1 \cdot x_3 \cdot x_6 \cdot x_5 = S_5$, a contradiction. So $x_4 \mid S_2$. By the symmetry of x_3, x_6 and x_5 in $\{S_1, S_5\}$, we may assume $x_3 \mid S_2$, so $S_2 = x_1 \cdot x_4 \cdot x_3$.

Note that $x_1 \nmid S_3S_4$, and thus we have

$$S_3 \mid x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6 \quad \text{and} \quad S_4 \mid x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_6\,.$$

Since $\mathsf{v}_{x_2}(T) \ge 2$, we have $\mathsf{v}_{x_2}(S_3S_4) \ge 1$. Let $x_2 \mid S_3$. If $x_6, x_5 \nmid S_3$, then $S_3 = x_2 \cdot x_3 \cdot x_4$, and thus $|\gcd(S_2, S_3)| = 2$, a contradiction. So $x_6 \mid S_3$ or $x_5 \mid S_3$. Without loss of generality, let $x_6 \mid S_3$. If $x_5 \mid S_3$, then $S_3 = x_2 \cdot x_6 \cdot x_5 = SS_2^{-1}$, so $\mathcal{A}_i$ is the dual class of itself, a contradiction. Then $x_5 \nmid S_3$. Therefore, $S_3 = x_2 \cdot x_6 \cdot x_3$ or $S_3 = x_2 \cdot x_4 \cdot x_6$.

First assume that $S_3 = x_2 \cdot x_6 \cdot x_3$. If $x_2 \nmid S_4$, we have $S_4 \mid x_3 \cdot x_4 \cdot x_5 \cdot x_6$. So $S_4 = x_3 \cdot x_6 \cdot x_5$ or $S_4 = x_3 \cdot x_4 \cdot x_6$ or $S_4 = x_3 \cdot x_5 \cdot x_4$ or $S_4 = x_4 \cdot x_5 \cdot x_6$. But

$$\begin{aligned} x_3 + x_6 + x_5 &\ne x_1 + x_3 + x_6 + x_5; \\ x_3 + x_4 + x_6 &\ne x_1 + x_4 + x_3; \\ x_3 + x_5 + x_4 &\ne x_1 + x_4 + x_3, \end{aligned}$$

so $S_4 = x_4 \cdot x_5 \cdot x_6$. Then $\mathcal{A}_i$ is of form $(c5)$. If $x_2 \mid S_4$, then $x_3, x_6 \nmid S_4$, so $S_4 = x_2 \cdot x_5 \cdot x_4$. Again, $\mathcal{A}_i$ is of form $(c6)$.

Next, assume that $S_3 = x_2 \cdot x_4 \cdot x_6$. If $x_2 \nmid S_4$, we have $S_4 \mid x_3 \cdot x_4 \cdot x_5 \cdot x_6$. So $S_4 = x_3 \cdot x_6 \cdot x_5$ or $x_3 \cdot x_4 \cdot x_6$ or $x_3 \cdot x_5 \cdot x_4$ or $S_4 = x_4 \cdot x_5 \cdot x_6$. Since

$$x_3 + x_6 + x_5 \ne x_1 + x_3 + x_6 + x_5;$$

$$x_3 + x_4 + x_6 \neq x_1 + x_4 + x_3;$$
$$x_3 + x_5 + x_4 \neq x_1 + x_4 + x_3;$$
$$x_4 + x_5 + x_6 \neq x_2 + x_4 + x_6,$$

none of the above cases are possible. So $x_2 \mid S_4$. Then $x_4, x_6 \nmid S_4$ and thus $S_4 = x_2 \cdot x_5 \cdot x_3$. By the symmetry of x_6 and x_5 in $\{S_1, S_2, S_5\}$, we have $\mathcal{A}_i$ is of form $(c6)$.

Subcase 3.2: $\mathcal{A}_i$ contains no sequence of length 4. Then $|S_2| = |S_3| = |S_4| = |S_5| = 3$.

Recall that $S_1 = x_1 \cdot x_2$. Since $\mathsf{v}_a(T) \geq 2$ for every $a \mid S$, we have $\mathrm{supp}(S_2S_3S_4S_5) = \mathrm{supp}(S)$.

We assert that $\mathsf{v}_a(S_2S_3S_4S_5) = 2$ for every $a \in S$.

If there exists $a \mid S$ such that $\mathsf{v}_a(S_2S_3S_4S_5) = 3$, we may assume $a \mid \gcd(S_2, S_3, S_4)$. Since

$$a^{-1}S_\nu \mid a^{-1}S \quad \text{for every} \quad \nu \in [2, 4]\,,$$

there exist $m, n \in [2, 4]$ such that $\left|\gcd\left(a^{-1}S_m, a^{-1}S_n\right)\right| \geq 1$. This implies that $|\gcd(S_m, S_n)| \geq 2$, a contradiction. Thus $\mathsf{v}_a(S_2S_3S_4S_5) \leq 2$ for every $a \in S$. Since $|S_2S_3S_4S_5| = 12$, we have $\mathsf{v}_a(S_2S_3S_4S_5) = 2$ for every $a \in S$. This proves the assert.

Re call that $T = S_1S_2S_3S_4S_5$. By the above assertion, we have $\mathsf{v}_{x_1}(T) = \mathsf{v}_{x_2}(T) = 3$. So we may assume $x_1 \mid \gcd(S_2, S_3)$ and $x_2 \mid \gcd(S_4, S_5)$. Then $x_2 \nmid \gcd(S_2, S_3)$ and $x_1 \nmid \gcd(S_4, S_5)$. Without loss of generality, let $S_2 = x_1 \cdot x_3 \cdot x_6$ and $S_3 = x_1 \cdot x_5 \cdot x_4$. Since $|\gcd(S_4, S_2)| \leq 1$ and $|\gcd(S_4, S_3)| \leq 1$, we may assume $S_4 = x_2 \cdot x_3 \cdot x_5$. Then $S_5 = x_2 \cdot x_4 \cdot x_6$ and therefore, $\mathcal{A}_i$ is of form $(c7)$.

Case 4: $\mathcal{A}_i$ contains no sequence of length 2. As before, we may assume $\mathcal{A}_i$ contains no sequence of length 4. Then $|S_1| = \cdots = |S_5| = 3$ and $|T| = 15$. Since $|S| = 6$, we must have $\mathsf{v}_a(T) = 3$ for some $a \mid S$. As in Subcase 3.2, there exist $m \neq n$ such that $\left|\gcd(S_m, S_n)\right| \geq 2$, giving a contradiction.

This completes the proof. ∎

Lemma 7.10. *If* $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 + x_1 + x_3 = x_5 + x_6 + x_2 + x_4 = x_5 + x_1 + x_4$, *then* $\mathsf{f}(S) \geq 19$.

Proof. Let

$a_1 = x_1 = x_2 + x_6 = x_4 + x_5 + x_6$,
$a_2 = x_2 = x_4 + x_5 = x_3 + x_5 + x_6$,
$a_3 = x_3 = x_1 + x_5 = x_2 + x_5 + x_6$,
$a_4 = x_4 = x_3 + x_6 = x_1 + x_5 + x_6$,
$a_5 = x_1 + x_2 = x_3 + x_4 = x_5 + x_6 + x_1 + x_3 = x_5 + x_6 + x_2 + x_4 = x_5 + x_1 + x_4$,
$a_6 = x_1 + x_3 = x_2 + x_4 = x_1 + x_2 + x_5 + x_6 = x_3 + x_4 + x_5 + x_6 = x_2 + x_3 + x_6$,
$a_7 = x_1 + x_2 + x_3 = x_1 + x_3 + x_4 + x_5 = x_2 + x_3 + x_4 + x_5 + x_6$,
$a_8 = x_1 + x_2 + x_4 = x_1 + x_2 + x_3 + x_6 = x_1 + x_3 + x_4 + x_5 + x_6$,
$a_9 = x_1 + x_3 + x_4 = x_2 + x_3 + x_4 + x_6 = x_1 + x_2 + x_4 + x_5 + x_6$,
$a_{10} = x_2 + x_3 + x_4 = x_1 + x_2 + x_4 + x_5 = x_1 + x_2 + x_3 + x_5 + x_6$,
$a_{11} = x_1 + x_2 + x_3 + x_4 = x_1 + x_2 + 2x_3 + x_6 = x_1 + x_3 + 2x_4 + x_5$,
$a_{12} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$,

$a_{13} = x_1 + x_4 = x_1 + x_3 + x_6 = x_2 + x_4 + x_6 = x_2 + x_3 + 2x_6,$
$a_{14} = x_2 + x_3 = x_3 + x_4 + x_5 = x_1 + x_2 + x_5,$
$a_{15} = x_1 + x_2 + x_6 = x_3 + x_4 + x_6 = x_1 + x_4 + x_5 + x_6 = x_1 + x_3 + x_5 + 2x_6,$
$a_{16} = x_1 + x_3 + x_5 = x_2 + x_4 + x_5 = x_2 + x_3 + x_5 + x_6,$
$a_{17} = x_5 + x_6,$
$a_{18} = x_5,$
$a_{19} = x_6,$
$a_{20} = x_1 + x_2 + x_3 + x_4 + x_5,$
$a_{21} = x_1 + x_6.$

By Lemma 2.4, we have

$$a_1, a_2, \ldots, a_{16}$$

are pairwise distinct.

In view of $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 + x_1 + x_3 = x_5 + x_6 + x_2 + x_4$, we obtain that $2(x_5 + x_6) = 0$. So $a_{17} \neq a_1, \ldots, a_{11}, a_{13}, a_{14}$. By Lemma 2.4, we have $a_{17} \neq a_{12}, a_{15}, a_{16}$. Therefore,

$$a_1, \ldots, a_{17}$$

are pairwise distinct.

By Lemma 2.4, we have

$$a_{18} \neq a_1, \ldots, a_{10}, a_{12}, a_{14}, \ldots, a_{17};$$
$$a_{19} \neq a_1, \ldots, a_{10}, a_{12}, a_{13}, a_{15}, \ldots, a_{18}.$$

If $a_{18} = a_{13}$, then $x_5 = x_1 + x_4 = x_1 + x_3 + x_6 = x_2 + x_4 + x_6$, so $x_2 = x_4 + x_5 = x_3 + x_5 + x_6 = x_1 + x_3 + x_4 + x_6$. It follows from Lemma 3.3 that $\mathsf{f}(S) \geq 19$. So, we may assume $a_{18} \neq a_{13}$. Similarly, we may assume that $a_{19} \neq a_{14}$, so $x_6 \neq x_2 + x_3$.

If $a_{18} \neq a_{11}$ and $a_{19} \neq a_{11}$, then $a_1, a_2, \ldots, a_{19}$ are pairwise distinct and we are done. Without loss of generality, let $a_{18} = a_{11}$. Then $a_{19} \neq a_{11}$ and thus $a_1, a_2, \ldots, a_{17}, a_{19}$ are pairwise distinct.

By Lemma 2.4, we have $a_{20} \neq a_1, \ldots, a_{14}, a_{16}$. Since $x_6 \neq x_2 + x_3$, we have $x_1 + x_2 + x_3 + x_4 + x_5 \neq x_1 + x_4 + x_5 + x_6$, that is $a_{20} \neq a_{15}$. Note that $x_1 + x_2 + x_3 + x_4 + x_5 = x_5 + x_5 \neq x_5 + x_6$. We have $a_{20} \neq a_{17}$. If $a_{20} \neq a_{19}$, then $a_1, \ldots a_{17}, a_{19}, a_{20}$ are pairwise distinct and we are done. So, we may assume that $a_{20} = a_{19}$, so $x_1 + x_2 + x_3 + x_4 + x_5 = x_6$. Then we have

$$a_{21} = x_1 + x_6 = x_2 + x_6 + x_6 = x_1 + x_1 + x_2 + x_3 + x_4 + x_5.$$

Since S has no elements of order 2, again, by Lemma 2.4, we have $a_{21} \neq a_1, \ldots, a_{17}, a_{19}$. Therefore

$$a_1, a_2, \ldots, a_{17}, a_{19}, a_{21}$$

are pairwise distinct and we are done. ∎

Lemma 7.11. *If* $x_1 + x_2 = x_1 + x_3 + x_4 = x_1 + x_5 + x_6 = x_2 + x_3 + x_5 = x_2 + x_4 + x_6$, *then* $\mathsf{f}(S) \geq 19$.

Proof. Let

$a_1 = x_1 = x_3 + x_5 = x_4 + x_6,$
$a_2 = x_2 = x_3 + x_4 = x_5 + x_6,$
$a_3 = x_3,$
$a_4 = x_4,$
$a_5 = x_5,$
$a_6 = x_6,$
$a_7 = x_1 + x_2 = x_1 + x_3 + x_4 = x_1 + x_5 + x_6 = x_2 + x_3 + x_5 = x_2 + x_4 + x_6,$
$a_8 = x_3 + x_4 + x_5 + x_6 = x_2 + x_5 + x_6 = x_2 + x_3 + x_4 = x_1 + x_4 + x_6 = x_1 + x_3 + x_5,$
$a_9 = x_1 + x_3 + x_4 + x_5 + x_6 = x_1 + x_2 + x_5 + x_6 = x_1 + x_2 + x_3 + x_4,$
$a_{10} = x_2 + x_3 + x_4 + x_5 + x_6 = x_1 + x_2 + x_4 + x_6 = x_1 + x_2 + x_3 + x_5,$
$a_{11} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6,$
$a_{12} = x_1 + x_3 = x_3 + x_4 + x_6 = x_2 + x_6,$
$a_{13} = x_1 + x_4 = x_3 + x_4 + x_5 = x_2 + x_5,$
$a_{14} = x_1 + x_5 = x_4 + x_5 + x_6 = x_2 + x_4,$
$a_{15} = x_1 + x_6 = x_3 + x_5 + x_6 = x_2 + x_3,$
$a_{16} = x_2 + x_4 + x_5 + x_6 = x_1 + x_2 + x_5 = x_1 + x_3 + x_4 + x_5,$
$a_{17} = x_2 + x_3 + x_5 + x_6 = x_1 + x_2 + x_6 = x_1 + x_3 + x_4 + x_6,$
$a_{18} = x_2 + x_3 + x_4 + x_6 = x_1 + x_2 + x_3 = x_1 + x_3 + x_5 + x_6,$
$a_{19} = x_2 + x_3 + x_4 + x_5 = x_1 + x_2 + x_4 = x_1 + x_4 + x_5 + x_6.$

Since S contains no elements of order 2, we have $a_3 \neq a_{14}$, $a_{12} \neq a_{19}$, $a_{13} \neq a_{18}$, $a_{14} \neq a_{17}$, $a_{15} \neq a_{16}$. This together with Lemma 2.4 shows that

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}$$

are pairwise distinct and we are done. ∎

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4.

By Lemma 5.1, $|\mathcal{A}_k| \leq 5$ for all $k \in [1, r]$. If $\mathcal{A}_i$ has the form $(c1)$ or $(c7)$ described in Lemma 7.9, then by Lemma 7.10 or Lemma 7.11, we have $\mathsf{f}(S) \geq 19$. Next, we may assume $\mathcal{A}_i$ has one of the forms $(c2)$, $(c3)$, $(c4)$, $(c5)$ and $(c6)$. Then we have one of the following holds correspondingly.

$$x_{\tau(2)} = x_{\tau(3)} + x_{\tau(5)} + x_{\tau(6)} = x_{\tau(4)} + x_{\tau(5)} = x_{\tau(1)} + x_{\tau(3)},$$
$$x_{\tau(3)} = x_{\tau(4)} + x_{\tau(5)} + x_{\tau(6)} = x_{\tau(1)} + x_{\tau(5)} = x_{\tau(2)} + x_{\tau(6)},$$
$$x_{\tau(1)} = x_{\tau(2)} + x_{\tau(5)} + x_{\tau(6)} = x_{\tau(3)} + x_{\tau(5)} = x_{\tau(4)} + x_{\tau(6)},$$
$$x_{\tau(2)} = x_{\tau(3)} + x_{\tau(5)} + x_{\tau(6)} = x_{\tau(1)} + x_{\tau(5)} = x_{\tau(3)} + x_{\tau(4)},$$

and

$$x_{\tau(2)} = x_{\tau(3)} + x_{\tau(5)} + x_{\tau(6)} = x_{\tau(1)} + x_{\tau(5)} = x_{\tau(3)} + x_{\tau(4)}.$$

It follows from Lemma 5.1, Lemma 7.2 that $\mathcal{A}_i$ induces a class $[x_{\tau(j)}]$ of form $(b3)$ described in Lemma 7.2, and therefore, the lemma follows from Lemma 7.4. ∎

8 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on Theorem 1.2, and it uses ideas of P. Erdős, W. Gao, A. Geroldinger, Y. ould Hamidoune et.al. (see [8, Sections 5.3 and 5.4]).

Let G be cyclic of order $n \ge 3$ and let $S \in \mathcal{F}(G)$ be zero-sum free with

$$|S| \ge \frac{6n+28}{19}\,.$$

Let $q \in \mathbb{N}_0$ be maximal such that S has a representation in the form $S = S_0 S_1 \cdot \ldots \cdot S_q$ with squarefree, zero-sum free sequences $S_1, \ldots, S_q \in \mathcal{F}(G)$ of length $|S_\nu| = 6$ for all $\nu \in [1, q]$. Among all those representations of S choose one for which $d = |\operatorname{supp}(S_0)|$ is maximal, and set $S_0 = g_1^{r_1} \cdot \ldots \cdot g_d^{r_d}$, where $g_1, \ldots, g_d \in G$ are pairwise distinct, $d \in \mathbb{N}_0$ and $r_1 \ge \cdots \ge r_d \in \mathbb{N}$. Since q is maximal, we have $d \in [0, 5]$.

Assume to the contrary that $r_1 \le 1$. Then either $d = 0$ or $r_1 = \ldots = r_d = 1$, and for convenience we set $\mathsf{F}(0) = 0$. By Theorem 1.2, Lemmas 2.1 and 2.2, it follows that

$$\begin{aligned} |\Sigma(S)| &\ge |\Sigma(S_0)| + \sum_{i=1}^{q} |\Sigma(S_i)| \ge |\Sigma(S_0)| + 19q \\ &\ge 19\frac{|S|-d}{6} + \mathsf{F}(d) = \frac{19|S| - 19d + 6\mathsf{F}(d)}{6} \ge \frac{19|S|-28}{6} \ge n\,, \end{aligned}$$

a contradiction.

Thus it follows that $r_1 \ge 2$, and we set $g = g_1$. We assert that $\mathsf{v}_g(S_i) \ge 1$ for all $i \in [1, q]$. Assume to the contrary that there exists some $i \in [1, q]$ with $g \nmid S_i$. Then there is an $h \in \operatorname{supp}(S_i)$ with $h \nmid S_0$. Since S may be written in the form

$$S = (hg^{-1}S_0) S_1 \cdot \ldots \cdot S_{i-1} (gh^{-1}S_i) S_{i+1} \cdot \ldots \cdot S_q\,,$$

and $|\operatorname{supp}(hg^{-1}S_0)| > |\operatorname{supp}(S_0)|$, we get a contradiction to the maximality of $|\operatorname{supp}(S_0)|$.

Clearly S_0 allows a product decomposition of the form

$$S_0 = \prod_{i=1}^{5} T_1^{(i)} \cdot \ldots \cdot T_{q_i}^{(i)}\,,$$

where all $T_\nu^{(i)} \in \mathcal{F}(G)$ are squarefree with $\mathsf{v}_g(T_\nu^{(i)}) = 1$, $q_1, \ldots, q_5 \in \mathbb{N}_0$ and $|T_1^{(i)}| = \ldots = |T_{q_i}^{(i)}| = i$ for all $i \in [1, 5]$. Thus we get

$$|S| = |S_0| + 6q = q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 6q\,,$$

$$\mathsf{v}_g(S_0) = q_1 + \ldots + q_5 \quad \text{and hence} \quad \mathsf{v}_g(S) \ge q + q_1 + \ldots + q_5\,.$$

Since

$$\begin{aligned} n - 1 \ge |\Sigma(S)| &\ge |\Sigma(S_0)| + \sum_{i=1}^{5} |\Sigma(T_1^{(i)} \cdot \ldots \cdot T_{q_i}^{(i)})| \\ &\ge q\mathsf{F}(6) + \sum_{i=1}^{5} q_i \mathsf{F}(i) = 19q + q_1 + 3q_2 + 5q_3 + 8q_4 + 13q_5\,, \end{aligned}$$

we infer that

$$\begin{aligned} 6|S| - (n-1) &\le 6(q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 6q) - (q_1 + 3q_2 + 5q_3 + 8q_4 + 13q_5 + 19q) \\ &= 17q + 17q_5 + 16q_4 + 13q_3 + 9q_2 + 5q_1 \\ &\le 17\mathsf{v}_g(S)\,. \end{aligned}$$

∎

We close the paper with a remark on Olson's constant. Let $\mathsf{ol}(G)$ denote the maximal length of a squarefree, zero-sum free sequence over G, and let $\mathsf{Ol}(G)$ be the smallest integer $l \in \mathbb{N}$ such that every squarefree sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ satisfies $0 \in \Sigma(S)$. Then $1 + \mathsf{ol}(G) = \mathsf{Ol}(G)$, and $\mathsf{Ol}(G)$ is called *Olson's constant*. If

$$\mathsf{F}(G,k) \ge 1 + c^{-2}k^2 \quad \text{for some} \quad k \in \mathbb{N} \text{ and } c \in \mathbb{R}_{>0}\,,$$

then a simple argument shows that $\mathsf{ol}(G) < c\sqrt{|G|-1}$ (see [8, Lemma 5.1.17] for details). A survey on Olson's constant can be found in [6, Section 10].

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