On Subsequence Sums of a Zero-sum Free Sequence II

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Abstract

Let G be an additive finite abelian group with exponent $\exp(G) = n$. For a sequence S over G, let $\mathsf{f}(S)$ denote the number of non-zero group elements which can be expressed as a sum of a nontrivial subsequence of S. We show that for every zero-sum free sequence S over G of length $|S| = n + 1$ we have $\mathsf{f}(S) \geq 3n - 1$.

1 Introduction and Main results

Let G be an additive finite abelian group with exponent $\exp(G) = n$ and let S be a sequence over G (we follow the conventions of [5] concerning sequences over abelian groups; details are recalled in Section 2). We denote by $\Sigma(S)$ the set of all subsums of S, and by $\mathsf{f}(G, S) = \mathsf{f}(S)$ the number of nonzero group elements which can be expressed as a sum of a nontrivial subsequence of S (thus $\mathsf{f}(S) = |\Sigma(S) \setminus \{0\}|$).

In 1972, R.B. Eggleton and P. Erdős (see [2]) first tackled the problem of determining the minimal cardinality of $\Sigma(S)$ for squarefree zero-sum free sequences (that is for zero-sum free subsets of G), see [7] for recent progress. For general sequences the problem was first studied by J.E. Olson and E.T. White in 1977 (see Lemma 2.5). In a recent new approach [16], the fourth author of this paper proved that every zero-sum free sequence S over G of length $|S| = n$ satisfies $\mathsf{f}(S) \geq 2n - 1$. A main result of the present paper runs as follows.

Theorem 1.1. *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 \mid \ldots \mid n_r$. If $r \geq 2$ and $n_{r-1} \geq 3$, then every zero-sum free sequence S over G of length $|S| = n_r + 1$ satisfies $\mathsf{f}(S) \geq 3n_r - 1$.*

This partly confirms a former conjecture of B. Bollobás and I. Leader, which is outlined in Section 6. All information on the minimal cardinality of $\Sigma(S)$ can successfully applied to the investigation of a great variety of problems in combinatorial and additive number theory. In the final section of this paper we will discuss applications to the study of $\Sigma_{|G|}(S)$, a topic which has been studied by many authors (see [14], [3], [13], [12], [10], [11] and the surveys [5, 8]). In particular, Theorem 1.1 and a result of B. Bollobás and I. Leader (see Theorem **A** in Section 6) has the following consequence.

Corollary 1.2. *Let G be a finite abelian group with exponent $\exp(G) = n$, and let S be a sequence over G of length $|S| = |G| + n$. Then, either $0 \in \sum_{|G|}(S)$ or $|\sum_{|G|}(S)| \geq 3n - 1$.*

This paper is organized as follows. In Section 2 we fix notation and gather the necessary tools from additive group theory. In Section 3 we prove a crucial result (Theorem 3.2) whose corollary answers a question of H. Snevily. In Section 4 we continue to present some more preliminary results which will be used in the proof of the main result 1.1, which will finally be given in Section 5. In Section 6 we briefly discuss some applications.

Throughout this paper, let G denote an additive finite abelian group.

2 Notation and some results from additive group theory

Our notation and terminology are consistent with [5] and [9]. We briefly gather some key notions and fix the notation concerning sequences over abelian groups. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

Throughout, all abelian groups will be written additively. For $n \in \mathbb{N}$, let C_n denote a cyclic group with n elements.

Let $A, B \subset G$ be nonempty subsets. Then $A + B = \{a + b \mid a \in A, b \in B\}$ denotes their *sumset*. The *stabilizer* of A is defined as $\mathrm{Stab}(A) = \{g \in G \mid g + A = A\}$, A is called *periodic* if $\mathrm{Stab}(A) \neq \{0\}$, and we set $-A = \{-a \mid a \in A\}$.

An s-tuple $(e_1, \ldots, e_s)$ of elements of G is said to be *independent* if $e_i \neq 0$ for all $i \in [1, s]$ and, for every s-tuple $(m_1, \ldots, m_s) \in \mathbb{Z}^s$,

$$m_1e_1 + \ldots + m_se_s = 0 \qquad \text{implies} \qquad m_1e_1 = \ldots = m_se_s = 0\,.$$

An s-tuple $(e_1, \ldots, e_s)$ of elements of G is called a *basis* if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_s \rangle$.

Let $\mathcal{F}(G)$ be the multiplicative, free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,, \quad \text{with} \quad \mathsf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all} \quad g \in G\,.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S *contains* g if $\mathsf{v}_g(S) > 0$. A sequence S_1 is called a *subsequence* of S if $S_1 \,|\, S$ in $\mathcal{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$). Given two sequences $S, T \in \mathcal{F}(G)$, we denote by $\gcd(S,T)$ the longest subsequence dividing both S and T. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S \;=\; g_1 \cdot \ldots \cdot g_l \;=\; \prod_{g \in G} g^{\mathsf{v}_g(S)} \;\in \mathcal{F}(G)\,,$$

we call

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \qquad \text{the } \textit{length} \text{ of } S\,,$$

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|]$$
$$\text{the } \textit{maximum of the multiplicities} \text{ of } S\,,$$

$$\operatorname{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \qquad \text{the } \textit{support} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \qquad \text{the } \textit{sum} \text{ of } S\,,$$

$$\Sigma_k(S) = \Big\{ \sum_{i \in I} g_i \;\Big|\; I \subset [1, l] \text{ with } |I| = k \Big\}$$
$$\text{the } \textit{set of } k\textit{-term subsums} \text{ of } S\,, \text{ for all } k \in \mathbb{N}\,,$$

$$\Sigma_{\le k}(S) = \bigcup_{j \in [1,k]} \Sigma_j(S)\,, \qquad \Sigma_{\ge k}(S) = \bigcup_{j \ge k} \Sigma_j(S)\,,$$

and

$$\Sigma(S) = \Sigma_{\ge 1}(S) \quad \text{the } \textit{set of (all) subsums} \text{ of } S\,.$$

The sequence S is called

- *zero-sum free* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if $1 \ne S$, $\sigma(S) = 0$, and every $S' | S$ with $1 \le |S'| < |S|$ is zero-sum free.

We denote by $\mathcal{A}(G) \subset \mathcal{F}(G)$ the set of all minimal zero-sum sequences over G. Every map of abelian groups $\varphi\colon G \to H$ extends to a homomorphism $\varphi\colon \mathcal{F}(G) \to \mathcal{F}(H)$ where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. If φ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \operatorname{Ker}(\varphi)$.

Let $\mathsf{D}(G)$ denote the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \ge l$ has a zero-sum subsequence. Equivalently, we have $\mathsf{D}(G) = \max\{|S| \mid S \in \mathcal{A}(G)\}$, and $\mathsf{D}(G)$ is called the *Davenport constant* of G.

We shall need the following results on the Davenport constant (proofs can be found in [9, Proposition 5.1.4 and Proposition 5.5.8.2.(c)]).

Lemma 2.1. *Let $S \in \mathcal{F}(G)$ be a zero-sum free sequence.*

1. *If $|S| = \mathsf{D}(G) - 1$, then $\Sigma(S) = G \setminus \{0\}$, and hence $\mathsf{f}(S) = |G| - 1$.*

2. *If G is a p-group and $|S| = \mathsf{D}(G) - 2$, then there exist a subgroup $H \subset G$ and an element $x \in G \setminus H$ such that $G \setminus \big(\Sigma(S) \cup \{0\}\big) \subset x + H$.*

Lemma 2.2. *Let $G = C_{n_1} \bigoplus C_{n_2}$ with $1 \le n_1 \mid n_2$, and let $S \in \mathcal{F}(G)$.*

1. *$\mathsf{D}(C_{n_1} \bigoplus C_{n_2}) = n_1 + n_2 - 1$.*

2. *If S has length $|S| = 2n_1 + n_2 - 2$, then S has a zero-sum subsequence T of length $|T| \in [1, n_2]$.*

3. *If S has length $|S| = n_1 + 2n_2 - 2$, then S has a zero-sum subsequence W of length $|W| \in \{n_2, 2n_2\}$.*

Proof. 1. and 2. follow from [9, Theorem 5.8.3].
3. See [5, Theorem 6.7]. □

Proofs of the two following classical addition theorems can be found in [9, Theorem 5.2.6 and Corollary 5.2.8].

Lemma 2.3. *Let $A, B \subset G$ be nonempty subsets.*

1. *(Cauchy-Davenport) If G is cyclic of order $|G| = p \in \mathbb{P}$, then $|A+B| \ge \min\{p, |A| + |B| - 1\}$.*

2. *(Kneser) If $H = \mathrm{Stab}(A + B)$ denotes the stabilizer of $A + B$, then $|A + B| \ge |A + H| + |B + H| - |H|$.*

We continue with some crucial definitions going back to R.B. Eggleton and P. Erdős. For a sequence $S \in \mathcal{F}(G)$ let

$$\mathsf{f}(G, S) = \mathsf{f}(S) = |\Sigma(S) \setminus \{0\}| \quad \text{be the } \textit{number of nonzero subsums} \text{ of } S\,.$$

Let $k \in \mathbb{N}$. We define

$$\mathsf{F}(G, k) = \min\big\{|\Sigma(S)| \;\big|\; S \in \mathcal{F}(G) \text{ is a zero-sum free and squarefree sequence of length } |S| = k\big\}\,,$$

and we denote by $\mathsf{F}(k)$ the minimum of all $\mathsf{F}(A, k)$ where A runs over all finite abelian groups A having a squarefree and zero-sum free sequence of length k. Furthermore, we set

$$\mathsf{f}(G, k) = \min\big\{|\Sigma(S)| \;\big|\; S \in \mathcal{F}(G) \text{ is zero-sum free of length } |S| = k\big\}\,.$$

By definition, we have $\mathsf{f}(G,k) \le \mathsf{F}(G,k)$. Since there is no zero-sum sequence S of length $|S| \ge \mathsf{D}(G)$, we have $\mathsf{f}(G,k) = 0$ for $k \ge \mathsf{D}(G)$. The following simple example provides an upper bound for $\mathsf{f}(G,\cdot)$ which will be used frequently in the sequel (see also Conjecture 6.2).

Example 1. Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \ge 2$, $1 < n_1 \mid \ldots \mid n_r$ and let $(e_1, \ldots, e_r)$ be a basis of G with $\operatorname{ord}(e_i) = n_i$ for all $i \in [1, r]$. For $k \in [0, n_{r-1} - 2]$ we set

$$S = e_r^{n_r - 1} e_{r-1}^{k+1} \in \mathcal{F}(G) \,.$$

Clearly, S is zero-sum free, $|S| = n_r + k$ and $\mathsf{f}(S) = (k+2)n_r - 1$. Thus we get $\mathsf{f}(G, n_r + k) \le (k+2)n_r - 1$.

Lemma 2.4. *[9, Theorem 5.3.1] If $t \in \mathbb{N}$ and $S = S_1 \cdot \ldots \cdot S_t \in \mathcal{F}(G)$ is zero-sum free, then*

$$\mathsf{f}(S) \ge \mathsf{f}(S_1) + \ldots + \mathsf{f}(S_t) \,.$$

Lemma 2.5. *[15] Let $S \in \mathcal{F}(G)$ be zero-sum free. If $\langle \operatorname{supp}(S) \rangle$ is not cyclic, then*

$$|\Sigma(S)| \ge 2|S| - 1 \,.$$

Lemma 2.6. *[7, Lemma 2.3] Let $S = S_1 S_2 \in \mathcal{F}(G)$, $H = \langle \operatorname{supp}(S_1) \rangle$ and let $\varphi \colon G \to G/H$ denote the canonical epimorphism. Then we have*

$$\mathsf{f}(S) \ge \big(1 + \mathsf{f}(\varphi(S_2))\big)\mathsf{f}(S_1) + \mathsf{f}\big(\varphi(S_2)\big) \,.$$

Lemma 2.7.

1. *$\mathsf{F}(1) = 1$, $\mathsf{F}(2) = 3$, $\mathsf{F}(3) = 5$ and $\mathsf{F}(4) = 8$.*
2. *If $S \in \mathcal{F}(G)$ is squarefree, zero-sum free of length $|S| = 3$ and contains no elements of order 2, then $\mathsf{f}(S) \ge 6$.*
3. *$\mathsf{F}(5) = 13$ and $\mathsf{F}(6) = 19$.*

Proof. 1. See [9, Corollary 5.3.4.1].
2. See [9, Proposition 5.3.2.2].
3. See [7]. □

The proof of the following lemma follows the lines of the proof of [7, Theorem 1.3].

Lemma 2.8. *Let $S \in \mathcal{F}(G)$ be zero-sum free of length $|S| \ge 2$. If $\mathsf{f}(S) \le 3|S| - 5$, then $\mathsf{h}(S) \ge \max\{2, \frac{3|S|+5}{17}\}$.*

Proof. Let $q \in \mathbb{N}_0$ be maximal such that S has a representation in the form $S = S_0 S_1 \cdot \ldots \cdot S_q$ with $S_0 \in \mathcal{F}(G)$ and squarefree, zero-sum free sequences $S_1, \ldots, S_q \in \mathcal{F}(G)$ of length $|S_\nu| = 6$ for all $\nu \in [1, q]$. Among all those representations of S choose one for which $d = |\operatorname{supp}(S_0)|$ is maximal, and set $S_0 = g_1^{r_1} \cdot \ldots \cdot g_d^{r_d}$, where $g_1, \ldots, g_d \in G$ are pairwise distinct, $d \in \mathbb{N}_0$ and $r_1 \geq \cdots \geq r_d \in \mathbb{N}$. Since q is maximal, we have $d \in [0, 5]$.

Assume to the contrary that $r_1 \leq 1$. Then either $d = 0$ or $r_1 = \ldots = r_d = 1$, and for convenience we set $\mathsf{F}(0) = 0$. By Lemmas 2.4 and 2.7, we obtain that

$$\mathsf{f}(S) \geq \mathsf{f}(S_1) + \ldots + \mathsf{f}(S_q) + \mathsf{F}(d) \geq 19q + \mathsf{F}(d) = 3|S| - 4 + q + \mathsf{F}(d) - 3d + 4 \geq 3|S| - 4\,,$$

a contradiction.

Therefore, $\mathsf{h}(S) \geq r_1 \geq 2$, and we set $g = g_1$. We assert that $\mathsf{v}_g(S_i) \geq 1$ for all $i \in [1, q]$. Assume to the contrary that there exists some $i \in [1, q]$ with $g \nmid S_i$. Since $|S_i| = 6 > d$, there is an $h \in \operatorname{supp}(S_i)$ with $h \nmid S_0$. Since S may be written in the form

$$S = (hg^{-1}S_0)S_1 \cdot \ldots \cdot S_{i-1}(gh^{-1}S_i)S_{i+1} \cdot \ldots \cdot S_q\,,$$

and $|\operatorname{supp}(hg^{-1}S_0)| > |\operatorname{supp}(S_0)|$, we get a contradiction to the maximality of $|\operatorname{supp}(S_0)|$. Therefore, $\mathsf{h}(S) \geq \mathsf{v}_g(S) = q + r_1 \geq 2$.

Clearly, S_0 allows a product decomposition

$$S_0 = \prod_{i=1}^{5} T_i^{q_i}\,,$$

where, for all $i \in [1, 5]$, $T_i = g_1 \cdot \ldots \cdot g_i$ and $q_i = r_i - r_{i+1}$, with $r_6 = 0$. Thus we get $q_1 + \ldots + q_5 = r_1 = \mathsf{v}_g(S_0)$, $q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 = |S_0|$ and

$$q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 6q = |S|\,.$$

By Lemma 2.4 and Lemma 2.7 we obtain that

$$q_1 + 3q_2 + 5q_3 + 8q_4 + 13q_5 + 19q \leq \mathsf{f}(S) \leq 3|S| - 5\,.$$

Using the last two relations we infer that

$$\begin{aligned} &17q + 17q_5 + 16q_4 + 13q_3 + 9q_2 + 5q_1 = \\ &6(q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 6q) - (q_1 + 3q_2 + 5q_3 + 8q_4 + 13q_5 + 19q) \geq 3|S| + 5\,, \end{aligned}$$

and therefore

$$\mathsf{h}(S) \geq \mathsf{v}_g(S) = q + r_1 = q + q_1 + \ldots + q_5 \geq \frac{3|S| + 5}{17}\,. \qquad \square$$

3 Sums and Element Orders

Theorem 3.2 in this section will be used repeatedly to deduce Theorem 1.1 and it also has its own interest. Moreover, its corollary answers a question of H. Snevily. We first prove a lemma.

Lemma 3.1. *Let $A \subset G$ be a finite nonempty subset.*

1. *If $x + A = A$ for some $x \in G$, then $|A|x = 0$.*
2. *Let $r \in \mathbb{N}$, $y_1, \ldots, y_r \in G$ and $k = \min\{\mathrm{ord}(y_i) \mid i \in [1, r]\}$. Then $|\sum(0y_1 \cdot \ldots \cdot y_r) + A| \geq \min\{k, r + |A|\}$.*

Proof. 1. Since $x + A = A$, we have that

$$|A|x + \sum_{a \in A} a = \sum_{a \in A}(x + a) = \sum_{a \in A} a.$$

Therefore, $|A|x = 0$.

2. We proceed by induction on r. Let $r = 1$. If $|\sum(0y_1) + A| \geq 1 + |A|$ then we are done. Otherwise, $\sum(0y_1) + A = (y_1 + A) \cup A = A$. This forces that $y_1 + A = A$. By 1., we have $|A|y_1 = 0$. Therefore, $k \leq \mathrm{ord}(y_1) \leq |A|$, and thus $|\sum(0y_1) + A| = |A| \geq \mathrm{ord}(y_1) \geq k$. So, $|\sum(0y_1) + A| \geq \min\{k, 1 + |A|\}$.

Suppose that $r \geq 2$ and that the assertion is true for $r - 1$. Let $B = \sum(0y_1 \cdot \ldots \cdot y_{r-1}) + A$. If $|\sum(0y_1 \cdot \ldots \cdot y_r) + A| \geq 1 + |B|$, then by induction hypothesis, we have that $|\sum(0y_1 \cdot \ldots \cdot y_r) + A| \geq 1 + |B| \geq 1 + \min\{k, r - 1 + |A|\} \geq \min\{k, r + |A|\}$ and we are done. So, we may assume that $|\sum(0y_1 \cdot \ldots \cdot y_r) + A| \leq |B|$. Note that $\sum(0y_1 \cdot \ldots \cdot y_r) + A = (y_r + (\sum(0y_1 \cdot \ldots \cdot y_{r-1}) + A)) \cup (\sum(0y_1 \cdot \ldots \cdot y_{r-1}) + A) = (y_r + B) \cup B$. We must have $y_r + B = B$. By 1., we have $|B|y_r = 0$, and thus $k \leq \mathrm{ord}(y_r) \leq |B|$. Therefore, $|\sum(0y_1 \cdot \ldots \cdot y_r) + A| \geq |B| \geq k$. This completes the proof. □

Theorem 3.2. *Let $S = a_1 \cdot \ldots \cdot a_k \in \mathcal{F}(G \setminus \{0\})$ be a sequence of length $|S| = k \geq 2$, and set $q = |\{0\} \cup \sum(S)|$.*

1. *If T is a proper subsequence of S such that $|\{0\} \cup \sum(U)| = |\{0\} \cup \sum(T)|$ for every subsequence U of S with $T|U$ and $|U| = |T| + 1$, then $\{0\} \cup \sum(T) = \{0\} \cup \sum(S)$.*
2. *For any nontrivial subsequence V_0 of S, there is a subsequence V of S with $V_0|V$, such that $|\{0\} \cup \sum(V)| - |V| \geq |\{0\} \cup \sum(V_0)| - |V_0|$ and $\{0\} \cup \sum(V) = \{0\} \cup \sum(S)$.*
3. *Suppose that $q \leq |S|$. Then there is a proper subsequence W of S such that $\{0\} \cup \sum(W) = \{0\} \cup \sum(S)$ and $|W| \leq q - 1$. Moreover, $qx = 0$ for every term $x \in SW^{-1}$.*
4. *If $q \leq |S|$ and $a_i \notin \{a_1, -a_1\}$ for some $i \in [2, k]$, then we can find a W with all properties stated in (3) such that $|W| \leq q - 2$.*
5. *Suppose that $q \leq |S|$. There is a subsequence T of S with $|T| \geq |S| - q + 2$ such that $|\langle \mathrm{supp}(T) \rangle| \mid q$.*

Proof. 1. Let $ST^{-1} = g_1 \cdot \ldots \cdot g_l$. By the assumption,

$$\{0\} \cup \sum(g_i T) = \{0\} \cup \sum(T)$$

holds for every $i \in [1, l]$, or equivalently,

$$\{0\} \cup \{g_i\} + \{0\} \cup \sum(T) = \{0\} \cup \sum(T)$$

for every $i \in [1, t]$. Therefore,

$$\{0\} \cup \sum(S) = \{0\} \cup \{g_1\} + \{0\} \cup \{g_2\} + \ldots + \{0\} \cup \{g_t\} + \{0\} \cup \sum(T) = \{0\} \cup \sum(T).$$

2. Let V be a subsequence of S with maximal length such that $V_0 | V$ and $|\{0\} \cup \sum(V)| - |V| \geq |\{0\} \cup \sum(V_0)| - |V_0|$. If $V = S$, then clearly the result holds. Next, we may assume that V is a proper subsequence. It is not hard to show that V satisfies the assumption in 1.. By 1. we conclude that $\{0\} \cup \sum(V) = \{0\} \cup \sum(S)$.

3. Let W be a subsequence of S with maximal length such that $|\{0\} \cup \sum(W)| \geq |W| + 1$. Then $|W| \leq |\{0\} \cup \sum(W)| - 1 \leq |\{0\} \cup \sum(S)| - 1 = q - 1 < |S|$. Therefore, W is a proper subsequence of S.

Using the maximality of W, we can easily verify that W satisfies the assumption in 1.. It follows from 1. that $\{0\} \cup \sum(W) = \{0\} \cup \sum(S)$. Since for each $x \in SW^{-1}$, $|x + \{0\} \cup \sum(S)| = |\{0\} \cup \sum(S)|$ and $x + \{0\} \cup \sum(S) = x + \{0\} \cup \sum(W) \subset \{0\} \cup \sum(S)$, we obtain that $x + \{0\} \cup \sum(S) = \{0\} \cup \sum(S)$. It now follows from Lemma 3.1 that $qx = 0$ holds for every $x \in SW^{-1}$.

4. Let $V_0 = a_1 a_i$. Then $|\{0\} \cup \sum(V_0)| - |V_0| = 4 - 2 = 2$. By 2., there exists a subsequence W such that $|\{0\} \cup \sum(W)| - |W| \geq 2$ and $\{0\} \cup \sum(W) = \{0\} \cup \sum(S)$. Thus $|W| \leq q - 2 \leq |S| - 2$, and therefore, clearly W is a proper subsequence of S. As in 3., we can prove that $qx = 0$ holds for every $x \in SW^{-1}$.

5. If $a_i \in \{a_1, -a_1\}$ holds for every $i \in [2, k]$, then by 3. we have that $qa_i = 0$ for some i. Since $a_i = \pm a_1$, we have $qa_1 = 0$ and $\mathrm{ord}(a_1)$ divides q. Let $T = S$. Then $|\langle \mathrm{supp}(T) \rangle| = |\langle a_1 \rangle| = \mathrm{ord}(a_1)$ divides q. Next we assume that $a_i \notin \{a_1, -a_1\}$ for some $i \in [2, k]$, by 4. there is a proper subsequence W of S with $\{0\} \cup \sum(W) = \{0\} \cup \sum(S)$ and $|W| \leq q - 2$. Let $T = SW^{-1}$. Then,

$$|T| = |S| - |W| \geq |S| - q + 2.$$

For every term y in T, as shown in 3. we have that

$$y + \{0\} \cup \sum(U) = \{0\} \cup \sum(U).$$

Therefore,

$$\langle \mathrm{supp}(T) \rangle + \{0\} \cup \sum(W) = \{0\} \cup \sum(W).$$

Since the left hand side is a union of some cosets of $\langle \mathrm{supp}(T) \rangle$, we conclude that $|\langle \mathrm{supp}(T) \rangle|$ divides $|\{0\} \cup \sum(U)| = q$ as desired. $\square$

The following result answers a question of H. Snevily, formulated in a private communication to the first author.

Corollary 3.3. *Let $S = a_1 \cdot \ldots \cdot a_r \in \mathcal{F}(G)$, and suppose that* $\mathrm{ord}(a_i) \ge r$ *holds for every* $i \in [1, r]$. *Then,* $|\{a_i\} \cup (a_i + \sum(Sa_i^{-1}))| \ge r$ *holds for every* $i \in [1, r]$.

Proof. Let $q = |0 \cup \sum(Sa_i^{-1})|$. If $q \le r - 1$, then by Theorem 3.2.3, $qa_j = 0$ for some $j \ne i$. Thus $q \ge \mathrm{ord}(a_j) \ge r$, giving a contradiction. Therefore, $q \ge r$ and thus $|\{a_i\} \cup (a_i + \sum(Sa_i^{-1}))| = |0 \cup \sum(Sa_i^{-1})| \ge r$ as desired. ☐

4 Zero-sum free sequences over groups of rank two

Lemma 4.1. *Let $G = C_m \oplus C_n$ with $1 < m \mid n$. Suppose that $\mathsf{f}(C_m \oplus C_m, m + k) = (k+2)m - 1$ for every positive integer $k \in [1, m-2]$ and $n \ge m(1 + \frac{km+3}{\mathsf{f}(N, m+k+1)+1-(k+2)m})$. Then $\mathsf{f}(G, n + k) = (k + 2)n - 1$.*

Proof. Clearly, we have $n \ge 2m$. Let $k \in [1, m-2]$ and let $S \in \mathcal{F}(G)$ be zero-sum free of length

$$|S| = n + k = (\frac{n}{m} - 3)m + (3m - 2) + 2 + k\,. \tag{$*$}$$

By Example 1, we obtain that $\mathsf{f}(G, n+k) \le (k+2)n - 1$, and so we need only show that $\mathsf{f}(S) = |\sum(S)| \ge (k + 2)n - 1$. Let $\varphi\colon G \to N$ be an epimorphism with $N \cong C_m \oplus C_m$ and $\mathrm{Ker}(\varphi) \cong C_{\frac{n}{m}}$.

By ($*$) and Lemma 2.2.1 (for details see [9, Lemma 5.7.10]), S allows a product decomposition $S = S_1 \cdot \ldots \cdot S_{n/m-2}T$, where $S_1, \ldots, S_{n/m-2}, T \in \mathcal{F}(G)$ and, for every $i \in [1, n/m - 2]$, $\varphi(S_i)$ has sum zero and length $|S_i| \in [1, m]$. Note that $|T| \ge 2m + k$. We distinguish two cases.

Case 1: $|T| \ge 3m - 2$.

Applying Lemma 2.2.1 to $\varphi(T)$, we can find a subsequence of T, say $S_{\frac{n}{m}-1}$, such that

$$1 \le |S_{\frac{n}{m}-1}| \le m \quad \text{ and } \quad \sigma(S_{\frac{n}{m}-1}) \in \mathrm{Ker}(\varphi)\,.$$

We claim that $\varphi(TS^{-1}_{\frac{n}{m}-1})$ is zero-sum free. Otherwise, if $\varphi(TS^{-1}_{\frac{n}{m}-1})$ is not zero-sum free, or equivalently, if $TS^{-1}_{\frac{n}{m}-1}$ has a nontrivial subsequence $S_{\frac{n}{m}}$ (say) such that $\sigma(S_{\frac{n}{m}}) \in \mathrm{Ker}(\varphi)$, then the sequence $\prod_{i=1}^{\frac{n}{m}} \sigma(S_i)$ of $\frac{n}{m}$ elements in $\mathrm{Ker}(\varphi)$ is not zero-sum free. Therefore, S is not zero-sum free, giving a contradiction. Hence, $\varphi(TS^{-1}_{\frac{n}{m}-1})$ is zero-sum free as claimed. Note that $|\varphi(TS^{-1}_{\frac{n}{m}-1})| \ge 2m + k - m = m + k$. By the hypothesis of the lemma,

$$\mathsf{f}(\varphi(TS^{-1}_{\frac{n}{m}-1})) \ge \mathsf{f}(N, m + k) \ge (k + 2)m - 1.$$

Let $R_1 = \prod_{i=1}^{\frac{n}{m}-1} \sigma(S_i)$. Then $|R_1| = \frac{n}{m} - 1$ and R_1 is zero-sum free. Therefore, $|\langle \mathrm{supp}(R_1)\rangle| \ge \mathsf{f}(R_1) + 1 \ge |R_1| + 1 = \frac{n}{m} = |\mathrm{Ker}(\varphi)|$ and then $\langle \mathrm{supp}(R_1)\rangle = \mathrm{Ker}(\varphi)$. Let $R_2 = TS^{-1}_{\frac{n}{m}-1}$. Now applying Lemma 2.6 to the sequence R_1R_2, we obtain that

$$\begin{aligned} \mathsf{f}(S) &\ge \mathsf{f}(R_1R_2) \ge (1 + \mathsf{f}(\varphi(R_2)))\mathsf{f}(R_1) + \mathsf{f}(\varphi(R_2)) \\ &\ge (1 + \mathsf{f}(\varphi(TS^{-1}_{\frac{n}{m}-1})))(\frac{n}{m} - 1) + \mathsf{f}(\varphi(TS^{-1}_{\frac{n}{m}-1})) \ge (k + 2)n - 1\,. \end{aligned}$$

Case 2: $|T| \in [2m+k, 3m-3]$.
If $\varphi(T)$ has a nontrivial zero-sum subsequence of length not exceeding m, then by repeating the argument used in the above case we can prove the result, i.e. $\mathsf{f}(S) \ge (k+2)n-1$. So, we may assume that $\varphi(T)$ has no nontrivial zero-sum subsequence of length not exceeding m.

Next, consider the sequence $T0^{3m-2-|T|}$ of $3m-2$ elements in G. Then $\varphi(T0^{3m-2-|T|})$ is a sequence of length $3m-2$ in $N = C_m \oplus C_m$. By applying Lemma 2.2.2 to $\varphi(T0^{3m-2-|T|})$, we obtain that $T0^{3m-2-|T|}$ has a subsequence W such that $\sigma(\varphi(W)) = 0$ and $|W| \in \{m, 2m\}$. If $|W| = m$, then $\varphi(T)$ has a nontrivial zero-sum subsequence $\varphi(W \cap T)$ of length not exceeding m, a contradiction. Therefore, $|W| = 2m$ and

$$\sigma(W) \in \mathrm{Ker}(\varphi)\,.$$

Let $W_1 = \gcd(W, T)$. Then $|W_1| \ge |W| - (3m-2-|T|) \ge m+k+2$, and $\varphi(W_1)$ is a minimal zero-sum sequence. Since $\varphi(T)$ has no nontrivial zero-sum subsequences of length not exceeding m, we can choose a subsequence W_2 of W_1 with $|W_2| = m+k+1$ such that the subgroup generated by $\varphi(TW_2^{-1})$ is not cyclic. Let $T_1 = TW_2^{-1}$. Clearly, $|T_1| \ge m-1$ and $\mathsf{f}(\varphi(W_2)) \ge \mathsf{f}(N, m+k+1)$. It follows from Lemma 2.4 , Lemma 2.5 and Lemma 2.6 that

$$\begin{aligned}
\mathsf{f}(S) &\ge \mathsf{f}(\prod_{i=1}^{\frac{n}{m}-2} \sigma(S_i)W_2T_1) \ge \mathsf{f}(\prod_{i=1}^{\frac{n}{m}-2} \sigma(S_i)W_2) + \mathsf{f}(T_1) \\
&\ge (1+\mathsf{f}(\varphi(W_2)))(\frac{n}{m}-2) + \mathsf{f}(\varphi(W_2)) + \mathsf{f}(T_1) \\
&\ge (1+\mathsf{f}(N, m+k+1))(\frac{n}{m}-2) + \mathsf{f}(N, m+k+1) + (2m-3) \\
&\ge (k+2)n-1.
\end{aligned}$$

□

Let $G = C_n \oplus C_n$ with $n \ge 2$. We say that G has Property **B** if every minimal zero-sum sequence $S \in \mathcal{F}(G)$ of length $|S| = \mathsf{D}(G) = 2n-1$ contains some element with multiplicity $n-1$. This property was first addressed in [4], and it is conjectured that every group (of the above form) satisfies Property **B**. The present state of knowledge on Property **B** is discussed in [8, Section 7]). In particular, if $n \in [4, 7]$, then G has Property **B**. Here we need the following characterization (for a proof see [9, Theorem 5.8.7]).

Lemma 4.2. *Let $G = C_n \oplus C_n$ with $n \ge 2$. Then the following statements are equivalent:*

1. *If $S \in \mathcal{F}(G)$, $|S| = 3n-3$ and S has no zero-sum subsequence T of length $|T| \ge n$, then there exists some $a \in G$ such that $0^{n-1}a^{n-2} \mid S$.*
2. *If $S \in \mathcal{F}(G)$ is zero-sum free and $|S| = 2n-2$, then $a^{n-2} \mid S$ for some $a \in G$.*
3. *If $S \in \mathcal{A}(G)$ and $|S| = 2n-1$, then $a^{n-1} \mid S$ for some $a \in G$.*

4. *If* $S \in \mathcal{A}(G)$ *and* $|S| = 2n-1$, *then there exists a basis* (e_1, e_2) *of* G *and integers* $x_1, \ldots, x_n \in [0, n-1]$ *with* $x_1 + \ldots + x_n \equiv 1 \mod n$ *such that*

$$S = e_1^{n-1} \prod_{\nu=1}^{n} (x_\nu e_1 + e_2)\,.$$

Lemma 4.3. *Let* $G = C_n \oplus C_n$ *with* $n \geq 2$ *and suppose that* G *satisfies Property* **B**. *Let* $S \in \mathcal{A}(G)$ *with length* $|S| = 2n-1$. *If* T *is a subsequence of* S *such that* $|T| = n+k$, *where* $1 \leq k \leq n-2$, *then*

$$\mathsf{f}(T) \geq (k+2)n - 1.$$

Furthermore, if W *is a zero-sum free sequence over* G *with* $|W| = 2n-3$, *then*

$$\mathsf{f}(W) \geq n^2 - n - 1.$$

Proof. Let $S \in \mathcal{A}(G)$ be of length $|S| = 2n-1$. Then by Lemma 4.2, there is a basis (e_1, e_2) of G such that $S = {e_1}^{n-1} \prod_{i=1}^{n} (e_1 + a_i e_2)$ with $\sum_{i=1}^{n} a_i \equiv 1 \mod n$. Without loss of generality, let $S = {e_2}^{n-1} \prod_{i=1}^{n} (e_1 + a_i e_2)$ and let $V = \prod_{i=1}^{n} (e_1 + a_i e_2)$. Then $T = {e_2}^{n+k-l} \prod_{i=1}^{l} (e_1 + a_i e_2)$, where $l \in [k+1, n]$. Let $\varphi\colon G \to \langle e_2 \rangle$ be the canonical epimorphism.

Case 1: $l = n$.

Then $T = {e_2}^k \prod_{i=1}^{n} (e_1 + a_i e_2) = {e_2}^k V$. Since $\sum_{i=1}^{n} a_i \equiv 1 \mod n$, we have $\sigma(V) = e_2$. Therefore, $|\langle e_2 \rangle \cap \Sigma(T)| \geq k+1$. Since $\sum_{i=1}^{n} a_i \equiv 1 \mod n$ we infer that $a_1, \ldots, a_n$ are not all equal to the same number modulo n. Without loss of generality, we may assume that $a_{n-1} \not\equiv a_n \mod n$. So, for every $i \in [1, n-1]$ we have $|(ie_1 + \langle e_2 \rangle) \cap \Sigma(V)| \geq |\{ie_1 + (a_1 + \ldots + a_{i-1} + a_{n-1})e_2, ie_1 + (a_1 + \ldots + a_{i-1} + a_n)e_2\}| = 2$. By Lemma 3.1.2, we have $|(ie_1 + \langle e_2 \rangle) \cap \Sigma(T)| \geq |(ie_1 + \langle e_2 \rangle) \cap \Sigma(V) + \Sigma(0{e_2}^k)| \geq k+2$. Therefore,

$$\begin{aligned}|\Sigma(T)| \geq& |\langle e_2 \rangle \cap \Sigma(T)| + |(e_1 + \langle e_2 \rangle) \cap \Sigma(T)| + \ldots + |((n-1)e_1 + \langle e_2 \rangle) \cap \Sigma(T)| \\ \geq& k+1 + (k+2) \times (n-1) = (k+2)n - 1.\end{aligned}$$

Case 2: $l \leq n-1$.

Then $k+2 \leq l+1 \leq n$. Let $S_1 = {e_2}^{n+k-l}$ and $S_2 = \prod_{i=1}^{l} (e_1 + a_i e_2)$. Then $\mathsf{f}(S_1) = n+k-l$ and $\mathsf{f}(\varphi(S_2)) = l$. By Lemma 2.6, we have

$$\begin{aligned}\mathsf{f}(T) &\geq (1 + \mathsf{f}(\varphi(S_2)))\mathsf{f}(S_1) + \mathsf{f}(\varphi(S_2)) \\ &= (n+k-l)(l+1) + l \\ &= (n+k-l+1)(l+1) - 1 \\ &\geq (k+2)n - 1.\end{aligned}$$

Next, suppose that $W \in \mathcal{F}(G)$ is zero-sum free of length $|S| = 2n-3$. If $G \setminus \{0\} \subset \Sigma(W)$, then $\mathsf{f}(W) \geq n^2 - 1 > n^2 - n - 1$ and we are done. So, we may assume there exists $g \in G \setminus \{0\}$, such that $-g \notin \Sigma(W)$. Then gW is zero-sum free, and thus, $gW(-g-\sigma(W))$ is a minimal zero-sum sequence of length $2n-1$. It follows from the first part of this lemma that $\mathsf{f}(W) \geq n^2 - n - 1$ as desired. □

Lemma 4.4. *Let G be cyclic of order $|G| = p \in \mathbb{P}$ and $T \in \mathcal{F}(G \setminus \{0\})$. If $a \in G \setminus \{0\}$, then*
$$|\Sigma(Ta) \setminus \{0\}| \geq \min\{p-1, 1 + |\Sigma(T) \setminus \{0\}|\}.$$

Proof. Let $A = \{0\} \cup (\Sigma(T) \setminus \{0\})$ and $B = \{0, a\}$. By Lemma 2.3.1, $|A+B| \geq \min\{p, |A| + |B| - 1\} = \min\{p, 2 + |\Sigma(T) \setminus \{0\}|\}$. Therefore, $|\Sigma(Ta) \setminus \{0\}| = |A + B| - 1 \geq \min\{p - 1, 1 + |\Sigma(T) \setminus \{0\}|\}$. □

Lemma 4.5. *If $G = C_n \oplus C_n$ with $n \in [4, 7]$, then $\mathsf{f}(G, n+2) = 4n - 1$.*

Proof. Let $S \in \mathcal{F}(G)$ be zero-sum free of length $|S| = n + 2$ with $n \in [4, 7]$. As noted above G satisfies Property **B**. By Example 1, it suffices to show that $\mathsf{f}(S) \geq 4n - 1$. If $n = 4$, then $n + 2 = 6 = \mathsf{D}(C_4 \oplus C_4) - 1$. By Lemma 2.1.1, $\mathsf{f}(S) = 16 - 1 = 15$ as desired. If $n = 5$, then $|S| = 2m - 3$, and thus, the result follows immediately from Lemma 4.3.

Now suppose that $n = 6$, and assume to the contrary that $\mathsf{f}(S) \leq 22$. Then, $|-\Sigma(S)| = |\Sigma(S)| = \mathsf{f}(S) \leq 22$ and $|G \setminus (\{0\} \cup (-\Sigma(S)))| \geq 13$. Let $A = \{x_1, \ldots, x_{13}\} \subset G \setminus (\{0\} \cup (-\Sigma(S)))$. Then $x_i S$ is zero-sum free for every $i \in [1, 13]$. If there exist $i, j \in [1, 13]$ such that $x_i x_j S$ is zero-sum free, then $x_i x_j S(-\sigma(x_i x_j S))$ is a minimal zero-sum sequence. Thus, the result follows from Lemma 4.3.

Next, assume that $x_i x_j S$ is not zero-sum free for any $i, j \in [1, 13]$. Since $x_i S$, $x_j S$ is zero-sum free, we must have $x_i + x_j \in -\Sigma(S)$. This implies $A + A \subset -\Sigma(S)$. Then
$$|A + A| \leq |-\Sigma(S)| = |\Sigma(S)| = \mathsf{f}(S) \leq 22.$$
We set $H = \mathrm{Stab}(A + A)$. Then, by Lemma 2.3.2, we have
$$|A + A| \geq 2|A + H| - |H|,$$
and since H is a subgroup of G, we get $|H| \in \{36, 18, 12, 9, 6, 4, 3, 2, 1\}$.

If $|H| \in \{18, 36\}$, then $|G/H| \in \{1, 2\}$, and thus $H \subset (A + H) + (A + H)$. Hence, $0 \in H \subset A + H + A + H = A + A \subset -\Sigma(S)$. Therefore, $0 \in \Sigma(S)$, a contradiction.

We now assume that $|H| \in \{12, 9, 6, 4, 3, 2, 1\}$. Note that
$$|A + H| \geq \left\lceil \frac{|A|}{|H|} \right\rceil |H|.$$
We have
$$|A + A| \geq 2|A + H| - |H| \geq \left(2 \left\lceil \frac{|A|}{|H|} \right\rceil - 1\right) |H| > 22,$$
giving a contradiction.

It remains to consider the case that $n = 7$.

Let S_1 be the maximal subsequence of S such that $\langle \mathrm{supp}(S_1) \rangle$ is cyclic. Then $N = \langle \mathrm{supp}(S_1) \rangle \cong C_7$. Since there are exactly 8 distinct subgroups of order 7 and $|S| = 9$, we must have $\mathsf{f}(S_1) \geq |S_1| \geq 2$. Let $S_2 = S S_1^{-1} = b_1 \cdot \ldots \cdot b_w$ and let $\varphi \colon G \to G/N$ be the canonical epimorphism. Then none of the terms of S_2 is in N, and thus $\varphi(S_2)$ a sequence of non-zero elements in G/N. Let $q = |\{0\} \bigcup \sum \varphi(S_2)|$.

If $\mathsf{f}(S_1) \geq 3$ and $q \geq 7$, then by Lemma 2.6 we have that $\mathsf{f}(S) \geq q\mathsf{f}(S_1) + q - 1 \geq 27$ and we are done. If $\mathsf{f}(S_1) \geq 3$ and $q \leq 6$, then by Theorem 3.2, $|S_2| + 1 \leq q \leq 6$, and thus $4 \leq |S_1| \leq 6$. Again by Lemma 2.6, we have that $\mathsf{f}(S) \geq q\mathsf{f}(S_1) + q - 1 \geq (10 - |S_1|)(|S_1| + 1) - 1 \geq 27$ as desired.

Next we may assume that $\mathsf{f}(S_1) = 2$. Choose a basis (f_1, f_2) of G with $f_2 \mid S_1$. Then, $S_1 = {f_2}^2$ and $\langle \mathrm{supp}(S_1) \rangle = \langle f_2 \rangle = N$. Now

$$S = {f_2}^2 \prod_{i=1}^{k} (a_i f_1 + b_i f_2)$$

with $a_i \neq 0$ for every $i \in [1, k]$, and $S_2 = \prod_{i=1}^{7} (a_i f_1 + b_i f_2)$. Let $r_j = |\Sigma(S) \cap (jf_1 + N)|$ and $s_j = |\Sigma(S_2) \cap (jf_1 + N)|$, where $j \in [0, 6]$. Then

$$\mathsf{f}(S) = \Sigma_{j=0}^{6} r_j.$$

By Lemma 4.4, we have $\Sigma\left(\prod_{i=1}^{7} a_i\right) \cong C_7$, so $s_j = |\Sigma(S_2) \cap (jf_1 + N)| \geq 1$ for every $j \in [0, 6]$. By Lemma 3.1.2, $r_j \geq \min\{\mathrm{ord}(f_2), 2 + s_j\} \geq 3$ for every $j \in [0, 6]$.

Case 1: $\mathsf{h}\left(\prod_{i=1}^{7} a_i\right) \geq 3$.

Without loss of generality, let $a = a_1 = a_2 = a_3$. Since $\mathsf{h}(S) = 2$, we may assume $b_1 \neq b_2$. Then $\left|(af_1 + N) \cap \Sigma(S_2)\right| \geq 2$. By Lemma 3.1.2, $r_a \geq 4$.

By Lemma 4.4, we have $\left|\Sigma\left(\prod_{i=3}^{7} a_i\right) \setminus \{0\}\right| \geq 5$. Assume that $\{x_1, x_2, \ldots, x_5\} \subset \Sigma\left(\prod_{i=3}^{7} a_i\right) \setminus \{0\}$. Then $|((a + x_j)f_1 + N) \cap \Sigma(S_2)| \geq 2$ for every $j \in [1, 5]$. By Lemma 3.1.2, $r_{a+x_j} \geq 4$.

Note that $a, a+x_1, \ldots, a+x_5$ are pairwise distinct, we have $\mathsf{f}(S) = \Sigma_{j=0}^{6} r_j \geq 6 \times 4 + 3 = 27$ as desired.

Case 2: $\mathsf{h}\left(\prod_{i=1}^{7} a_i\right) \leq 2$.

Since $a_i \neq 0$ for every $i \in [1, 7]$ we infer that $\mathsf{h}\left(\prod_{i=1}^{7} a_i\right) = 2$. So, we may assume a_1, a_2, a_3, a_4 are pairwise distinct and $a_1 + a_2 = 0$. Therefore, $(a_1 f_1 + b_1 f_2) + (a_2 f_1 + b_2 f_2) = (b_1 + b_2) f_2 \in N$. By Lemma 2.3.1, we have $\Sigma\left(\prod_{i=3}^{7} a_i\right) \geq 6$. Let $\{x_1, x_2, \ldots, x_5, x_6\} \subset \Sigma\left(\prod_{i=3}^{7} a_i\right)$. For every $j \in [1, 6]$, by Lemma 3.1.2, $r_{x_j} \geq |\sum(0S_1((b_1 + b_2)f_2)) + (x_j f_1 + N) \cap \Sigma(S_2)| \geq 3 + |(x_j f_1 + N) \cap \Sigma(S_2)| \geq 4$. Therefore $\mathsf{f}(S) = \Sigma_{j=1}^{7} r_j \geq 6 \times 4 + 3 = 27$.

Lemma 4.6. *Let $G = C_4 \oplus C_8$. Then $\mathsf{f}(G, 9) = 23$.*

Proof. Assume to the contrary that $\mathsf{f}(G, 9) \neq 23$. By Example 1, there is a zero-sum free sequence $S \in \mathcal{F}(G)$ of length $|S| = 9$ such that $\mathsf{f}(S) = |\sum(S)| \leq 22$. By Lemma 2.1.2, $G \setminus (\sum(S) \cup \{0\}) \subset x + H$ for some subgroup $H \subset G$ and some $x \in G \setminus H$. Therefore,

$$22 \geq |\sum(S)| \geq |G| - 1 - |x + H| = 31 - |H|,$$

and hence, $|H| \geq 9$. Since $|H|$ divides $|G| = 32$, it follows that $|H| = 16$. Therefore, $G = H \cup (x + H)$. From $G \setminus (\sum(S) \cup \{0\}) \subset x + H$ we infer that

$$H \setminus \{0\} \subset \sum(S).$$

Hence,

$$|\sum(S) \cap H| = 15.$$

Since $\mathsf{D}(H) \le 8 + 2 - 1 = 9 = |S|$, we infer that there is at least one term of S is not in H. Let $y \in S$ with $y \in G \setminus H$. Let $T = Sy^{-1}$. Then, $\mathsf{f}(T) \ge \mathsf{f}(G, 8) \ge 2 \times 8 - 1 = 15$. Note that $G = H \cup (x + H)$. We obtain that, $|\sum(T) \cap (x + H)| \ge 8$ or $|\sum(T) \cap H| \ge 8$. This together with $S = Ty$ and $y \in G \setminus H$ implies $|\sum(S) \cap (x + H)| \ge 8$. Therefore, $|\sum(S)| = |\sum(S) \cap H| + |\sum(S) \cap (x + H)| \ge 15 + 8 = 23$, a contradiction. □

5 Proof of Theorem 1.1.

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$, $r \ge 2$, $n_{r-1} \ge 3$, and we set $n = \exp(G) = n_r$. Let $S = a_1 \cdot \ldots \cdot a_{n+1} \in \mathcal{F}(G)$ be a zero-sum free sequence of length $|S| = n + 1$. By Example 1, we need only prove that $\mathsf{f}(S) \ge 3n - 1$. Assume to the contrary that

$$\mathsf{f}(S) \le 3n - 2.$$

By Lemma 2.8, we have

$$\mathsf{h}(S) \ge \max\{2, \frac{3|S| + 5}{17}\} = \max\{2, \frac{3n + 8}{17}\}. \tag{1}$$

Let S_1 be a subsequence of S with maximal length such that $\langle \mathrm{supp}(S_1) \rangle$ is cyclic. We set $N = \langle \mathrm{supp}(S_1) \rangle$ and $S_2 = SS_1^{-1}$. As before, we have $S = S_1 S_2$, and all terms of S_1 are in N, but none of the terms of S_2 is in N. Clearly, $|S_1| \ge \mathsf{h}(S) \ge \frac{3n+8}{17}$. Let $\varphi \colon G \to G/N$ denote the canonical epimorphism, and put

$$S_2 = b_1 \cdot \ldots \cdot b_w \quad \text{and} \quad q = |\{0\} \cup \sum(\varphi(S_2))|\,.$$

By Theorem 3.2, there is a subsequence W_0 of S_2 with $|W_0| \le q - 1$ such that

$$|\{0\} \bigcup \sum(\varphi(W_0))| = q\,.$$

From (1) we have that $|S_1| \ge \max\{2, \frac{3n+8}{17}\} \ge 2$. By Lemma 2.6, we can prove that $q \le |S_2|$. Therefore, $|W_0| \le q - 1 \le |S_2| - 1$. It follows from Theorem 3.2 that

$$\gcd(q, n) > 1 \text{ and } 2 \le q \le \min\{|S_2|, n - 2\}. \tag{2}$$

Using Lemma 2.4 and Lemma 2.6, we obtain that

$$\begin{aligned} 3n - 2 \ge \mathsf{f}(S) &\ge \mathsf{f}(S_1 W_0) + \mathsf{f}(S_2 W_0^{-1}) \\ &\ge q\mathsf{f}(S_1) + q - 1 + \mathsf{f}(S_2 W_0^{-1}) \\ &\ge q\mathsf{f}(S_1) + q - 1 + |S_2| - |W_0| \\ &\ge q|S_1| + q - 1 + |S_2| - |W_0| \\ &\ge q|S_1| + |S_2| \\ &= (q - 1)|S_1| + n + 1. \end{aligned}$$

This gives that $|S_1| \le \frac{2n-3}{q-1}$. Therefore

$$\frac{2n-3}{q-1} \ge |S_1| \ge \frac{3n+8}{17}. \tag{3}$$

Hence, $q \le 12$. Next we distinguish cases according to the value of $q \in [1, 12]$.

Case 1: $9 \le q \le 12$.

We distinguish subcases according to the value taken by n.

Subcase 1.1: $n \ge 15$.

Then $|S_2W_0^{-1}| \ge n+1-\frac{2n-3}{q-1}-(q-1) > \frac{2n-3}{q-1} \ge |S_1|$ (since $n \ge 15$). By the maximality of $|S_1|$, the subgroup generated by $\mathrm{supp}(S_2W_0^{-1})$ is not cyclic. By Lemma 2.5 we have $\mathsf{f}(S_2W_0^{-1}) \ge 2|S_2| - 2|W_0| - 1$. It follows from Lemma 2.4 and Lemma 2.6 that

$$\begin{aligned}\mathsf{f}(S) &\ge \mathsf{f}(S_1W_0) + \mathsf{f}(S_2W_0^{-1}) \ge q\mathsf{f}(S_1) + q - 1 + 2|S_2| - 2|W_0| - 1\\ &\ge q|S_1| + q - 1 + 2(n+1-|S_1|) - 2(q-1) - 1 = (q-2)(|S_1|-1) + 2n\\ &\ge 7(\tfrac{3n+8}{17} - 1) + 2n > 3n - 2 (\text{since } n \ge 10),\end{aligned}$$

a contradiction.

Subcase 1.2: $n = 14$.

By (3) we obtain that $|S_1| = 3$ and $q = 9$. In a similar way to above we derive that $\langle \mathrm{supp}(S_2W_0^{-1})\rangle$ is not cyclic and $\mathsf{f}(S_2W_0^{-1}) \ge 2|S_2| - 2|W_0| - 1$, and

$$\begin{aligned}\mathsf{f}(S) &\ge \mathsf{f}(S_1W_0) + \mathsf{f}(S_2W_0^{-1}) \ge q\mathsf{f}(S_1) + q - 1 + 2|S_2| - 2|W_0| - 1\\ &\ge q|S_1| + q - 1 + 2(n+1-|S_1|) - 2(q-1) - 1 = (q-2)(|S_1|-1) + 2n\\ &\ge 7(3-1) + 2n \ge 3n - 1,\end{aligned}$$

a contradiction.

Subcase 1.3: $n \in \{11, 12, 13\}$.

By (3) we have that $2 \ge |S_1| \ge 3$, a contradiction.

Subcase 1.4: $n \le 10$.

By (2), $q \le n - 2 \le 8$, a contradiction.

Case 2: $q = 8$.

By (2), n is even and $n \ge 10$. We distinguish subcases according to the value of n.

Subcase 2.1: $n \ge 21$.

By (3), $|S_1| \le \frac{2n-3}{7}$. Hence, $|S_2W_0^{-1}| \ge n + 1 - \frac{2n-3}{7} - 7 > \frac{2n-3}{7} \ge |S_1|$ (since $n \ge 13$). By the maximality of $|S_1|$ we know that the subgroup generated by $\mathrm{supp}(S_2W_0^{-1})$ is not cyclic. By Lemma 2.5 we have $\mathsf{f}(S_2W_0^{-1}) \ge 2|S_2| - 2|W_0| - 1$. Therefore,

$$\begin{aligned}3n - 2 \ge \mathsf{f}(S) &\ge \mathsf{f}(S_1W_0) + \mathsf{f}(S_2W_0^{-1})\\ &\ge q\mathsf{f}(S_1) + q - 1 + \mathsf{f}(S_2W_0^{-1})\\ &\ge q\mathsf{f}(S_1) + q - 1 + 2|S_2| - 2|W_0| - 1\\ &\ge q|S_1| + q - 1 + 2|S_2| - 2(q-1) - 1|\\ &= q|S_1| + 2(n+1-|S_1|) - (q-1) - 1 = (q-2)|S_1| + 2n + 2 - q\\ &= 6(|S_1| - 1) + 2n > 3n - 2 \text{ (since } n \ge 21\text{)}\,,\end{aligned}$$

a contradiction.

Subcase 2.2: $10 \le n \le 20$ and $n \ne 16$.

By (3) we have that $|S_1| \ge \frac{3n+8}{17}$. If $\varphi(b_i) \in \{\varphi(b_1), -\varphi(b_1)\}$ holds for every $i \in [2, w]$, then $\varphi(b_i) \in \langle \varphi(b_1) \rangle$ for every $i \in [1, w]$, and by Theorem 3.2 we have $8\varphi(b_1) = 0$. This together with $n\varphi(b_1) = 0$ gives that $\gcd(8, n)\varphi(b_1) = 0$. Therefore, $8 = q = |\{0\} \cup \sum(\varphi(S_2))| \le |\langle \varphi(b_1) \rangle| \le \gcd(8, n) < 8$, a contradiction. Thus, $\varphi(b_i) \notin \{\varphi(b_1), -\varphi(b_1)\}$ for some $i \in [2, w]$. By Theorem 3.2, we can take W_0 such that $|W_0| \le q - 2$ and $\{0\} \cup \sum(\varphi(W_0)) = \{0\} \cup \sum(\varphi(S_2))$. As above, we derive that $\langle \mathrm{supp}(S_2 W_0^{-1}) \rangle$ is not cyclic and $\mathsf{f}(S_2 W_0^{-1}) \ge 2|S_2| - 2|W_0| - 1$. Then

$$\begin{aligned}
\mathsf{f}(S) &\ge \mathsf{f}(S_1 W_0) + \mathsf{f}(S_2 W_0^{-1}) \ge q\mathsf{f}(S_1) + q - 1 + 2|S_2| - 2|W_0| - 1 \\
&\ge q|S_1| + q - 1 + 2(n + 1 - |S_1|) - 2(q - 2) - 1 = (q - 2)(|S_1| - 1) + 2n + 2 \\
&\ge 6(\frac{3n + 8}{17} - 1) + 2n + 2 \ge 3n - 1\,,
\end{aligned}$$

a contradiction.

Subcase 2.3: $n = 16$.

By (3) we have that $|S_1| = 4$. As above, we can take W_0 such that $|W_0| \le q - 1$, and derive that $\langle \mathrm{supp}(S_2 W_0^{-1}) \rangle$ is not cyclic and thus, $\mathsf{f}(S_2 W_0^{-1}) \ge 2|S_2| - 2|W_0| - 1$. Therefore,

$$\begin{aligned}
\mathsf{f}(S) &\ge \mathsf{f}(S_1 W_0) + \mathsf{f}(S_2 W_0^{-1}) \ge q\mathsf{f}(S_1) + q - 1 + 2|S_2| - 2|W_0| - 1 \\
&\ge q|S_1| + q - 1 + 2(n + 1 - |S_1|) - 2(q - 1) - 1 = (q - 2)(|S_1| - 1) + 2n \\
&\ge 6(4 - 1) + 2n \ge 3n - 1\,,
\end{aligned}$$

a contradiction.

Case 3: $q \le 7$.

So, we must have that for every subsequence W of S_2,

$$|\{0\} \bigcup \sum \varphi(W)| \le q \le 7. \tag{4}$$

By Theorem 3.2, there is a subsequence U of S_2 with $|U| \ge |S_2| - q + 1$ such that

$$|\langle \varphi(U) \rangle| \mid q. \tag{5}$$

Let $K = \langle \mathrm{supp}(S_1 U) \rangle$. It follows from (5) that

$$|K| = |N||\langle \varphi(U) \rangle| \mid q|N|. \tag{6}$$

As before, write $S = T_1 T_2$ where all terms of T_1 are in K, but none of the terms of T_2 is in K. Then $\langle \mathrm{supp}(T_1) \rangle = \langle \mathrm{supp}(S_1 U) \rangle = K$, and $|T_1| \ge |S_1 U| \ge n + 2 - q$. Therefore,

$$|T_1| \ge n + 2 - q \ge n - 5. \tag{7}$$

Let $\psi\colon G \to G/K$ be the canonical epimorphism and let $T_2 = c_1 \cdot \ldots \cdot c_{t_2}$.

We distinguish two subcases.

Subcase 3.1: $|T_2| = 0$.

Then

$$K = \langle \mathrm{supp}(S_1U)\rangle = \langle \mathrm{supp}(S)\rangle.$$

Set $\ell = \exp(K)$. Then $|N| \mid \ell \mid n$. Let $K = C_\ell \oplus R$ where R is a finite abelian group with $\exp(R) \mid \ell$. By (6) we have

$$|R| \mid q.$$

Assume to the contrary, that R is not cyclic. Since $|R| \mid q \le 7$, we must have $R = C_2^2$ and $K = C_\ell \oplus C_2 \oplus C_2$. From $\mathsf{D}(K) = \ell + 2 \ge n+1$ we infer that $\ell = n$. Hence, $\mathsf{D}(K) = n+2$. By Lemma 2.1.1, $\mathsf{f}(S) = |K| - 1 = 4n - 1 > 3n - 1$, a contradiction.

Therefore, R is cyclic. If $n = q$, since $|S_1| \ge 2$, by Lemma 2.6 we have that $\mathsf{f}(S) \ge q|S_1| + q - 1 \ge 3q - 1 = 3n - 1$, a contradiction. Therefore, $n = fq$ for some $f \ge 2$.

Since $n + 1 \le \mathsf{D}(K) - 1 = \ell + |R| - 2$, $|R| \mid |q|$, $\ell \mid n$ and $n \ge 2q$, we infer that $\ell = n$ and $|R| \ge 3$. If $|R| < q$, then we must have $|R| = 3$. It follows from Lemma 2.1.1 that $\mathsf{f}(S) = |K| - 1 = 3n - 1$, a contradiction. Therefore, $|R| = q \ge 4$ and $K = C_n \oplus C_q$. By Lemma 4.5 and Lemma 4.1, we have that $n \in \{q, 2q\}$, and therefore, $n = 2q$. We distinguish subcases according to the value $q \le 7$.

Subcase 3.1.1: $q \in \{5, 6, 7\}$.

By (3), $|S_1| \in \{3, 4\}$. Since $|S_2| = |S| - |S_1| \ge 2q + 1 - 4 \ge q > |S_1|$, $\langle \mathrm{supp}(S_2)\rangle$ is not cyclic. By Lemma 2.5, we have $|\Sigma(S_2)| \ge 2|S_2| - 1$.

From $|N||n$, $|K| = nq$ and (6), we obtain that $N \cong C_n$ and $K/N \cong C_q$. Let $K = (g_0 + N) \cup \ldots \cup (g_{q-1} + N)$ be the decomposition of cosets of N, where $g_i \in K$ and $g_0 \in N$. Let $r_i = |(g_i + N) \cap \Sigma(S_2)|$ and $s_i = |(g_i + N) \cap \Sigma(S)|$. Then $|\Sigma(S_2)| = \Sigma_{i=0}^{q-1} r_i$ and $|\Sigma(S)| = \Sigma_{i=0}^{q-1} s_i$. Since $\Sigma(\varphi(S_2)) = G/N \cong C_q$, we have $r_i \ge 1$.

Subcase 3.1.1.1: $|S_1| = 4$.

If $\mathsf{f}(S_1) \ge 5$, then by Lemma 2.6, $\mathsf{f}(S) \ge 5q + q - 1 = 6q - 1 = 3n - 1$, and we are done. So we may assume $S_1 = h^4$, where $\mathrm{ord}(h) = |N| = 2q$. By Lemma 3.1.2, $s_i \ge \min\{2q, r_i + 4\} \ge 5$ for every $i \in [0, q-1]$. If $r_i + 4 \ge 2q$ for some $i \in [0, q-1]$, then

$$|\Sigma(S)| = \Sigma_{i=0}^{q-1} s_i \ge 2q + 5(q-1) = 7q - 5 \ge 6q - 1 = 3n - 1,$$

a contradiction. Next, we may assume $r_i + 4 < 2q$ for all $i \in [0, q-1]$. We have

$$\begin{aligned}|\Sigma(S)| &= \Sigma_{i=0}^{q-1} s_i \ge \Sigma_{i=0}^{q-1}(r_i + 4) = |\Sigma(S_2)| + 4q \ge 2|S_2| - 1 + 4q \\ &= 2(2q + 1 - 4) - 1 + 4q = 8q - 7 \ge 6q - 1,\end{aligned}$$

a contradiction again.

Subcase 3.1.1.2: $|S_1| = 3$.

Since $\mathsf{h}(S) \ge \lceil \frac{3n+8}{17} \rceil \ge 3$, we may assume that $S_1 = h^3$, where $\mathrm{ord}(h) = 2q$. By Lemma 3.1.2, $s_i \ge \min\{2q, r_i + 3\} \ge 4$ for every $i \in [0, q-1]$.

If $r_i + 3 > 2q$ holds for at least two distinct indices $i \in [0, q-1]$, then

$$|\Sigma(S)| = \Sigma_{i=0}^{q-1} s_i \ge 2q + 2q + 4(q-2) = 8q - 8 \ge 6q - 1,$$

a contradiction. If $r_i+3\le 2q$ for every $i\in[0,q-1]$, we have

$$\begin{aligned}|\Sigma(S)| &= \Sigma_{i=0}^{q-1}s_i \ge \Sigma_{i=0}^{q-1}(r_i+3) = |\Sigma(S_2)|+3q \ge 2|S_2|-1+3q\\ &= 2(2q+1-3)-1+3q = 7q-5 \ge 6q-1,\end{aligned}$$

a contradiction. So we may assume that $r_i+3>2q$ holds exactly for one $i\in[0,q-1]$.

If $\varphi(b_i)\in\{\varphi(b_1),-\varphi(b_1)\}$ for every $i\in[1,2q-2]$. We may assume that $\varphi(b_1)=\ldots=\varphi(b_t)$ and $\varphi(b_{t+1})=\ldots=\varphi(b_{2q-2})=-\varphi(b_1)$. Since $\mathsf{v}_g(S_2)\le 3$, and $q-1\ge 4$, we may assume $b_1\ne b_2$. Next, we show that

$$|(b+N)\cap\Sigma(S_2)|\ge 2 \tag{8}$$

holds for every $b\in\{g_0,g_1,\ldots,g_{q-1}\}$.

Note that $\mathrm{ord}(\varphi(b_1))=q$, we have that $N,b_1+N,\ldots,(q-1)b_1+N$ are pairwise disjoint. Therefore, $b+N=jb_1+N=(q-j)(-b_1)+N$ for some $j\in[1,q]$. We may assume that $t\ge q-1$. If $1\le j\le q-2$, then $\{b_3+\ldots+b_{3+j-1}+b_1, b_3+\ldots+b_{3+j-1}+b_2\}\subset (jb_1+N)\cap\Sigma(S_2)=(b+N)\cap\Sigma(S_2)$. Hence, $|(b+N)\cap\Sigma(S_2)|\ge 2$. If $j=q-1$ and $t\ge q$ then $|\Sigma(S_2)|=|(b+N)\cap\Sigma(S_2)|\ge|\{b_3+\ldots+b_q+b_1, b_3+\ldots+b_q+b_2\}|=2$. If $j=q-1$ and $t=q-1$ then $\varphi(b_q)=\ldots=\varphi(b_{2q})=-\varphi(b_1)$. Since $q-1\ge 4$ we may assume that $b_q\ne b_{q+1}$. We now have $|(b+N)\cap\Sigma(S_2)|\ge|\{b_q,b_{q+1}\}|=2$ as desired. Next, assume that $j=q$. If $t\ge q+1$, then as above we can prove that $|(b+N)\cap\Sigma(S_2)|\ge 2$. Otherwise, $t\le q$ and $\varphi(b_{q+1})=-\varphi(b_1)$. Thus, we have that $|(b+N)\cap\Sigma(S_2)|\ge|\{b_{q+1}+b_1,b_{q+1}+b_2\}|=2$. This proves (8). Therefore

$$|\Sigma(S)|=\Sigma_{i=0}^{q-1}s_i\ge(2+3)(q-1)+2q=7q-5\ge 6q-1,$$

a contradiction.

Next, we may assume $\varphi(b_j)\notin\{\varphi(b_1),-\varphi(b_1)\}$ for some $j\in[1,2q-2]$. Then we can choose a subsequence W_0 of S_2 with $|W_0|\le q-2$ such that $|\{0\}\cup\sum(\varphi(W_0))|=q$, so $\Sigma(W_0)\cap(g_i+N)\ne\emptyset$ for every $i\in[1,q-1]$. Since $|S_2W_0^{-1}|\ge q=|\varphi(G)|$, $S_2W_0^{-1}$ has a nontrivial subsequence W_1 with $\sigma(W_1)\in N=\mathrm{Ker}(\varphi)$. Thus, $r_i\ge 2$ for every $i\in[1,q-1]$, and therefore,

$$|\Sigma(S)|=\Sigma_{i=0}^{q-2}s_i\ge 4+(2+3)(q-2)+2q=7q-6\ge 6q-1,$$

a contradiction.

Subcase 3.1.2: $q=4$.

Then S is a zero-sum free sequence of length 9 in $K\cong C_4\oplus C_8$, a contradiction to Lemma 4.6.

Subcase 3.2: $|T_2|\ge 1$.

If $\varphi(b_i)\in\{\varphi(b_1),-\varphi(b_1)\}$ for every $i\in[1,w]$, then we can take $U=S_2$, and this reduces to Subcase 3.1. Next, assume that $\varphi(b_i)\notin\{\varphi(b_1),-\varphi(b_1)\}$ for some $i\in[1,w]$. By Theorem 3.2, we can choose W_0 such that $|W_0|\le q-2$, so $|T_1|\ge n+3-q$.

We first assume that $n\ge 3q-9$. By the maximality of S_1, we know that K is not cyclic. By Lemma 2.5, $\mathsf{f}(T_1)\ge 2|T_1|-1$. It follows from Lemma 2.6 and Lemma 2.4 that

$\mathsf{f}(S) \geq 2\mathsf{f}(T_1)+1+|T_2|-1 \geq 4|T_1|-2+|T_2| = 3|T_1|+n-1 \geq 3(n+3-q)+n-1 \geq 3n-1$ (since $n \geq 3q-9$), giving a contradiction.

Next, we assume that $n \leq 3q-10$. It follows from (2) that

$$q+2 \leq n \leq 3q-10. \tag{9}$$

Thus, $q \geq 6$. Hence, $q \in \{6,7\}$. Let

$$\lambda = |\{0\} \cup \sum(\psi(T_2))|.$$

By Theorem 3.2, there is a subsequence X of T_2 with $|X| \leq \lambda - 1$ such that

$$|\{0\} \bigcup \sum(\psi(X))| = \lambda.$$

We next distinguish subcases according to the possible value of $q \in \{6,7\}$.

Subcase 3.2.1: $q = 6$.

From (9), we obtain that $n = 8$. By Lemma 2.6, we obtain that $q|S_1| + q - 1 \leq 3 \times 8 - 2$. This gives that $|S_1| \leq 2$, so $|S_1| = 2$. Again, by Lemma 2.6, we obtain that $\lambda\mathsf{f}(T_1) + \lambda - 1 \leq 22$. By Lemma 2.5, $\mathsf{f}(T_1) \geq 2|T_1| - 1$. Since $\lambda \geq 2$, $4|T_1| - 1 \leq 22$, and thus $|T_1| \leq 5$. Note that $|T_1| \geq n + 3 - q = 5$. We have $|T_1| = 5$ and $\lambda = 2$. Therefore, $|X| = 1$. By Lemma 2.6 and Lemma 2.4, we obtain that $\mathsf{f}(S) \geq 2\mathsf{f}(T_1) + 1 + \mathsf{f}(T_2X^{-1})$. Since $|T_2X^{-1}| = 3$ and $|S_1| = 2$, by the maximality of S_1 we infer that no element could occur more than two times in T_2X^{-1}. It now follows from Lemma 2.7 and Lemma 2.4 that $\mathsf{f}(T_2X^{-1}) \geq 4$. Therefore, $\mathsf{f}(S) \geq 2\mathsf{f}(T_1)+1+\mathsf{f}(T_2X^{-1}) \geq 4|T_1|-1+4 = 23 = 3n-1$, giving a contradiction.

Subcase 3.2.2: $q = 7$.

From (9), we obtain that $n \in \{9, 10, 11\}$. So, we have $\gcd(q,n) = 1$, giving a contradiction to (2). In all cases, we are able to find a contradiction. Therefore, we must have $\mathsf{f}(S) \geq 3n-1$, so $\mathsf{f}(G, n+1) = 3n-1$ as desired. □

6 On $\Sigma_{|G|}(S)$ and proof of Corollary 1.2.

We briefly point out the relationship between the invariants $\mathsf{f}(G,k)$ and the study of $|\Sigma_{|G|}(S)|$ for suitable $S \in \mathcal{F}(G)$. To do so we need the following result, conjectured in [1] and proved by W. Gao and I. Leader in [6].

Theorem A. Let $S \in \mathcal{F}(G)$ be a sequence. If $0 \notin \Sigma_{|G|}(S)$, then there is a zero-sumfree sequence $T \in \mathcal{F}(G)$ of length $|T| = |S| - |G| + 1$ such that $|\Sigma_{|G|}(S)| \geq |\Sigma(T)|$.

Note that for $S = 0^{|G|-1}T$, where $T \in \mathcal{F}(G)$ is zero-sum free, we have $|\Sigma_{|G|}(S)| = |\Sigma(T)|$. Thus for every $k \in [1, \mathsf{D}(G)-1]$ we have

$$\begin{aligned}&\min\{\Sigma_{|G|}(S) \mid S \in \mathcal{F}(G), |G|+k-1, 0 \notin \Sigma_{|G|}(S)\} = \\ &\min\{|\Sigma(T)| \mid T \in \mathcal{F}(G) \text{ is zero-sum free of length } |T| = k\} = \mathsf{f}(G,k)\,.\end{aligned}$$

Now we are in a position to prove Corollary 1.2.

Proof of Proposition 1.2. Let $\exp(G) = n$ and let $S \in \mathcal{F}(G)$ be a sequence of length $|S| = |G| + n$. Suppose that $0 \notin \Sigma_{|G|}(S)$. Then [9, Theorem 5.8.3]) implies that G is neither cyclic nor congruent to $C_2 \oplus C_n$. Thus it follows that $n+1 \le \mathsf{D}(G) - 1$. Therefore the above considerations (applied with $k = n+1$) show that $|\Sigma_{|G|}(S)| \ge \mathsf{f}(G, n+1)$, and by Theorem 1.1 we have $\mathsf{f}(G, n+1) \ge 3n - 1$. $\square$

We recall a conjecture by B. Bollobás and I. Leader, stated in [1].

Conjecture 6.1. *Let $G = C_n \oplus C_n$ with $n \ge 2$ and let (e_1, e_2) be a basis of G. If $k \in [0, n-2]$ and $S = e_1^{n-1} e_2^{k+1} \in \mathcal{F}(G)$, then $\mathsf{f}(G, n+k) = \mathsf{f}(S)$.*

If S is as above, then clearly $\mathsf{f}(S) = (k+2)n - 1$. Thus [16], Theorem 1.1 and Lemma 4.3 imply that conjecture for $k \in \{0, 1, n-2\}$. We generalize this conjecture as follows (see Example 1).

Conjecture 6.2. *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $r \ge 2$ and $1 < n_1 \mid \ldots \mid n_r$. Let $(e_1, \ldots, e_r)$ be a basis of G with $\mathrm{ord}(e_i) = n_i$ for all $i \in [1, r]$, $k \in [0, n_{r-1} - 2]$ and*

$$S = e_r^{n_r - 1} e_{r-1}^{k+1} \in \mathcal{F}(G) \,.$$

Then we have $\mathsf{f}(G, n_r + k) = \mathsf{f}(S) = (k+2)n_r - 1$.

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