New Graphs of Finite Mutation Type

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Abstract

To a directed graph without loops or 2-cycles, we can associate a skew-symmetric matrix with integer entries. Mutations of such skew-symmetric matrices, and more generally skew-symmetrizable matrices, have been defined in the context of cluster algebras by Fomin and Zelevinsky. The mutation class of a graph Γ is the set of all isomorphism classes of graphs that can be obtained from Γ by a sequence of mutations. A graph is called mutation-finite if its mutation class is finite. Fomin, Shapiro and Thurston constructed mutation-finite graphs from triangulations of oriented bordered surfaces with marked points. We will call such graphs "of geometric type". Besides graphs with 2 vertices, and graphs of geometric type, there are only 9 other "exceptional" mutation classes that are known to be finite. In this paper we introduce 2 new exceptional finite mutation classes.

Cluster algebras were introduced by Fomin and Zelevinsky in [5, 6] to create an algebraic framework for total positivity and canonical bases in semisimple algebraic groups.

An $n \times n$ matrix $B = (b_{i,j})$ is called *skew symmetrizable* if there exists nonzero $d_1, d_2, \ldots, d_n$ such that $d_i b_{i,j} = -d_j b_{j,i}$ for all i, j. An *exchange matrix* is a skew-symmetrizable matrix with integer entries.

A *seed* is a pair $(\mathbf{x}, B)$ where B is an exchange matrix and $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a set of n algebraically independent elements. For any k with $1 \leq k \leq n$ we define another

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seed $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$ as follows. The matrix $B' = (b'_{i,j})$ is given by

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k; \\ b_{i,j} + [b_{i,k}]_+[b_{k,j}]_+ - [-b_{i,k}]_+[-b_{k,j}]_+ & \text{otherwise.} \end{cases}$$

Here, $[z]_+ = \max\{z, 0\}$ denotes the positive part of a real number z. Define

$$\mathbf{x}' = \{x_1, x_2, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n\}$$

where x'_k is given by

$$x'_k = \frac{\prod_{i=1}^n x_i^{[b_{i,k}]_+} + \prod_{i=1}^n x_i^{[-b_{i,k}]_+}}{x_k}.$$

Note that μ_k is an involution. Starting with an initial seed $(\mathbf{x}, B)$ one can construct many seeds by applying sequences of mutations. If $(\mathbf{x}', B')$ is obtained from the initial seed $(\mathbf{x}, B)$ by a sequence of mutations, then $\mathbf{x}'$ is called a cluster, and the elements of $\mathbf{x}'$ are called *cluster variables*. The *cluster algebra* is the commutative subalgebra of $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ generated by all cluster variables. A cluster algebra is called *of finite type* if there are only finitely many seeds that can be obtained from the initial seed by sequences of mutations. Cluster algebras of finite type were classified in [6].

Example 1 (Cluster algebra of type $\mathbf{A}_1$) *If we start with the initial seed* $(\mathbf{x}, B)$ *where* $\mathbf{x} = \{x_1, x_2\}$ *and*

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Using mutations we get

$$\{x_1, x_2\}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \{\frac{1+x_2}{x_1}, x_2\}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \{\frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1, x_2}\}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \{\frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1, x_2}\}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \{\frac{1+x_1}{x_2}, x_1\} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \{x_2, x_1\}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The last seed

$$\{x_2, x_1\}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is considered the same as the initial seed. We just need to exchange x_1 *and* x_2 *(and accordingly swap the 2 rows and swap the 2 columns in the exchange matrix) to get the initial seed.*

A cluster algebra is called *mutation-finite* if only finitely many exchange matrices appear in the seeds. Obviously a cluster algebra of finite type is mutation-finite. But the converse is not true. For example, the exchange matrix

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

gives a cluster algebra that is not of finite type. However, the only exchange matrix that appears is B (and $-B$, but $-B$ is the same as B after swapping the 2 rows and swapping the 2 columns).

In this paper we will only consider exchange matrices that are already skew-symmetric. To a skew-symmetric $n \times n$ matrix $B = (b_{i,j})$ we can associate a directed graph $\Gamma(B)$ as follows. The vertices of the graph are labeled by $1, 2, \ldots, n$. If $b_{i,j} > 0$, draw $b_{i,j}$ arrows from j to i. Any finite directed graph without loops or 2 cycles can be obtained from a skew-symmetric exchange matrix in this way. We can understand mutations in terms of the graph. If $\Gamma = \Gamma(B)$ then $\mu_k\Gamma := \Gamma(\mu_k B)$ is obtained from Γ as follows. Start with Γ. For every incoming arrow $a : i \to k$ at k and every outgoing arrow $b : k \to j$, draw a new composition arrow $ba : i \to j$. Then, revert every arrow that starts or ends at k. The graph now may have 2-cycles. Discard 2-cycles until there are now more 2-cycles left. The resulting graph is $\mu_k\Gamma$. Two graphs are called *mutation-equivalent*, if one is obtained from the other by a sequence of mutations and relabeling of the vertices. The *mutation class* of a graph Γ is the set of all isomorphism classes of graphs that are mutation equivalent to Γ. A graph is mutation-finite if its mutation class is finite.

Convention 2 *In this paper, a subgraph of a directed graph Γ will always mean a* full *subgraph, i.e., for every two vertices x, y in the subgraph, the subgraph also will contain all arrows from x to y.*

1 Known mutation-finite connected graphs

It is easy to see that a graph Γ is mutation-finite if and only if each of its connected components is mutation finite. We will discuss all known examples of graphs of finite mutation type.

1.1 Connected graphs with 2 vertices

Let $\Theta(m)$ be the graph with two vertices $1, 2$ and $m \geq 1$ arrows from 1 to 2. The mutation class of $\Theta(m)$ is just the isomorphism class of $\Theta(m)$ itself. So $\Theta(m)$ is mutation-finite.

$$\Theta(3) : \bullet \Rrightarrow \bullet$$

1.2 Graphs from cluster algebras of finite type.

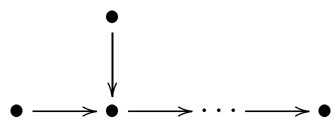
An exchange matrix of a cluster algebra of finite type is mutation finite. The cluster algebras of finite type were classified in [6]. This classification goes parallel to the classification of simple Lie algebras, there are types

$$\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E_8}, \mathbf{F}_4, \mathbf{G}_2.$$

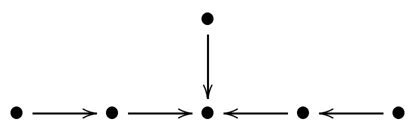
The types with a skew-symmetric exchange graph correspond to the simply laced Dynkin diagrams $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$:

$\mathbf{A}_n$:

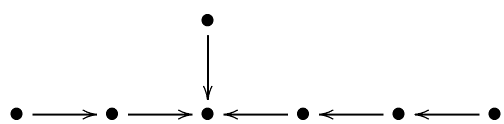
$\mathbf{D}_n$:



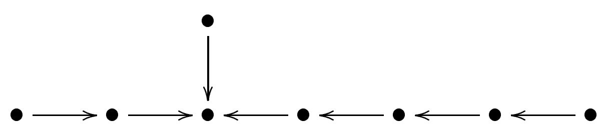
$\mathbf{E}_6$:



$\mathbf{E}_7$:



$\mathbf{E}_8$:

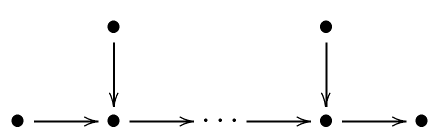


The orientation of the arrows here were chosen somewhat arbitrarily. For each diagram, a different choice of the orientation will still give the same mutation class.

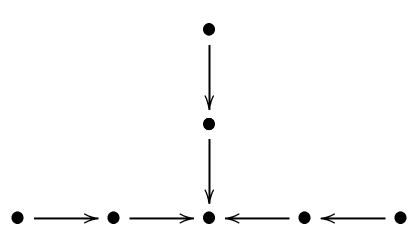
1.3 Graphs from extended Dynkin diagrams

In [2] it was shown that a connected directed graph without oriented cycles is of finite mutation type if and only if it has at 2 vertices (the graphs $\Theta(m)$, $m \geq 1$) or the underlying undirected graph is an extended Dynkin diagrams. The type $\mathbf{D}$ and $\mathbf{E}$ extended Dynkin diagrams give rise to the following finite mutation classes:

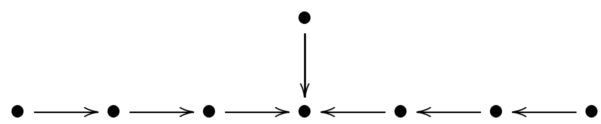
$\widehat{\mathbf{D}}_n$:



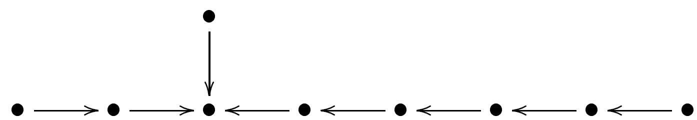
$\widehat{\mathbf{E}}_6$:



$\widehat{\mathbf{E}}_7$:



$\widehat{\mathbf{E}}_8$:

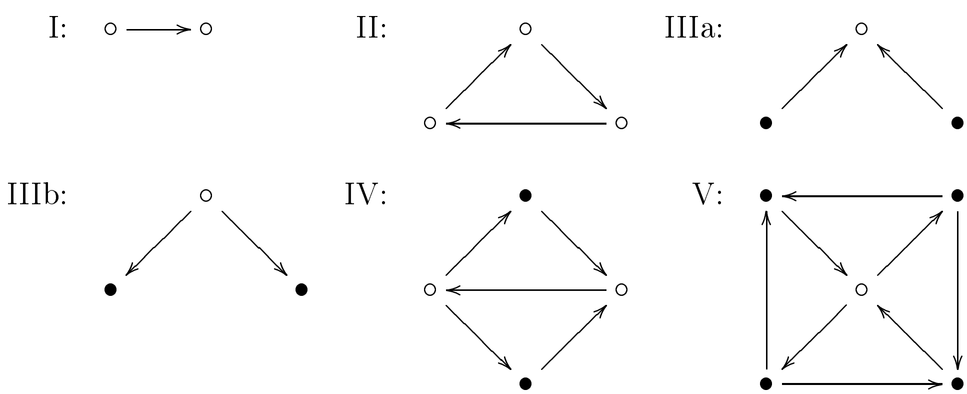


Again, for these types, a different choice for the orientations of the arrows still give the same mutation class. The diagram for $\widehat{\mathbf{A}}_n$ is an $(n+1)$-gon. If all arrows go clockwise or all arrows go counterclockwise, then we get the mutation class of $\mathbf{D}_n$. Let $\widehat{\mathbf{A}}_{p,q}$ be the mutation class of the graph where p arrows go counterclockwise and q arrows go clockwise, where $p \geq q \geq 1$. For the mutation class it does not matter which arrows are chosen to be counterclockwise and with ones are chosen counterclockwise.

1.4 Graphs coming from triangulations of surfaces

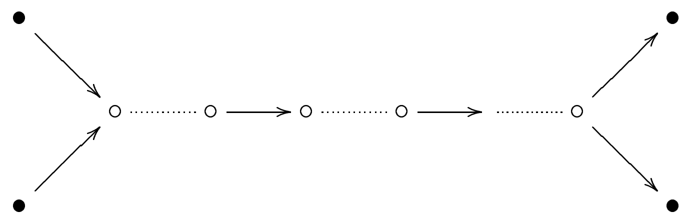
In [4] the authors construct cluster algebras from bordered oriented surfaces with marked points. These cluster algebras are always of finite mutation type. The exchange matrices for these types are skew-symmetric. The mutation-finite graphs that come from oriented bordered surfaces with marked points will be called *of geometric type*. In §13 of that paper, the authors give a description of the graphs of geometric type.

A *block* is one of the diagrams below:

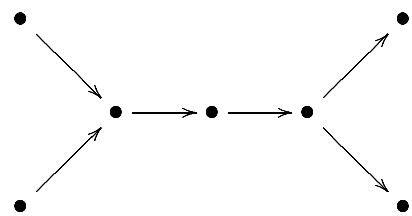


Start with a disjoint union of blocks (every type may appear several times) and choose a partial matching of the open vertices ($\circ$). No vertex should be matched to a vertex of the same block. Then construct a new graph by identifying the vertices that are matched to each other. If in the resulting graph there are two vertices x and y with an arrow from x to y and an arrow from y to x, then we omit both arrows (they cancel each other out). A graph constructed in this way is called *block decomposable*. Fomin, Shapiro and Thurston prove in [4, §13] that a graph is block decomposable if and only if the graph is of geometric type.

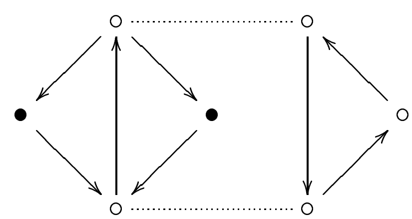
For example



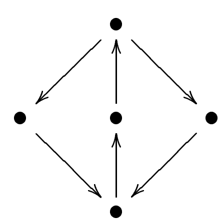
gives



of type $\widehat{\mathbf{D}}_6$. It is easy to see that all graphs of type $\mathbf{A}_n, \mathbf{D}_n, \widehat{\mathbf{A}}_{p,q}, \widehat{\mathbf{D}}_n$ are block decomposable. The partial matching



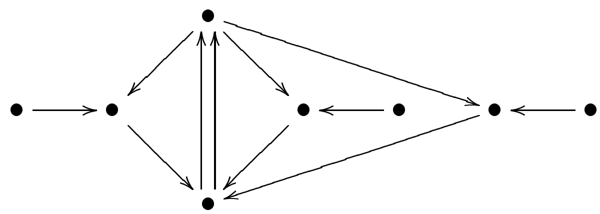
yields the block decomposable graph



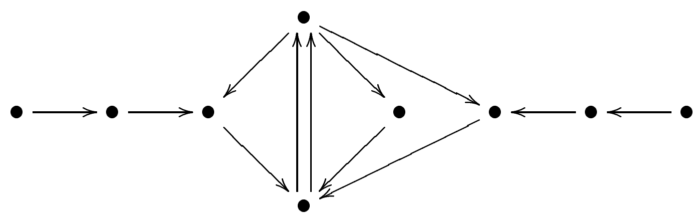
1.5 Graphs of extended affine types

The following graphs are also of finite mutation type:

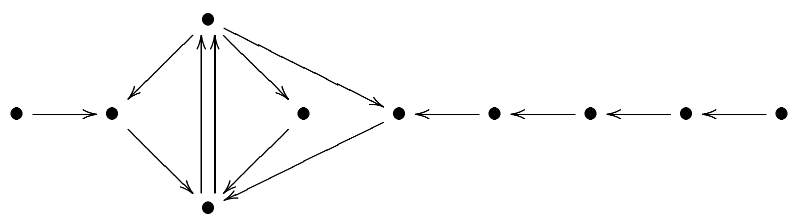
$\mathbf{E}_6^{(1,1)}$:



$\mathbf{E}_7^{(1,1)}$:



$\mathbf{E}_8^{(1,1)}$:



These graphs are orientations of the Dynkin diagrams of extended affine roots systems first described by Saito (see [10, Table 1]). The connection between extended affine root systems and cluster combinatorics was first noticed by Geiss, Leclerc, and Schroër in [8]. It was shown using that these graphs are of finite mutation type by an exhaustive computer search using the `Java` applet for matrix mutations written by Bernhard Keller ([9]) and Lauren Williams.

1.6 Summary

All the known quivers of finite mutation type can be summarized as follows:

1. graphs of geometric type,
2. graphs with 2 vertices,
3. graphs in the 9 exceptional mutation classes

$$\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \widehat{\mathbf{E}}_6, \widehat{\mathbf{E}}_7, \widehat{\mathbf{E}}_8, \mathbf{E}_6^{(1,1)}, \mathbf{E}_7^{(1,1)}, \mathbf{E}_8^{(1,1)}.$$

Fomin, Shapiro and Thurston asked to following question (see [4, Problem 12.10])[1].

Problem 3 *Are these* all *connected graphs of finite mutation type? If not, are there only finitely many exceptional finite mutation classes?*

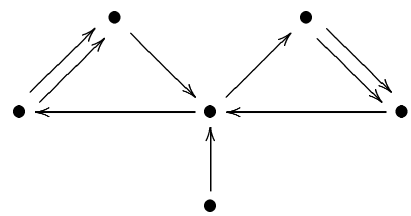
In the next section, we will introduce 2 new mutation classes of finite type.

[1] In the statement of Problem 12.10 in [4], the authors accidentally wrote $n \geq 2$ instead of $n \geq 3$.

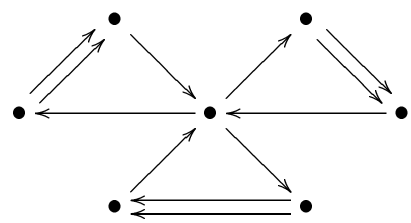
2 New exceptional graphs of finite mutation-type

Proposition 4 *The following two graphs are of finite mutation type:*

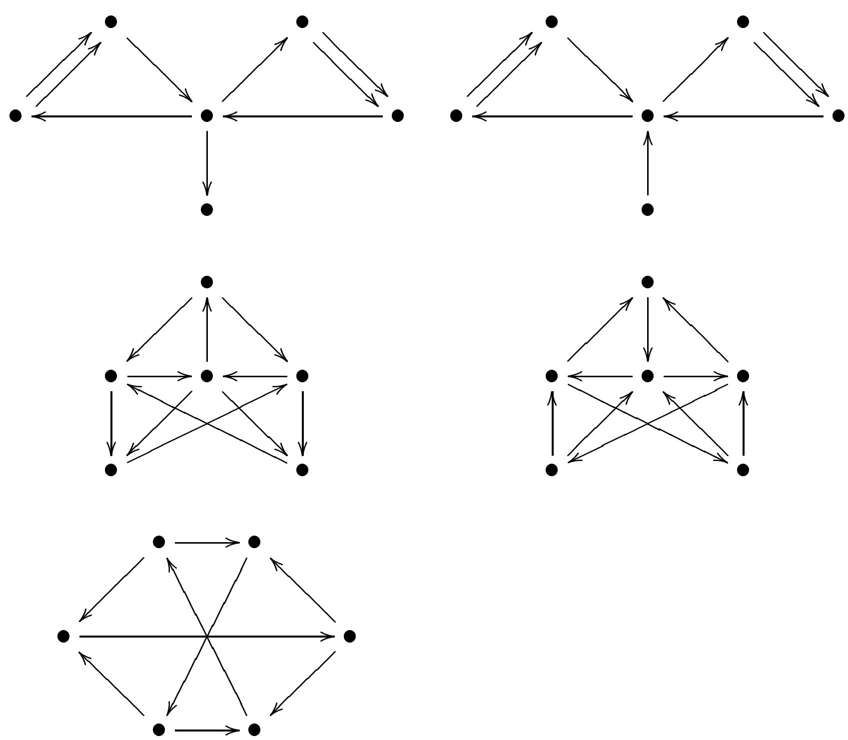
$\mathbf{X}_6$:



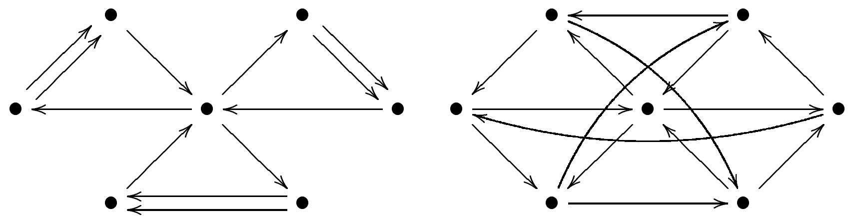
$\mathbf{X}_7$:



Proof. This is easy to verify by hand or by using the applet [9]. The mutation classes for $\mathbf{X}_6$ and $\mathbf{X}_7$ are surprisingly small. The mutation class of X_6 consists of the following 5 graphs:



The mutation class of $\mathbf{X}_7$ consists of the following 2 graphs:



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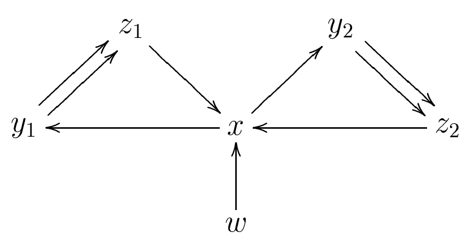
Corollary 5 *The graphs* $\mathbf{X}_6$ *and* $\mathbf{X}_7$ *are not mutation-equivalent to*

$$\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \widehat{\mathbf{E}}_6, \widehat{\mathbf{E}}_7, \widehat{\mathbf{E}}_8, \mathbf{E}_6^{(1,1)}, \mathbf{E}_7^{(1,1)}, \mathbf{E}_8^{(1,1)}.$$

Proof. The reader easily verifies that none of these 9 graphs are in the mutation classes of $\mathbf{X}_6$ or $\mathbf{X}_7$. •

Proposition 6 *The graphs* $\mathbf{X}_6$ *and* $\mathbf{X}_7$ *are not block decomposable. In particular, these graphs do not come from oriented surfaces with marked points.*

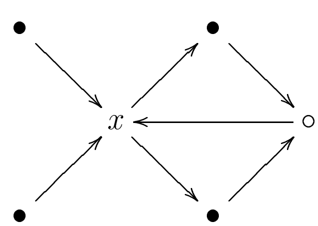
Proof. Suppose that $\mathbf{X}_6$ is block decomposable. None of the blocks will be of type V, since the block V contains a 4-cycle which will not vanish after the matching, and $\mathbf{X}_6$ does not contain a 4-cycle. We label the vertices of $\mathbf{X}_6$ as follows:



(1)

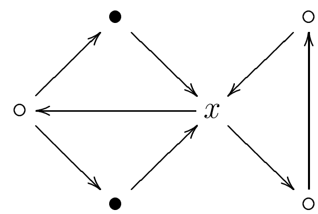
At vertex x there are 2 arrows going out and 3 arrows coming in. To form a graph with this property from the blocks, we must either glue blocks II and IV along x, or glue blocks IIIa and IV along x.

If we glue blocks IIIa and IV we get the following graph:



where the open vertex (◦) may be matched further with other blocks. Even after further matching, x will have at least 2 neighbors which are only connected to x.

If blocks II and IV are matched to form a vertex x, then we get the following graph



(2)

Here the open vertices can be matched further. However, they cannot be matched among themselves, because this would change the number of incoming and outgoing arrows at x. In $\mathbf{X}_6$, x has incoming arrows from w, z_1, z_2. So in (2), one of the vertices marked with $\bullet$ has to correspond to z_1 or z_2. But it is clear that even after further matching, the vertices marked with $\bullet$ will only have 1 incoming arrow. Contradiction. This shows that $\mathbf{X}_6$ is not block decomposable. Therefore, $\mathbf{X}_6$ does not come from an a triangulation of an oriented surface with marked points.

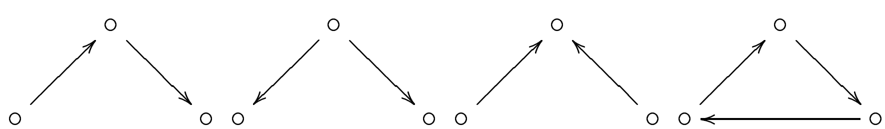
Since $\mathbf{X}_6$ is a subgraph of $\mathbf{X}_7$, $\mathbf{X}_7$ does not come from a triangulation of an oriented surface with marked points either. $\bullet$

3 Mutation-finite graphs containing $\mathbf{X}_6$ or $\mathbf{X}_7$

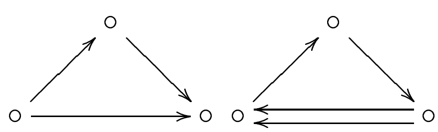
The following result was proven in [1]. We include the short proof for the reader's convenience.

Theorem 7 *The finite mutation classes of connected quivers with 3 vertices are:*

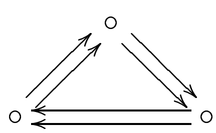
$\mathbf{A}_3$:



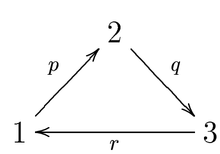
$\widehat{\mathbf{A}}_2$:



$\mathbf{Z}_3$:



Proof. Suppose that Γ is a connected graph of finite mutation type with 3 vertices. Assume that Γ, among all the graphs in its mutation class, has the largest possible number of arrows. Without loss of generality we may assume that Γ is of the form



(3)

where $p, q, r \geq 0$ denote the number of arrows. If Γ is not of the form (3), then it is of the form

$$\begin{tikzcd} & 2 \arrow[dr, "q"] & \\ 1 \arrow[ur, "p"] \arrow[rr, "r"'] & & 3 \end{tikzcd} \quad . \tag{4}$$

and mutation at vertex 2 will not decrease the number of arrows, and we obtain a graph of the form (3). Without loss of generality we may assume that

$$p, q \geq r. \tag{5}$$

In particular,

$$p, q \geq 1 \tag{6}$$

otherwise the graph would not be connected. After mutation at vertex 2, we get

$$\begin{tikzcd} & 2 \arrow[dl, "p"'] & \\ 1 \arrow[rr, "pq-r"'] & & 3 \arrow[ul, "q"'] \end{tikzcd} \tag{7}$$

Since Γ had the maximal number of arrows, we have $pq - r \leq r$, so

$$pq \leq 2r. \tag{8}$$

From $r^2 \leq pq \leq 2r$ follows that $r = 0, 1, 2$. If $r = 0$, then $pq = 0$ by (8) which contradicts (6). If $r = 1$, then $pq = 1$ or $pq = 2$ by (8). If $pq = 1$, then $p = q = 1$ which yields type $\mathbf{A}_3$. If $pq = 2$ then $p = 2$, $q = 1$ or $p = 1$, $q = 2$. In either case we get type $\widehat{\mathbf{A}}_2$. If $r = 2$, then (5) and (8) imply that $p = q = 2$ and we obtain type $\mathbf{Z}_3$. $\bullet$

Corollary 8 *If Γ is a graph of finite mutation type with ≥ 3 vertices Then the number of arrows between any 2 vertices is at most 2.*

Proof. Suppose x and y are vertices of Γ with $p \geq 1$ arrows from x to y. Since Γ is connected, there exists a vertex z that is connected to x or y.The subgraph with vertices x, y, z is also of finite mutation type. From the classification in Theorem 7 it is clear that $p \leq 2$. $\bullet$

Definition 9 *An* obstructive sequence *for a graph Γ is a sequence of vertices $x_1, \ldots, x_\ell$ such that the mutated graph*

$$\mu_{x_\ell} \cdots \mu_{x_2} \mu_{x_1} \Gamma$$

has two vertices with at least 3 arrows between them.

By Corollary 8, a graph for which an obstructive sequence exists cannot be of finite mutation type.

Lemma 10 *If Γ is a mutation-finite connected graph with ≥ 4 vertices, then Γ cannot contain $\mathbf{Z}_3$ as a subgraph.*

Proof. Suppose Γ is a mutation-finite connected graph with ≥ 4 vertices containing $\mathbf{Z}_3$. Then Γ has a mutation-finite connected subgraph with exactly 4 vertices containing $\mathbf{Z}_3$. Without loss of generality we may assume that Γ has 4 vertices. We label the vertices of $\mathbf{Z}_3$ as follows

$$\begin{tikzcd} & x_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ x_1 \arrow[ur, shift left] \arrow[ur, shift right] & & x_3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

Let y be the 4-th vertex of Γ. Now y is connected to at least one of the vertices of $\mathbf{Z}_3$, say to x_1. Because the subgraph with vertices $\{y, x_1, x_2\}$ is of finite mutation type, all arrows must go from y to x_1 and not in the opposite direction by Theorem 7. But because the subgraph with vertices $\{y, x_1, x_3\}$ is of finite mutation type, all arrows must go from x_1 to y. Contradiction. $\bullet$

Corollary 11 *If Γ is a mutation-finite graph with ≥ 4 vertices, then every connected subgraph with 3 vertices must be of type $\mathbf{A}_3$ or $\widehat{\mathbf{A}}_2$.*

Theorem 12 *The only connected mutation-finite quiver with 7 vertices containing $\mathbf{X}_6$ is $\mathbf{X}_7$.*

Proof. Suppose that Γ is a mutation finite quiver containing $\mathbf{X}_6$. We label the vertices of $\mathbf{X}_6$ as shown in (1), and denote the other vertex by u. Since Γ is connected, u is connected to at least 1 other vertex.

Case 1: Suppose that u is connected to 1 vertex of $\mathbf{X}_6$. If u is connected to y_1 or z_1 then the subgraph with vertices $\{u, y_1, z_1\}$ is not of type $\mathbf{A}_3$ or $\widehat{A}_2$, contradicting Corollary 11. Similarly u is not connected to y_2 or z_2.

Suppose u is connected to x. After mutation at u we may assume that arrows go from u to x. From Corollary 11 applied to the subgraph with vertices $\{u, x, w\}$ follows that there can be only 1 arrow.

Now w, x, y_1, z_1, w, u is a obstructive sequence.

Suppose that u is connected to w. Without loss of generality we may assume that arrows go from u to w. By Corollary 11 applied to the subgraph with vertices $\{u, w, x\}$ there is at most 1 arrow. Now $x, w, y_1, y_2, z_1, z_2, x, w, y_2$, is a obstructive sequence.

attached to X_6 by only one set of arrows.

Case 2: Suppose that u is connected to 2 vertices. If u is connected to y_1 or z_1, then the only possibility is that there is one arrow from u_1 to y_1 and one arrow from z_1 to u_1. An obstructive sequence is $u, x, y_1, y_2.x, w, x, z_2, y_1$. Similarly, u cannot be connected to y_2 or z_2.

Therefore, u must be connected to w and x. The only cases that avoid a connected subgraph with 3 vertices not of type $\mathbf{A}_3$ or $\widehat{\mathbf{A}}_2$ are:

$$(\mathbf{a}): \begin{tikzcd} & u \arrow[dr] & \\ x \arrow[ur] & & w \arrow[ll] \end{tikzcd} \qquad (\mathbf{b}): \begin{tikzcd} & u \arrow[dl] \arrow[dr] & \\ x & & w \arrow[ll] \end{tikzcd} \qquad (\mathbf{c}): \begin{tikzcd} & u & \\ x \arrow[ur] & & w \arrow[ul] \arrow[ll] \end{tikzcd} \qquad (\mathbf{d}): \begin{tikzcd} & u \arrow[dr, shift left] \arrow[dr, shift right] & \\ x \arrow[ur] & & w \arrow[ll] \end{tikzcd}$$

Case (a) reduces to Case 1 after mutation at vertex u. Cases (b) and (c) give isomorphic graphs. Case (b) has an obstructive sequence w, x, u. Case (d) gives us the graph $\mathbf{X}_7$.

Case 3: 3 attachments

Vertex u must be attached to either vertices y_1 and z_1 or vertices y_2 and z_2. Without loss of generality, we may assume that u is connected to both y_1 and z_1. There is one arrow from u to y_1 and one arrow from z_1 to u. The vertex u is also connected to either w or x. There are 4 cases:

$$(\mathbf{a}): \begin{tikzcd} u \arrow[d] \\ x \end{tikzcd} \quad (\mathbf{b}): \begin{tikzcd} u \arrow[d] \\ w \end{tikzcd} \quad (\mathbf{c}): \begin{tikzcd} u \\ x \arrow[u] \end{tikzcd} \quad (\mathbf{d}) \begin{tikzcd} u \\ w \arrow[u] \end{tikzcd}$$

Case (a) has the obstructive sequence u, x, z_1. Case (b) has the obstructive sequence u, x, z_1, w, x. In case (c) we have the obstructive sequence x, w, u, y_1. Case (d) has the obstructive sequence u, x, z_2, y_1, w.

Case 4: Suppose that u is connected to 4 vertices. If u is connected to y_1, z_1, y_2, z_2 then there is an arrow from u to y_1 and to y_2 and arrows from z_1 and z_2 to u. An obstructive sequence is u, x, w, y_2, z_1.

Otherwise, u must be connected to either y_1 and z_1, or to y_2 and z_2, but not both. Without loss of generality we may assume that u is connected to y_1 and z_1. Now u is also connected to w and x. The only cases that avoid a connected subgraph with 3 vertices not of type $\mathbf{A}_3$ or $\widehat{\mathbf{A}}_2$ are:

$$(\mathbf{a}): \begin{tikzcd} & u \arrow[dr] & \\ x \arrow[ur] & & w \arrow[ll] \end{tikzcd} \quad (\mathbf{b}): \begin{tikzcd} & u \arrow[dl] \arrow[dr] & \\ x & & w \arrow[ll] \end{tikzcd} \quad (\mathbf{c}): \begin{tikzcd} & u & \\ x \arrow[ur] & & w \arrow[ll] \arrow[ul] \end{tikzcd} \quad (\mathbf{d}): \begin{tikzcd} & u \arrow[dl] & \\ x & & w \arrow[ll] \arrow[ul] \end{tikzcd}$$

Case (a) has the obstructive sequence u, x, y_1. Case (b) has the obstructive sequence u, x, z_1. In case (c) we have the obstructive sequence u, x. And case (d) is mutation equivalent to case (c) via the mutation at w.

Case 5: Suppose that u is connected to 5 of the vertices of $\mathbf{X}_6$. Then u must be connected to y_1, z_1, y_2, z_2, with arrows going from u to y_1 and y_2 and arrows going from z_1 and z_2 to u. Now u must be connected to either x or w. There are 4 subcases:

$$(\mathbf{a}): \begin{tikzcd} u \arrow[d] \\ x \end{tikzcd} \quad (\mathbf{b}): \begin{tikzcd} u \arrow[d] \\ w \end{tikzcd} \quad (\mathbf{c}): \begin{tikzcd} u \\ x \arrow[u] \end{tikzcd} \quad (\mathbf{d}) \begin{tikzcd} u \\ w \arrow[u] \end{tikzcd}$$

Case (a) has the obstructive sequence u, x. Case (b) has the obstructive sequence u, x, w. In case (c) we have the obstructive sequence u, x. Case (d) has the obstructive sequence u, x, y_1, y_2.

Case 6: The vertex u is connected to all 6 vertices of $\mathbf{X}_6$. There must be arrows from u to y_1 and y_2 and arrows from z_1 and z_2 to u. The only possibilities to connect u to x and w that avoid a connected subgraph with 3 vertices not of type $\mathbf{A}_3$ or $\widehat{\mathbf{A}}_2$ are:

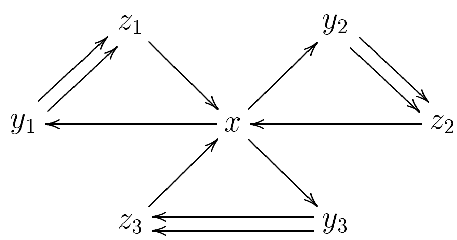
$$(\mathbf{a}): \begin{tikzcd} & u \arrow[dr] & \\ x \arrow[ur] & & w \arrow[ll] \end{tikzcd} \quad (\mathbf{b}): \begin{tikzcd} & u \arrow[dl] \arrow[dr] & \\ x & & w \arrow[ll] \end{tikzcd} \quad (\mathbf{c}): \begin{tikzcd} & u & \\ x \arrow[ur] & & w \arrow[ll] \arrow[ul] \end{tikzcd} \quad (\mathbf{d}): \begin{tikzcd} & u \arrow[dl] & \\ x & & w \arrow[ll] \arrow[ul] \end{tikzcd}$$

Case (a) has the obstructive sequence x, y_1, y_2. Case (b) has the obstructive sequence x, z_1, z_2. Case 3 has the obstructive sequence x, y_1, y_2. Case 4 has the obstructive sequence x, z_1, z_2.

Therefore the only connected mutation-finite quiver with 7 vertices containing X_6 is X_7. ●

Theorem 13 *There is no connected mutation-finite quiver with ≥ 8 vertices containing $\mathbf{X}_7$.*

Proof. Suppose that Γ is a connected graph with ≥ 8 vertices containing $\mathbf{X}_7$. We will show that this will lead to a contradiction. The graph Γ contains a connected subgraph with exactly 8 vertices which contains $\mathbf{X}_7$. So without loss of generality we may assume that Γ has exactly 8 vertices. Let us label the vertices of $\mathbf{X}_7$ as follows:



Suppose that u is the 8-th vertex of Γ. Because of the symmetry, we may assume, without loss of generality that u is connected to x, y_1 or z_1. The subgraph with vertices $\{x, y_1, z_1, y_2, z_2, z_3, u\}$ is connected and contains $\mathbf{X}_7$. By Theorem 12 this graph must be isomorphic to $\mathbf{X}_7$. This means that there must be 2 arrows from u to z_3. Now the subgraph with vertices y_3, z_3, u cannot be of finite mutation type because of Theorem 7. ●

Corollary 14 *No graph of one of the 9 exceptional types contains X_6 or X_7 as a subgraph.*

4 Conclusion

We have exhibited two graphs which are of finite mutation type, but which are not of geometric type or isomorphic to the 9 known exceptional cases. It is natural to ask whether there are any more exceptional graphs of finite mutation type.

Problem 15 *Is it true that every finite mutation class of graphs with ≥ 3 vertices which is not of geometric type is one of the following 11 mutation classes:*

$$\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \widehat{\mathbf{E}}_6, \widehat{\mathbf{E}}_7, \widehat{\mathbf{E}}_8, \mathbf{E}_6^{(1,1)}, \mathbf{E}_7^{(1,1)}, \mathbf{E}_8^{(1,1)}, \mathbf{X}_6, \mathbf{X}_7.$$

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