Dihedral f-Tilings of the Sphere by Equilateral and Scalene Triangles - III

A. M. d'Azevedo Breda*
Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal
ambreda@ua.pt

Patrícia S. Ribeiro*
Department of Mathematics
E.S.T. Setúbal
2910-761 Setúbal, Portugal
pribeiro@est.ips.pt

Altino F. Santos†
Department of Mathematics
U.T.A.D.
5001-801 Vila Real, Portugal
afolgado@utad.pt

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Abstract

The study of spherical dihedral f-tilings by equilateral and isosceles triangles was introduced in [3]. Taking as prototiles equilateral and scalene triangles, we are faced with three possible ways of adjacency. In [4] and [5] two of these possibilities were studied. Here, we complete this study, describing the f-tilings related to the remaining case of adjacency, including their symmetry groups. A table summarizing the results concerning all dihedral f-tilings by equilateral and scalene triangles is given in **Table 2**.

Keywords: dihedral f-tilings, combinatorial properties, symmetry groups

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1 Introduction

Dihedral spherical folding tilings or *dihedral f-tilings* for short, are edge-to-edge decompositions of the sphere by geodesic polygons, such that all vertices are of even valency, the sums of alternate angles around each vertex are π and every tile is congruent to one of two fixed sets X and Y (prototiles).

We shall denote by $\Omega(X,Y)$ the set, up to isomorphism, of all dihedral f-tilings of S^2 whose prototiles are X and Y.

The classification of all dihedral spherical folding tilings by rhombi and triangles was obtained in 2005, [7]. However the analogous study considering two triangular (non-isomorphic) prototiles, T_1 and T_2 is not yet completed. This is not surprising, since it is much harder.

The case corresponding to prototiles given by an equilateral and an isosceles triangle was already described in [3].

When the prototiles are an equilateral and a scalene triangle, there are three distinct possibilities of adjacency, as shown in Figure 1.

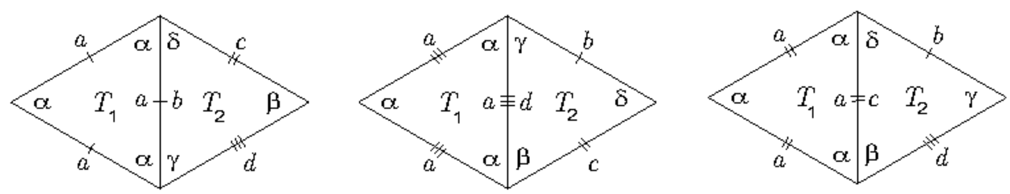

Figure 1: Distinct cases of adjacency.

We have already studied the cases corresponding to adjacency of Type I and II, see [4] and [5]. An interesting fact is that any tiling with adjacency of Type I or Type II can be seen as a subdivision of the sphere in $2n, n \geq 2$ lunes with a pattern whose orbit under the action of a specific group covers the all sphere. Here, our interest is focused in spherical triangular dihedral f-tilings with adjacency of type III. As we shall see in this case we will find two families of tilings, $\mathcal{E}_\alpha$ and $\mathcal{G}^k$, with the same particularity, and four apparent sporadic tilings ($\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, $\mathcal{L}$). However, these tilings can be seen, respectively, as new members of the following families (described in [5]) $\mathcal{F}_p$ and $\mathcal{D}_p$ allowing p to be 3, in both cases, and $\mathcal{E}^m$ allowing m to be 3 or 4.

From now on, T_1 denotes an equilateral spherical triangle of angle α $\left(\alpha > \dfrac{\pi}{3}\right)$ and side a and T_2 a scalene spherical triangle of angles δ, γ, β, with the order relation $\delta < \gamma < \beta$ $(\delta + \gamma + \beta > \pi)$ and with sides b (opposite to β), c (opposite to γ) and d (opposite to δ). The type III edge-adjacency condition can be analytically described by the equation

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\gamma + \cos\delta\cos\beta}{\sin\delta\sin\beta} \tag{1.1}$$

In order to get any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we find it useful to start by considering one of its *representations*, beginning with a vertex common to an equilateral triangle and a scalene triangle in adjacent positions. In the diagrams that follows, it is convenient to label the tiles according to the following procedures:

(i) The tiles by which we begin the local configuration of a tiling $\tau \in \Omega(T_1, T_2)$ are labelled by 1 and 2, respectively;

(ii) For $j \geq 2$, the presence of a tile j as shown can be deduced from the configuration of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete local configuration of a f-tiling (except in the cases indicated).

2 Triangular Dihedral F-Tilings with Adjacency of Type III

Starting a local configuration of $\tau \in \Omega(T_1, T_2)$ with two adjacent cells congruent to T_1 and T_2 respectively (see Figure 2), a choice for angle $x \in \{\gamma, \beta\}$ must be made. We shall consider and study separately each one of the choices $\alpha + x = \pi$ and $\alpha + x < \pi$, $\ x \in \{\gamma, \beta\}$.

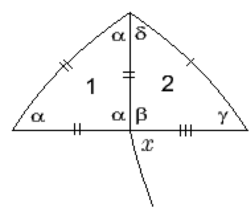

Figure 2: Local configuration.

With the above terminology one has:

Proposition 2.1. *If $x = \gamma$ and $\alpha + x = \pi$, then $\Omega(T_1, T_2) \neq \emptyset$ if and only if $\beta + \delta = \pi$.*

Proof. Suppose $x = \gamma$ and that $\alpha + x = \pi$. We may add some new cells to the configuration started in Figure 2 and get the one illustrated in Figure 3, with $\theta_1 \in \{\beta, \gamma\}$.

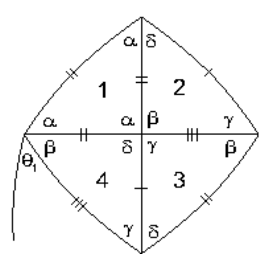

Figure 3: Local configuration.

If $\theta_1 = \beta$, then $\alpha + \theta_1 \leq \pi$, but since $\alpha + \gamma = \pi$ and $\gamma < \beta$, one has $\alpha + \theta_1 > \pi$, which is a contradiction.

If $\theta_1 = \gamma$, we can expand the configuration in Figure 3 and obtain a global representation of a tiling $\tau_\alpha \in \Omega(T_1, T_2)$ as is shown in Figure 4. This family of tilings is composed by two equilateral and six scalene triangles and is denoted by $\mathcal{E}_\alpha$.

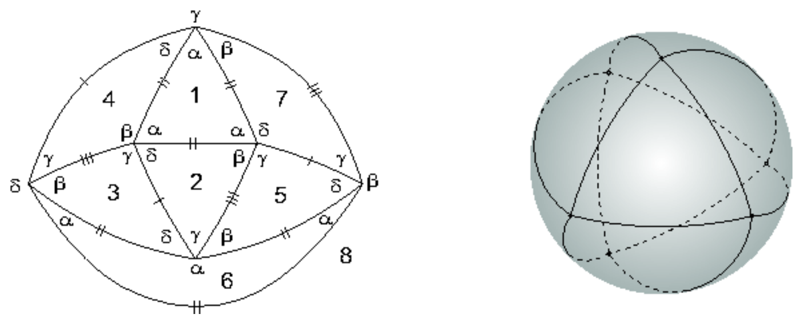

Figure 4: 2D and 3D representation of $\mathcal{E}_\alpha$.

By the adjacency condition (1.1), the condition $\alpha + \gamma = \pi = \beta + \delta$ and the order relation between the angles, we may conclude that $\beta > \alpha > \frac{\pi}{2}$. $\square$

Proposition 2.2. *If $x = \gamma$ and $\alpha + x < \pi$, then $\Omega(T_1, T_2) \neq \emptyset$ if and only if $\alpha + \gamma + k\delta = \pi$, $\beta + \gamma = \pi$ and $\beta + (k+1)\delta = \pi$, for some $k \geq 1$. In this situation, for each $k \geq 1$, there is a single f-tiling denoted by $\mathcal{G}^k$.*

Proof. Suppose that $\alpha + x < \pi$, with $x = \gamma$ (see Figure 2). We are led to the configuration illustrated in Figure 5 and a decision must be taken about the angle labelled $\theta_2 \in \{\gamma, \delta\}$:

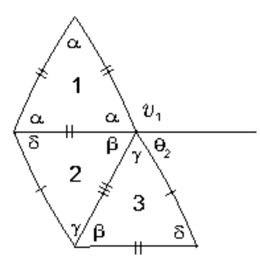

Figure 5: Local configuration.

1. If $\theta_2 = \gamma$, then $\beta + \theta_2 < \pi$ and since $\gamma < \beta$, we get $\delta < \gamma < \frac{\pi}{2}$. Consequently $\alpha \geq \frac{\pi}{2}$ or $\beta \geq \frac{\pi}{2}$, since vertices of valency four must exist (see [6]).

1.1 If $\alpha \geq \frac{\pi}{2}$, from the adjacency condition (1.1), $\beta > \frac{\pi}{2}$ and so the sum $\beta + \theta_2 + \lambda$ does not satisfy the angle folding relation for each $\lambda \in \{\alpha, \delta, \gamma, \beta\}$.

1.2 If $\beta \geq \frac{\pi}{2}$, the configuration in Figure 5 ends up in a contradiction since, in order to satisfy the angle folding relation, the sum of alternate angles containing β and $\theta_2 = \gamma$

must be $\beta + \gamma + \alpha = \pi$ and the other sum is $\alpha + 2\gamma = \pi$ leading to $\gamma = \beta$, which is impossible.

2. Suppose now that $\theta_2 = \delta$. As $\alpha + \gamma < \pi$, then $\beta + \theta_2 < \pi$ and consequently $\delta < \frac{\pi}{2}$. Additionally, $\gamma < \frac{\pi}{2}$, otherwise $\beta > \gamma \geq \frac{\pi}{2}$, $\alpha \leq \frac{\pi}{2}$ and the adjacency condition (1.1) is not fulfilled. Accordingly, $\delta < \gamma < \frac{\pi}{2}$ and vertices of valency four occur if and only if $\alpha \geq \frac{\pi}{2}$ or $\beta \geq \frac{\pi}{2}$.

2.1 If $\alpha = \frac{\pi}{2}$, by the adjacency condition (1.1), $\beta > \frac{\pi}{2}$. We may add some new cells to the configuration shown in Figure 5, obtaining the following one:

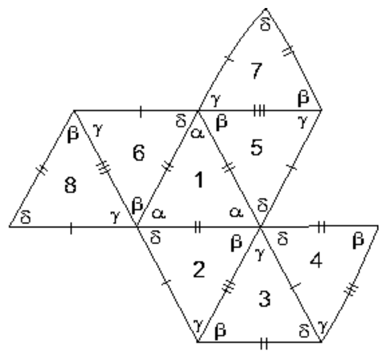

Figure 6: Local configuration.

The sum containing alternate angles β and δ must satisfy $\beta + k\delta = \pi$, for some $k > 1$ and taking into account the edge compatibility, we conclude that the other sum is $\alpha + \gamma + (k-1)\delta = \pi$. Therefore, $\beta + \delta = \frac{\pi}{2} + \gamma$ and by the adjacency condition (1.1),

$$\begin{aligned} \cos\gamma = -\cos\beta\cos\delta &\Leftrightarrow \sin(\beta+\delta) = -\cos\beta\cos\delta \\ &\Leftrightarrow \sin(\pi - k\delta + \delta) = \cos(k\delta)\cos\delta \\ &\Leftrightarrow -\sin(k\delta - \delta) = \cos(k\delta)\cos\delta. \end{aligned}$$

Taking into account that $k\delta < \frac{\pi}{2}$, then $\sin(k\delta - \delta) < 0$ and so $k\delta - \delta > \pi$, which is an impossibility.

2.2 If $\alpha > \frac{\pi}{2}$, from the adjacency condition (1.1), we conclude that $\delta < \gamma < \frac{\pi}{2} < \beta$. Since $\alpha + \gamma < \pi$, $\alpha + \delta < \pi$ and $\beta + \delta < \pi$, vertices of valency four are surrounded by alternate angles β and γ, which violates the adjacency condition.

2.3 If $\beta = \frac{\pi}{2}$, then $\alpha < \frac{\pi}{2}$ and vertices of valency four are surrounded exclusively by angles β.

Since $\gamma + \delta > \frac{\pi}{2}$ and $\gamma > \frac{\pi}{4}$, the angular sum containing α and γ must be $2\alpha + \gamma = \pi$, $\alpha + 2\gamma = \pi$ or $\alpha + \gamma + p\delta = \pi$, for some $p \geq 1$. We shall study each case separately.

2.3.1 The vertices of valency six in which one of the sums of alternate angles is $2\alpha+\gamma=\pi$ are surrounded by the angular sequence $(\alpha,\alpha,\alpha,\beta,\gamma,\delta)$. By the adjacency condition, we conclude that $\alpha=\dfrac{\pi}{3}$ or approximately $128,17^\circ$, which is impossible in both cases.
2.3.2 In case $\alpha+2\gamma=\pi$, the angle arrangement around vertex v_1, in Figure 5 (valency six) is impossible since $\theta_2=\delta$.

2.3.3 Assume now that $\alpha+\gamma+p\delta=\pi$, for some $p\geq 1$. Extending the configuration in Figure 5, we get the one below:

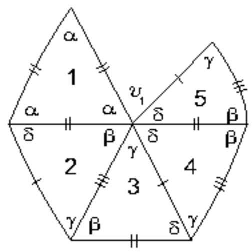

Figure 7: Local configuration.

The sum of the alternate angles, at vertex v_1, containing β and δ must satisfy $\beta+t\delta=\pi$, for some $t>1$. Then, $\beta+t\delta=\pi=\alpha+\gamma+(t-1)\delta=\pi$ and so $\beta+\delta=\alpha+\gamma$. Consequently, $\delta>\dfrac{\pi}{12}$ and $\delta=\dfrac{\pi}{2t}$, $t=2,3,4,5$. By the adjacency condition (1.1), one has

$$-\cos(\gamma+(t-1)\delta)\sin\delta=\cos\gamma\,(1+\cos(\gamma+(t-1)\delta))$$

and for $t=2,3,4,5$ we get, respectively, $\gamma\approx 66.26^\circ$, $\gamma=\dfrac{\pi}{3}$, $\gamma\approx 57,98^\circ$, $\gamma\approx 57.44^\circ$ and $\alpha\approx 68.74^\circ$, $\alpha=\dfrac{\pi}{3}$, $\alpha\approx 54.52^\circ$, $\alpha\approx 50.56^\circ$. Taking into account that $\alpha>\dfrac{\pi}{3}$, then $t=2$. However, extending the configuration in Figure 7, we get a vertex surrounded by three consecutive angles γ, whose sum $2\gamma+\mu$ violates the angle folding relation, where μ denotes a sum of angles containing α,δ,γ or β (see Figure 8).

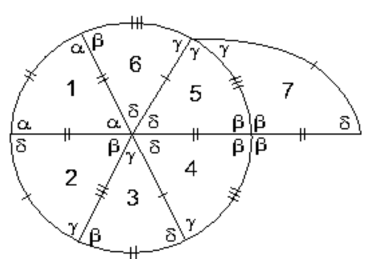

Figure 8: Local configuration.

2.4 Consider $\beta>\dfrac{\pi}{2}$. If $\alpha>\dfrac{\pi}{2}$, the vertices of valency four are surrounded by alternate angles β and γ. But, since $\beta+\delta<\pi$, $\alpha+\delta<\alpha+\gamma<\pi$, the sum $\beta+\gamma=\pi$ violates the adjacency condition (1.1) and so $\alpha\leq\dfrac{\pi}{2}$.

2.4.1 If $\alpha = \frac{\pi}{2}$, then $\beta + \gamma \neq \pi$, otherwise, by the adjacency condition (1.1) $\delta = 0$. The configuration started in Figure 5, with $\theta_2 = \delta$, extends to the one shown in the next figure.

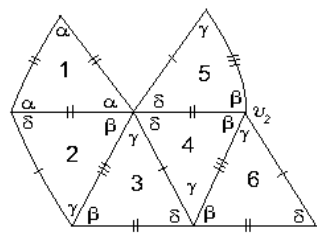

Figure 9: Local configuration.

Looking at vertex labelled v_2, we observe that the sum containing the alternate angles β and γ is of the form $\beta + \gamma + \lambda$, which does not satisfy the angle folding relation for any $\lambda \in \{\alpha, \beta, \gamma\}$.

2.4.2 Assume now that $\alpha < \frac{\pi}{2}$. Adding a new cell in the configuration of Figure 5, a decision must be taken about the angle $\theta_3 \in \{\alpha, \delta, \beta\}$ as is illustrated in Figure 10:

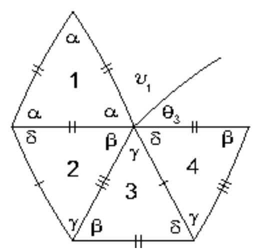

Figure 10: Local configuration.

2.4.2.1 Suppose $\theta_3 = \alpha$. Then, $2\alpha + \gamma \leq \pi$ and consequently $\gamma < \frac{\pi}{3}$. If $2\alpha + \gamma = \pi$, then the other sum of alternate angles at vertex v_1 must be $\beta + \delta + \alpha = \pi$ and so $\alpha + \gamma = \beta + \delta$. Taking into account that $\beta + \gamma + \delta > \pi$, we conclude that $2\gamma + \alpha > \pi$ and consequently $\gamma > \alpha > \frac{\pi}{3}$, contradicting $\gamma < \frac{\pi}{3}$.
If $2\alpha + \gamma < \pi$, we can add some cells to the configuration illustrated in Figure 10 and obtain the one in Figure 11.

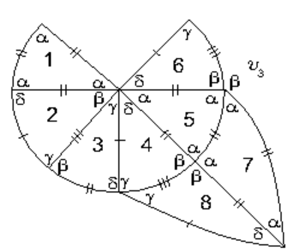

Figure 11: Local configuration.

Observe that if tile 6 is an equilateral triangle, the sum $\alpha+\delta+\beta$ implies that vertices of valency four must be surrounded by alternate angles β and γ. Consequently $\beta > \frac{2\pi}{3}$, contradicting $\beta+\delta+\alpha \leq \pi$. Still, note that in the construction of the configuration, vertex v_3 is of valency four, otherwise these types of vertices would be surrounded by alternate angles β and γ leading to the same contradiction above.
Since $\alpha+\beta = \pi$ and $\beta+\gamma+\delta > \pi$, one has $\gamma+\delta > \alpha > \frac{\pi}{3}$ and $\gamma > \frac{\pi}{6}$. Then, $2\alpha+\gamma+\lambda > \pi$, for any $\lambda \in \{\alpha, \delta, \gamma, \beta\}$, which is an impossibility.
2.4.2.2 Suppose now that $\theta_3 = \delta$. Then, $\alpha+\gamma+\delta \leq \pi$. If $\alpha+\gamma+\delta = \pi$, the configuration in Figure 10 ends up to the one illustrated in Figure 12.

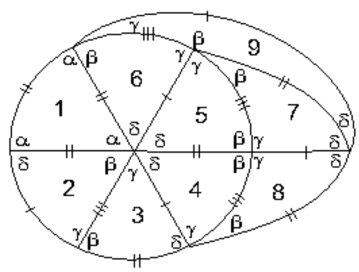

Figure 12: Local configuration.

From the adjacency condition (1.1), $\delta \approx 32.31°$, $\gamma \approx 64.63°$, $\beta \approx 115.38°$ and $\alpha \approx 83.07°$ and the configuration extends to a tiling $\tau \in \Omega(T_1, T_2)$. It is composed of two equilateral and eighteen scalene triangles and will be denoted by $\mathcal{G}^1$, Figure 13.

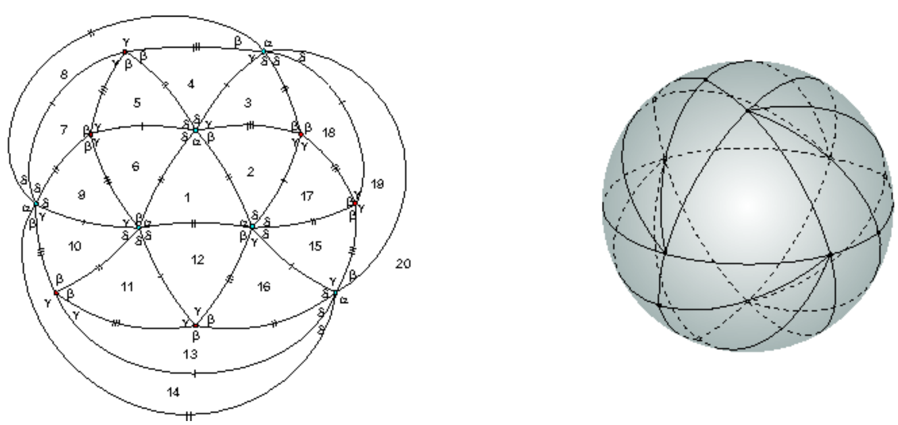

Figure 13: 2D and 3D representation of $\mathcal{G}^1$.

Assume now that $\alpha+\gamma+\delta < \pi$ (see Figure 10). Adding new cells to the configuration we conclude that $\beta+\gamma \leq \pi$, Figure 14. In case $\beta+\gamma < \pi$, then $\beta+\alpha = \pi$, since vertices of valency four must exist. Taking into account that $\beta+\gamma+\delta > \pi$, we conclude that $\gamma > \frac{\pi}{6}$ and consequently $\beta+\gamma+\lambda > \pi$, for each $\lambda \in \{\alpha, \gamma, \beta, \delta\}$. Therefore, the configuration cannot be expanded.

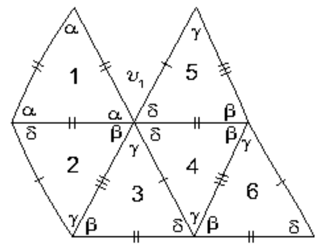

Figure 14: Local configuration.

At vertex v_1, the sum of alternate angles containing β and δ satisfies $\beta + k\delta = \pi$ or $\beta + \alpha + t\delta = \pi$, for $k \geq 2$ and $t \geq 1$.

2.4.2.2.1 Assuming that $\beta + k\delta = \pi,\ k \geq 2$, then the other sum of angles at the same vertex satisfies $\alpha + \gamma + (k-1)\delta = \pi$, as is shown in Figure 15.

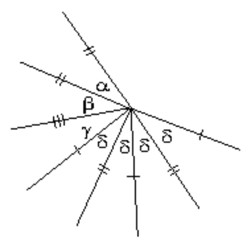

Figure 15: Angle arrangement around vertices surrounded by alternate β and δ.

We may now expand the configuration in Figure 10 getting a tiling $\tau \in \Omega(T_1, T_2)$. In Figure 16 we present a 2D and 3D representation of this tiling with $k = 2$, which is denoted by $\mathcal{G}^2$. The corresponding f-tiling is composed by two equilateral triangles and thirty scalene triangles, $\delta \approx 19.08°$, $\gamma \approx 57.24°$, $\beta \approx 122.76°$ and $\alpha \approx 84.60°$. Generalizing, for $k \geq 1$, the corresponding f-tiling, $\mathcal{G}^k$ is composed by two equilateral triangles and $6(2k+1)$ scalene triangles.

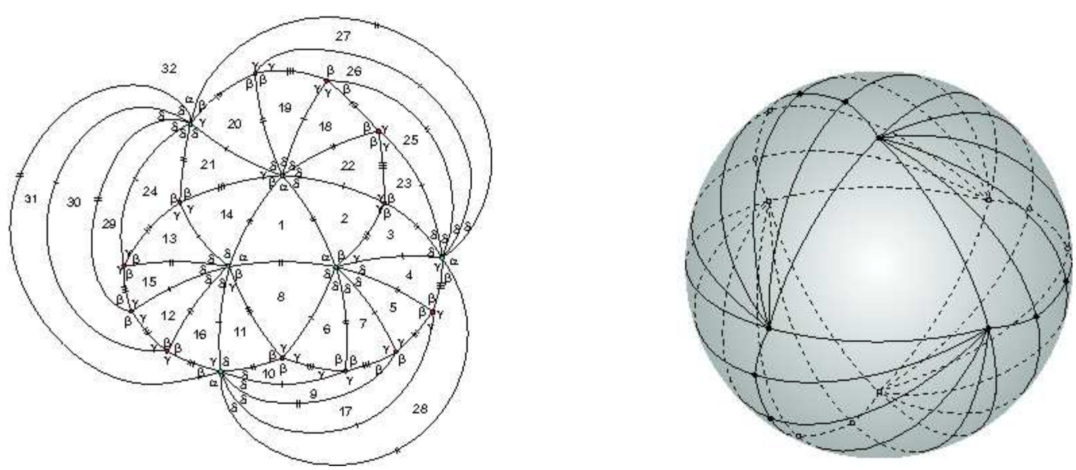

Figure 16: 2D and 3D representation of $\mathcal{G}^2$.

If the restriction of edge-to edge tiling was removed it would not be difficult to construct new tilings, starting from $\mathcal{G}^k$, with a similar pattern as the Dawson's swirl tiling illustrated in Figure 10 of [8].

2.4.2.2.2 If $\beta+\alpha+t\delta=\pi$, then $t\geq 2$, otherwise $\beta=\gamma$. Taking into account that $\beta+\gamma=\pi$, we get $\gamma>\alpha>\frac{\pi}{3}$ and so the vertices surrounded by the alternate angles α,γ and δ satisfy $\alpha+\gamma+t\delta=\pi$. Consequently, at vertex v_1, both sums of the alternate angles are of the form $\alpha+\gamma+t\delta=\pi=\beta+\alpha+t\delta$, which is an impossibility, since $\gamma<\beta$.

2.4.2.3 Suppose finally that $\theta_3=\beta$ (see Figure 10). Since vertices of valency four must be surrounded by alternate angles β and α or β and γ, then the sequence of alternate angles around vertex v_1 is impossible. $\square$

Proposition 2.3. *If $x=\beta$ and $\alpha+x=\pi$, then $\Omega(T_1,T_2)$ is composed of four isolated dihedral triangles f-tilings $\mathcal{E}$, $\mathcal{F},\mathcal{H}$ and $\mathcal{L}$, such that the sum of alternate angles around vertices are respectively of the form:*

$$\alpha+\beta=\pi,\ \alpha+2\delta=\pi\ \textit{and}\ \gamma=\frac{\pi}{3},\ \textit{for}\ \mathcal{E};$$
$$\alpha+\beta=\pi,\ 2\alpha+\delta=\pi\ \textit{and}\ \gamma=\frac{\pi}{3},\ \textit{for}\ \mathcal{F};$$
$$\alpha+\beta=\pi,\ \alpha+2\delta+\gamma=\pi\ \textit{and}\ \gamma=\frac{\pi}{3},\ \textit{for}\ \mathcal{H};$$
$$\alpha+\beta=\pi,\ \alpha+2\delta+\gamma=\pi\ \textit{and}\ \gamma=\frac{\pi}{4},\ \textit{for}\ \mathcal{L}.$$

Proof. Let us assume that $x=\beta$ and $\alpha+x=\pi$ in Figure 2. Then, $\gamma+\delta>\alpha>\frac{\pi}{3}$ and $\gamma>\frac{\pi}{6}$. The configuration started in Figure 2 extends to the one illustrated in Figure 17.

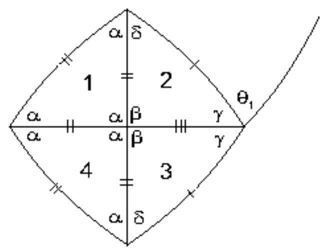

Figure 17: Local configuration.

A decision must be taken about the angle labelled $\theta_1\in\{\gamma,\delta\}$.

1. Assuming that $\theta_1=\gamma$, then $\gamma\leq\frac{\pi}{2}$. If $\gamma=\frac{\pi}{2}$, then $\beta>\frac{\pi}{2}$, $\delta<\frac{\pi}{2}$ and $\alpha<\frac{\pi}{2}$, which is impossible by the adjacency condition (1.1).
Therefore, $\delta<\gamma<\frac{\pi}{2}$ and again, by the adjacency condition, we conclude that $\alpha<\frac{\pi}{2}<\beta$. Since we are assuming that $\theta_1=\gamma$, the configuration extends a bit more to the one shown in Figure 18 and angle θ_2 must be γ, otherwise the sum containing $\theta_2=\beta$ and γ would be simply $\beta+\gamma$ or $\beta+\gamma+\lambda$.

In the first case, the other sum of angles would satisfy $2\gamma = \pi$, which is impossible and in the second case the angle folding relation is violated, for any $\lambda \in \{\alpha, \delta, \gamma, \beta\}$.

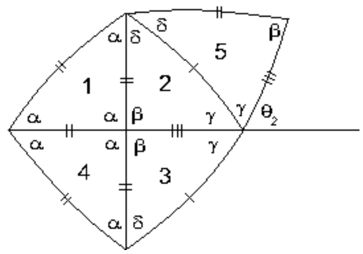

Figure 18: Local configuration.

Adding one new cell to the configuration in Figure 18, we get the following one:

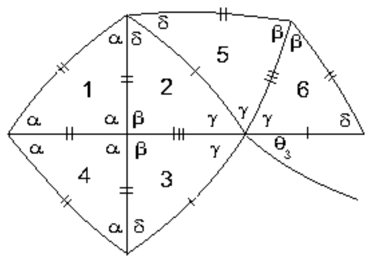

Figure 19: Local configuration.

1.1 Suppose firstly, that $\theta_3 = \gamma$ and $\gamma = \dfrac{\pi}{3}$. We may extend the configuration in Figure 19 and a decision must be taken about the angle $\theta_4 \in \{\beta, \alpha\}$, as is shown in Figure 20.

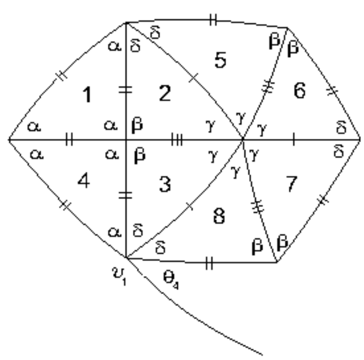

Figure 20: Local configuration.

1.1.1 If $\theta_4 = \beta$, then the sum of the alternate angles containing $\theta_4 = \beta$ and δ at vertex v_1 satisfies $\beta + t\delta = \pi$ and the other $\alpha + t\delta = \pi$ or $\alpha + \gamma + (t-1)\delta = \pi$ or $2\alpha + (t-1)\delta = \pi$, for some $t \geq 2$.

In the first case, we get $\alpha = \beta$, which is impossible. In the second case, by the adjacency condition (1.1), we conclude that

$$\frac{\cos\alpha\,(1+\cos\alpha)}{\sin^2\alpha} = \frac{\dfrac{1}{2}+\cos\alpha\cos\left(2\alpha+\dfrac{\pi}{3}\right)}{-\sin\alpha\sin\left(2\alpha+\dfrac{\pi}{3}\right)}.$$

Since $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$, then $\alpha \approx 69.12°$, $\beta \approx 110.84°$ and $\delta \approx 18.31°$, which is impossible for any $t \geq 2$.
In the third case, by the adjacency condition (1.1),

$$\frac{\cos\alpha\,(1+\cos\alpha)}{\sin^2\alpha} = \frac{\dfrac{1}{2}+\cos\alpha\cos(3\alpha)}{-\sin\alpha\sin(3\alpha)},$$

and so $\alpha = \frac{2\pi}{5}$, $\beta = \frac{3\pi}{5}$ and $\delta = \frac{\pi}{5}$. Therefore, $t = 2$, but by edge compatibility, we conclude that it is impossible to pursuing the configuration.

1.1.2 Suppose $\theta_4 = \alpha$. The sum of the alternate angles containing $\theta_4 = \alpha$ and δ satisfies $\alpha + t\delta = \pi$, for some $t \geq 2$ or $2\alpha + p\delta = \pi$, for some $p \geq 1$ or $\alpha + \gamma + q\delta = \pi$, for some $q \geq 1$.

1.1.2.1 In the first case, we have $\alpha = \pi - t\delta$, $\beta = t\delta$ and $\gamma = \frac{\pi}{3}$. For $t = 2$ the configuration extends globally to the one illustrated in Figure 21 and is denoted by $\mathcal{E}$.

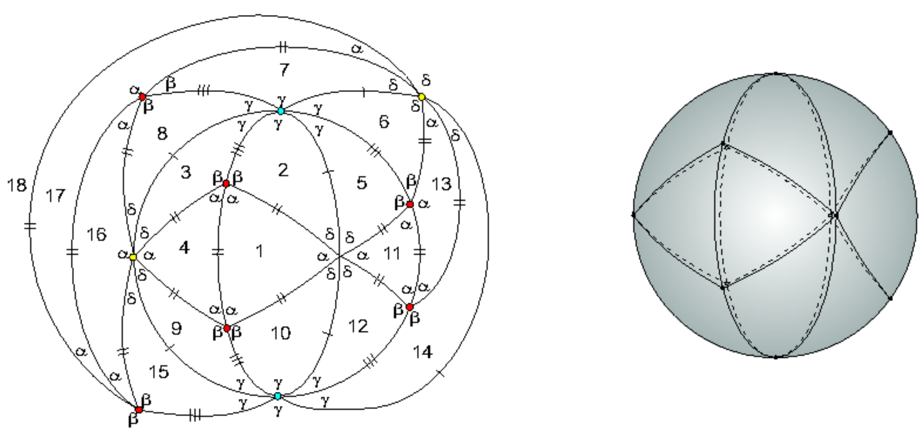

Figure 21: 2D and 3D representation of $\mathcal{E}$.

This tiling has six equilateral triangles and twelve scalene triangles and it was expanded in an unique way. By the adjacency condition (1.1), we conclude that $\alpha \approx 72,75°$, $\beta \approx 107,25°$ and $\delta \approx 53,63°$.

For $t > 2$, the local representation ends up at a vertex v_2 surrounded by angles β, β, γ, whose sum $\beta + \gamma$ does not satisfy the angle folding relation.

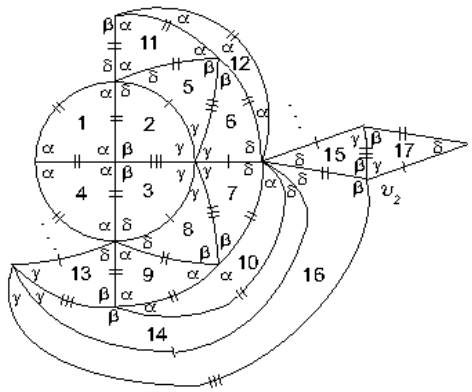

Figure 22: Local configuration.

1.1.2.2 In the second case, we have $2\alpha + p\delta = \pi$ and for $p = 1$, we get a global representation of a tiling $\tau \in \Omega(T_1, T_2)$, where $\alpha = \dfrac{2\pi}{5}$, $\delta = \dfrac{\pi}{5}$ and $\beta = \dfrac{3\pi}{5}$. It has twelve equilateral triangles and twelve scalene triangles and is denoted by $\mathcal{F}$. In Figure 23 we present a 2D and 3D representation of this f-tiling and this construction corresponds to a choice of the sides of tile 15, in order to avoid vertices surrounded by the angular sequence $(\alpha, \alpha, \alpha, \beta, \delta, ...)$, which does not satisfy the angle folding relation.

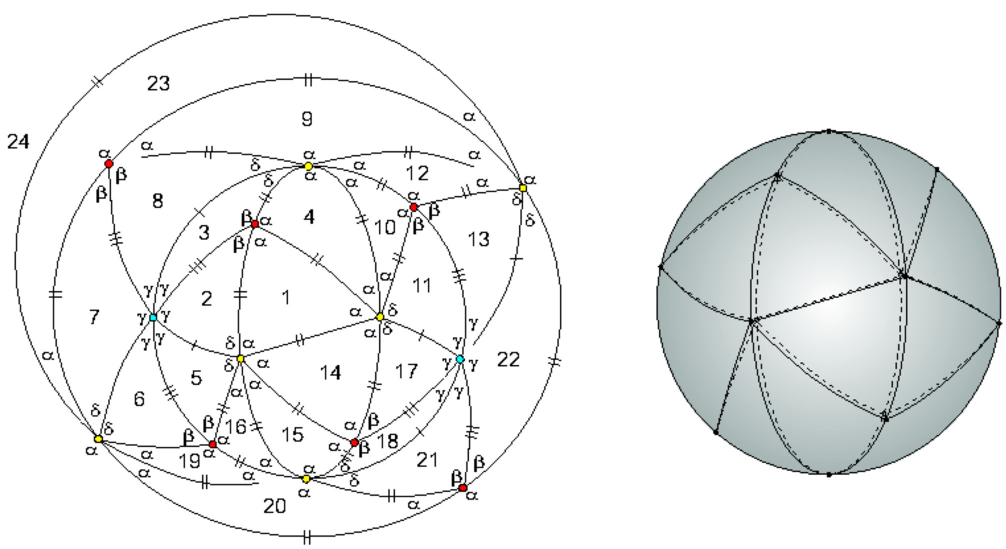

Figure 23: 2D and 3D representation of $\mathcal{F}$.

For $p > 1$ and assuming that tile 10 is an equilateral triangle in the positions illustrated below, we always get vertices surrounded by alternate angles β and γ (see Figure 24-I, II and III), whose sum does not satisfy the angle folding relation. Note that to avoid vertices surrounded by angles β, α, β (whose sum 2β does not satisfy the angle folding relation), tile 17 in 24-I must be an equilateral triangle and to avoid vertices surrounded by angles $(\beta, \alpha, \delta, \delta, ...)$ (which is incompatible with the edge sides), tiles 15 in 24-II and 18 in 24-III must be the ones illustrated.

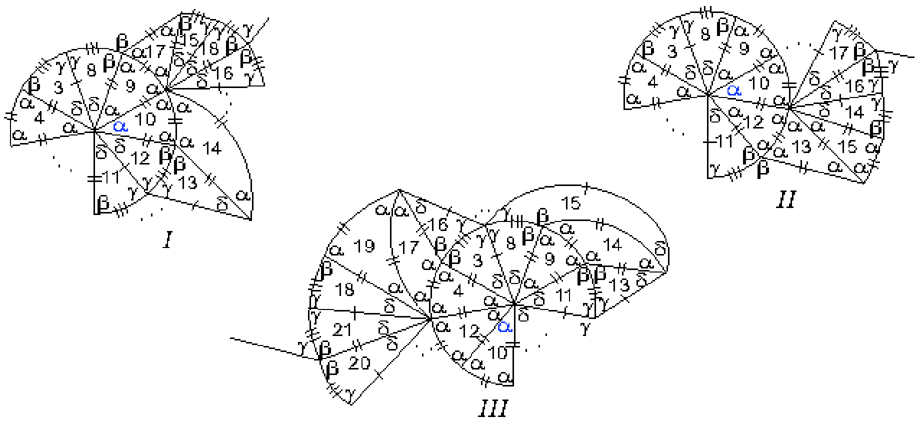

Figure 24: Local configuration.

The other position for the equilateral triangle in tile 10 is shown in Figure 25 and once again, we end up at a vertex surrounded by angles β, β, γ or β, γ, γ, whose sum $\beta + \gamma$ does not satisfy the angle folding relation.

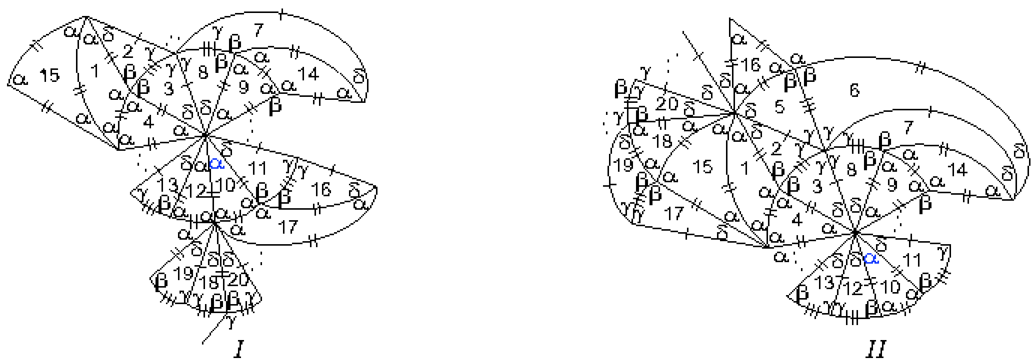

Figure 25: Local configuration.

1.1.2.3 In the third case, $\alpha + \gamma + q\delta = \pi$, for $q \geq 1$ and if $q = 1$, we get an impossibility due to the edge compatibility of the triangles. For $q = 2$, we get $\alpha \approx 70.52°$, $\delta \approx 24.74°$ and $\beta \approx 109.48°$ and we may expand globally the configuration obtaining a representation of a tiling $\tau \in \Omega(T_1, T_2)$, which is denoted by $\mathcal{H}$, see Figure 26. It is composed of twelve equilateral triangles and twenty four scalene triangles.

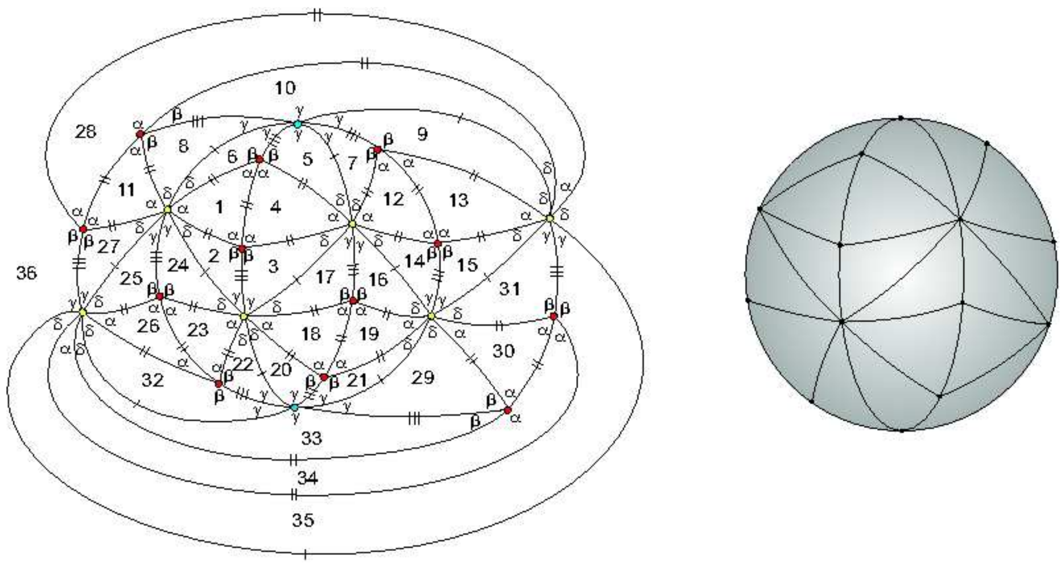

Figure 26: 2D and 3D representation of $\mathcal{H}$.

For $q > 2$, we observe that the angle arrangement at vertices whose sum of alternate angles satisfy $\alpha + \gamma + q\delta = \pi$ has always three consecutive angles δ leading to a vertex surrounded by angles β, β, γ, as is illustrated in Figure 27 for cases $q = 3$ and $q = 4$.

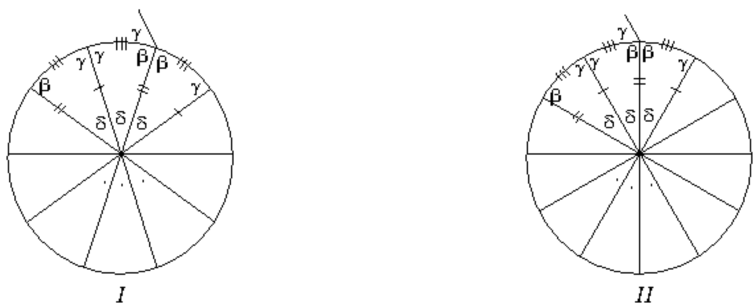

Figure 27: Angle arrangement at vertices with the sum $\alpha + q\delta + \gamma = \pi,\ q = 3, 4$.

1.2 Suppose now that $\gamma < \dfrac{\pi}{3}$, with $\theta_3 = \gamma$ (Figure 19). Then, in order to fulfill the angle folding relation, the sum 3γ must contain another parameter ρ being a sum of angles, which does not contain β and α (since the angular sequence $(\gamma, \gamma, \gamma, \gamma, \gamma, \beta, \alpha, \gamma)$ does not satisfy the angle folding relation). We shall study the cases $\rho = k\gamma,\ k = 1, 2,\ \rho = \gamma + \delta$ and $\rho = \delta, \rho = 2\delta$ separately.

1.2.1 Suppose $\rho = \gamma$. If $4\gamma = \pi$, then $\delta > \dfrac{\pi}{12}$, since $\gamma + \delta > \dfrac{\pi}{3}$.
The sum of the alternate angles α and δ must satisfy $\alpha + t\delta = \pi,\ t = 2, \ldots, 7$ or $2\alpha + p\delta = \pi,\ p = 1, 2, 3$ or $\alpha + \delta + 2\gamma = \pi$ or $\alpha + k\delta + \gamma = \pi,\ k = 2, 3, 4$ (observe that if $k = 1$, then $\delta > \dfrac{\beta}{4} = \gamma$). By the adjacency condition (1.1), the first case is valid for $t = 3, ..., 7$, but expanding the angle arrangement, we always end up at a vertex surrounded by angles β, β, γ, whose sum $\beta + \gamma$ does not satisfy the angle folding relation,

since $\gamma < \alpha$.
In the second case, we conclude that for $p = 1$, $\delta \approx 46.62°$, which is impossible since $\delta < \gamma$. Therefore, $p = 2, 3$ and once again the angle arrangement leads us to a vertex surrounded by angles β, β, γ (whether $p = 2$ or $p = 3$) and so it is impossible to extend the configuration.
In the third case, the angles arrangement is $(\alpha, \delta, \delta, \beta, \gamma, \gamma, \gamma, \delta)$ and the sum $\delta + \beta + \gamma + \delta$ violates the angle folding relation. It remains the last case and if $\alpha + k\delta + \gamma = \pi$, $k = 2, 3, 4$, respectively, we get $\alpha \approx 65.56°$, $\delta \approx 34.72°$, $\beta \approx 114.44°$ or $\alpha \approx 63.27°$, $\delta \approx 23.91°$, $\beta \approx 116.73°$ or $\alpha \approx 61.43°$, $\delta \approx 18.39°$, $\beta \approx 118.57°$. The other sum of alternate angles at vertices surrounded by angles α, δ and γ is always (independently of the position of the $k\delta's$) $\alpha + k\delta + \gamma = \pi$ or $\beta + (k+1)\delta = \pi$.
Taking into account the angles obtained by the adjacency condition, we conclude that the only possible sum is $\alpha + k\delta + \gamma = \pi$, $k = 2, 3, 4$. Assuming that $k = 2$, we may expand the configuration in Figure 19 and obtain a global representation of a f-tiling $\tau \in \Omega(T_1, T_2)$ (see Figure 28). Note that in the construction of the global representation, we must avoid the appearance of one angle β at vertices that already have two angles β, since it leads to a configuration with a vertex in which one of its sum of alternate angles contains two angles β. This avoidance obliges tile 11 to be an equilateral triangle. The corresponding tiling has sixteen equilateral triangles and thirty-two scalene triangles and is denoted by $\mathcal{L}$.

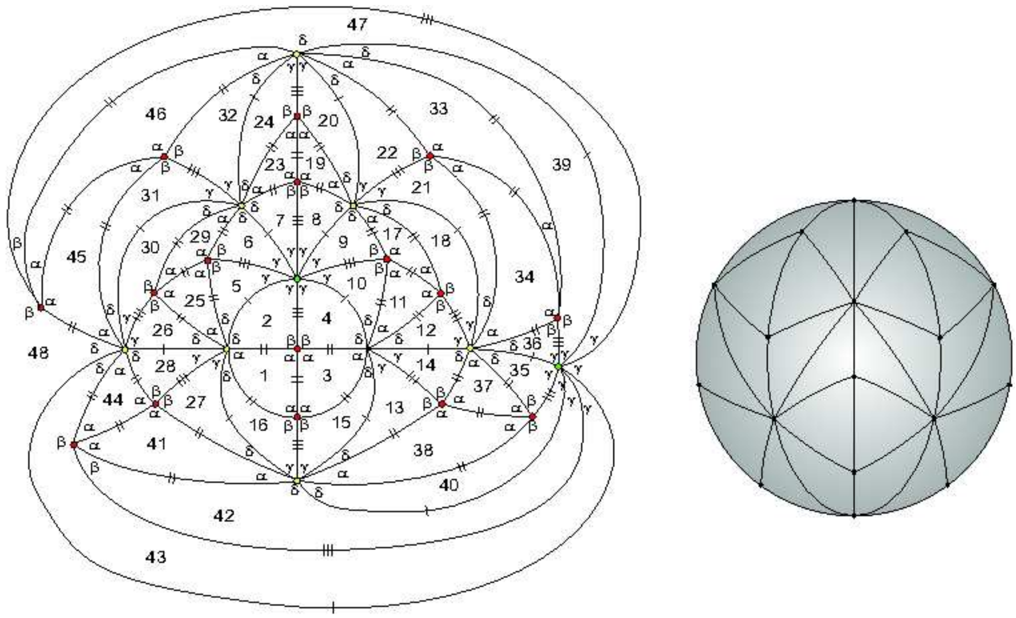

Figure 28: 2D and 3D representation of $\mathcal{L}$.

If $\alpha + k\delta + \gamma = \pi$, for $k = 3$ and $k = 4$, we always end up at a vertex surrounded by angles β, β, γ, since the angle arrangement at vertices of valency ten and twelve with this type of alternate sum has always three angles δ in consecutive positions, as in the case **1.1.2.3**.

1.2.2 If $5\gamma = \pi$, then $\delta > \dfrac{2\pi}{15}$ and again one of the sums at vertices surrounded by alternate angles α and δ must satisfy $\alpha + t\delta = \pi$, $t = 2, 3, 4$ or $2\alpha + p\delta = \pi$, $p = 1, 2$ or $\alpha + \delta + 2\gamma = \pi$ or $\alpha + k\delta + \gamma = \pi$, $k = 1, 2, 3$.

1.2.2.1 In the first case, since $\gamma > \delta$, we get $t = 4$ and so $\delta \approx 29.61°$, $\alpha \approx 61.56°$, $\beta \approx 118.44°$. However, expanding the configuration in Figure 19, we end up at a vertex surrounded by a sequence of angles of the form $(..., \beta, \beta, \gamma, ...)$ and so the sum $\beta + \gamma + \mu$ violates the angle folding relation, where μ is a sum of angles.

1.2.2.2 In the second case, for $p = 1$, we get $\alpha \approx 64.29°$ and $\delta \approx 51.43°$ which is impossible (since $\delta < \gamma$); for $p = 2$ we get $\alpha \approx 61.31°$, $\delta \approx 28.69°$, $\beta \approx 118.69°$. The configuration extends a bit more and the next figure shows the possible positions of the angles arrangement surrounding vertices in which one of sums of alternate angles is $2\alpha + 2\delta$. A contradiction is achieved in the configuration in Figure 29-I, II and III, since it always reaches at a vertex surrounded by angles β, β, γ or a vertex surrounded by angles β, α, β, whose sum $\beta + \gamma$ or 2β does not satisfy the angle folding relation.

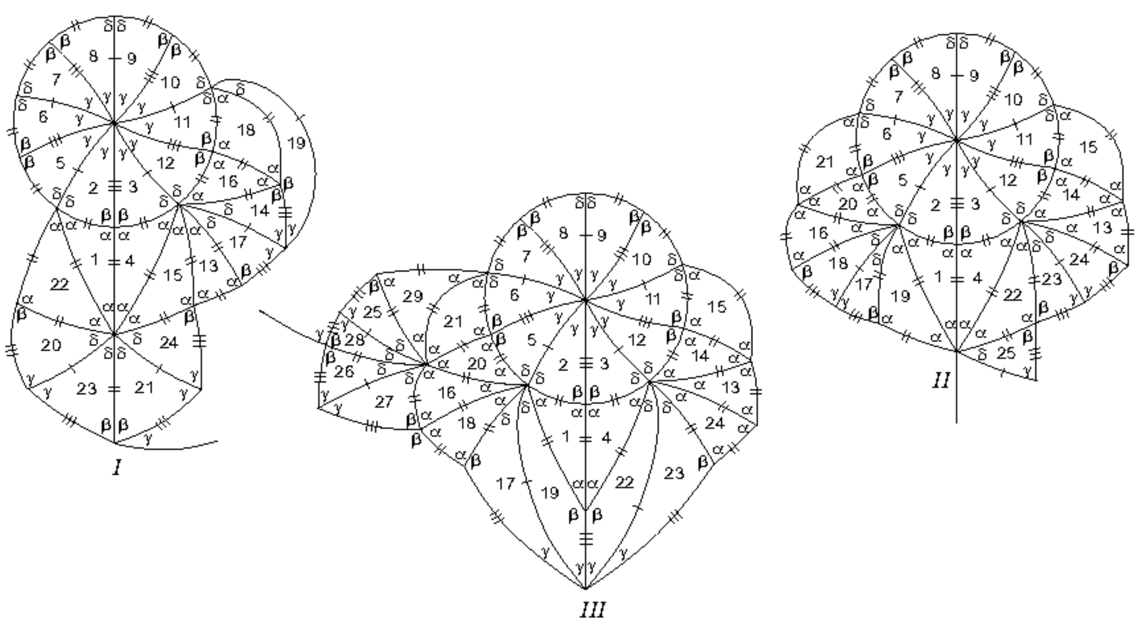

Figure 29: Local configuration.

1.2.2.3 If $\alpha + \delta + 2\gamma = \pi$, then, by the adjacency condition (1.1), $\delta \approx 44.1°$, which is impossible.

1.2.2.4 If $\alpha + k\delta + \gamma = \pi$, for $k = 1, 2, 3$, respectively, we get $\delta \approx 81.19°$ or $\delta \approx 40.28°$ $\delta \approx 27.62°$. Thus, for $k = 1, 2$, $\delta > \gamma$, which is a contradiction. Summarizing, $k = 3$ and $\alpha \approx 61.15°$, $\beta \approx 118.85°$, $\delta \approx 27.62°$. Extending the configuration in Figure 19 and choosing for tile 24 one of its two possible positions, we end up at a vertex surrounded by the angular sequence $(\beta, \beta, \gamma, ...)$, whose sum is $\beta + \gamma$ or $\beta + \gamma + \mu$, where μ is a sum of angles. In the first case, we conclude that $\gamma = \alpha$, which is impossible and in the second case, the sum violates the angle folding relation, Figure 30-I, II. The other position for tile 24 ends up in a similar impossibility.

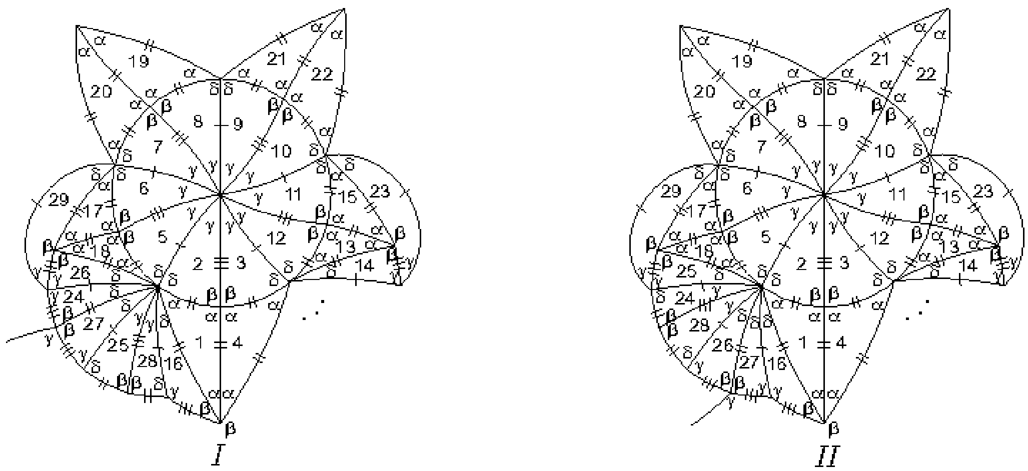

Figure 30: Local configuration.

1.2.3 The vertices of valency ten in which one of the sums of alternate angles is $4\gamma+\delta=\pi$ gives rise to another vertex surrounded by one alternate angle β and one angle γ, whose sum does not satisfy the angle folding relation, see Figure 31-I and II.

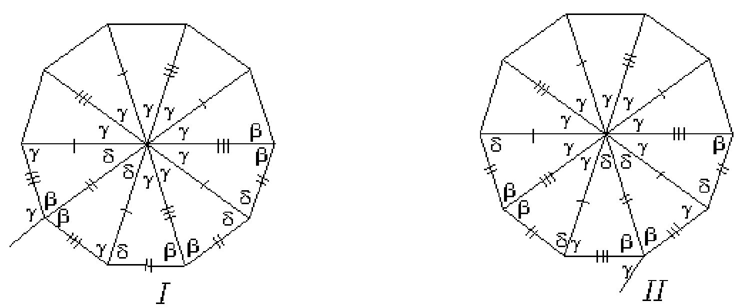

Figure 31: Angle arrangement around vertices satisfying $4\gamma+\delta=\pi$.

1.2.4 Suppose $\rho=\delta$. Then, $3\gamma+\delta=\pi$ and the configuration in Figure 19 ends up in the one shown in Figure 32-I. Once again we get an impossibility at vertex v_3.

1.2.5 If $\rho=2\delta$, then $3\gamma+2\delta=\pi$. The configuration ends up in a vertex surrounded by angles β and γ, similar to the one in Figure 31 and a contradiction is achieved as is shown in Figure 32-II.

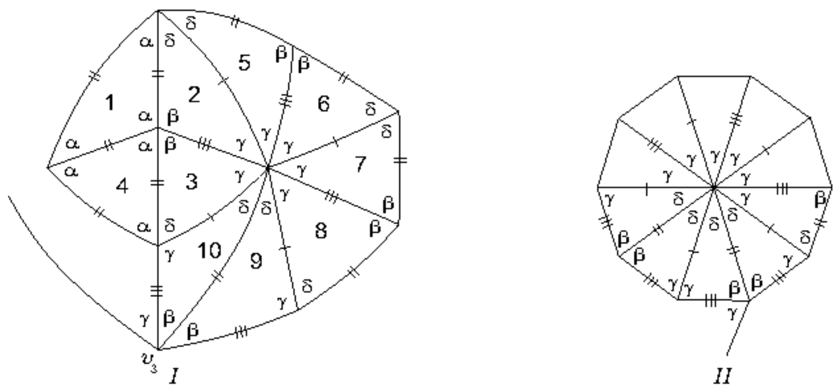

Figure 32: Local configuration.

1.3 Suppose now that $\theta_3 = \delta$ (see Figure 19). Then, $2\gamma + \delta \leq \pi$ and if $2\gamma + \delta = \pi$, the configuration is given in Figure 33.

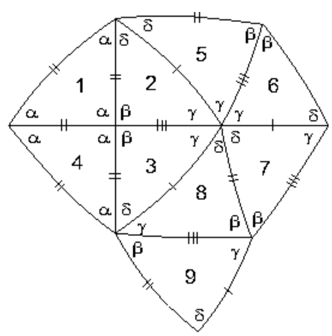

Figure 33: Local configuration.

The vertices surrounded by angles β and γ must be of valency four, so $\gamma = \alpha$ and consequently by the adjacency condition (1.1), $\alpha = \dfrac{\pi}{2}$, which is impossible. Therefore, $2\gamma + \delta < \pi$ and a decision must be taken about the angle $\theta_5 \in \{\delta, \alpha\}$ (see Figure 34).

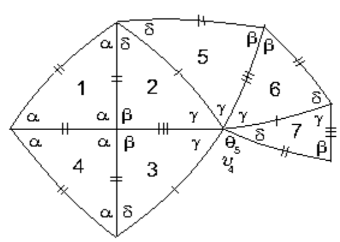

Figure 34: Local configuration.

In case $\theta_5 = \delta$, the configuration extends to the one shown in Figure 35.

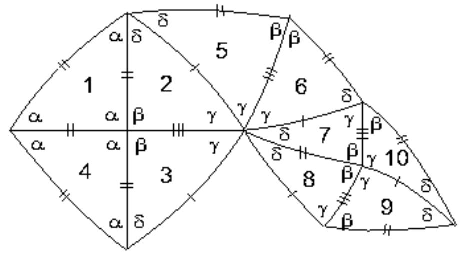

Figure 35: Local configuration.

The vertex surrounded by angles β and γ must be of valency four and once again $\gamma = \alpha$, which is impossible by the adjacency condition (1.1). Accordingly, $\theta_5 = \alpha$ and since $2\gamma + \alpha < \pi$ (due to edge compatibility), one has $3\gamma + \alpha = \pi$ or $2\gamma + \alpha + \delta = \pi$, taking into account that $\gamma > \dfrac{\pi}{6}$ and $\gamma + \delta > \alpha$.

If $3\gamma + \alpha = \pi$, then $\delta > \dfrac{\pi}{9}$ and the other sum of alternate angles at vertex v_4 is of the form $\beta + \delta + 2\gamma = \pi$, which is impossible.

If $2\gamma + \alpha + \delta = \pi$, we may add some new cells to the local configuration illustrated in Figure 35 and obtain the one in Figure 36.

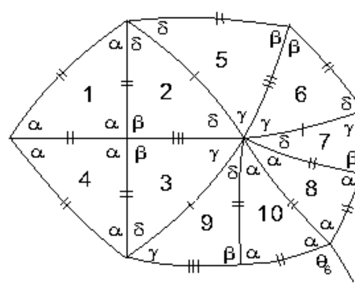

Figure 36: Local configuration.

The angle θ_6 must be α or β. In case $\theta_6 = \alpha$, then the sum containing two alternate angles α is $2\alpha + \gamma = \pi$ or $2\alpha + p\delta = \pi, p \geq 1$. However, by the assumption $2\gamma + \alpha + \delta = \pi$, it is impossible that $2\alpha + \gamma = \pi$ (note that $\gamma + \delta > \alpha$). Therefore, $2\alpha + p\delta = \pi$ for some $p \geq 1$. The configuration extends and we obtain the one in Figure 37-I.
The vertices surrounded by alternate angles β and γ, once again are of valency four, which is impossible since $\gamma = \alpha$ does not satisfy the condition $\alpha + \delta + 2\gamma = \pi$.

If $\theta_6 = \beta$, then the configuration extends a bit more, but we end up again at a vertex surrounded by alternate angles β and γ, which must be of valency four and consequently $\gamma = \alpha > \dfrac{\pi}{3}$, contradicting the assumption $2\gamma + \alpha + \delta = \pi$ (Figure 37-II).

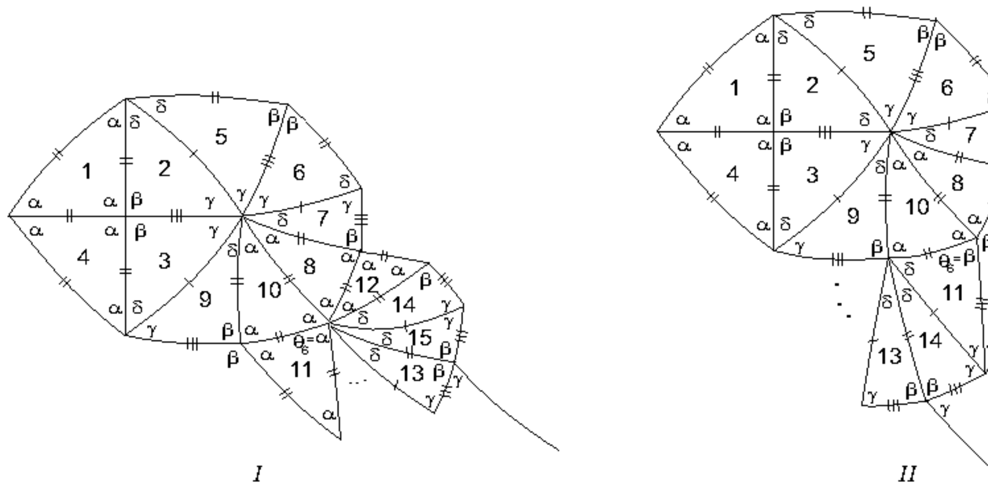

Figure 37: Local configuration.

2. Suppose now that $\theta_1 = \delta$ (see Figure 17). If $\gamma + \delta = \pi$, then $\beta > \gamma > \dfrac{\pi}{2}$ and from the assumption $\alpha + \beta = \pi$, then $\delta, \alpha < \dfrac{\pi}{2}$. However, the configuration can not be expanded since the sum $\rho + \beta$ (see Figure 38) does not satisfy the angle folding relation, for any $\rho \in \{\gamma, \beta\}$.

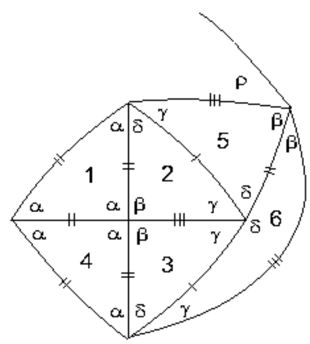

Figure 38: Local configuration.

As $\gamma + \delta < \pi$, then $\delta < \dfrac{\pi}{2}$. If $\gamma \geq \dfrac{\pi}{2}$, then $\beta > \dfrac{\pi}{2}$ and $\alpha < \dfrac{\pi}{2}$, which is impossible by the adjacency condition (1.1). Therefore, $\gamma < \dfrac{\pi}{2}$ and also $\alpha < \dfrac{\pi}{2} < \beta$, by the adjacency condition (1.1).

The configuration started in Figure 17 extends to the one in Figure 39.

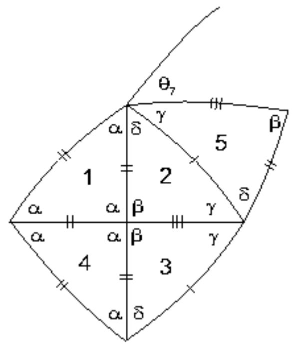

Figure 39: Local configuration.

The angle labelled θ_7 is either β or γ.

2.1 Suppose firstly, that θ_7 is β. Therefore, in order to satisfy the angle folding relation, the sum containing alternate angles β and δ is $\beta + r\delta = \pi$, for some $r > 1$. The other sum of alternate angles at the same vertex is $\alpha + \gamma + (r-1)\delta$, as is illustrated in the Figure 40.

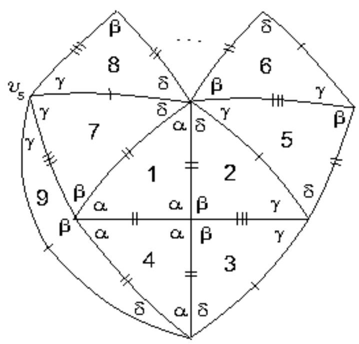

Figure 40: Local configuration.

Looking at vertex v_5 surrounded only by angles γ, one of the sums of alternate angles is $2\gamma + \lambda$, where the parameter λ is a sum of angles not containing any β, due to the angle folding relation.

2.1.1 Suppose that $\lambda = \alpha$. Then, $2\gamma + \alpha \leq \pi$. However, the case $2\gamma + \alpha = \pi$ is impossible, since the other sum of alternate angles is $\beta + \gamma + \delta$, not satisfying the angle folding relation, as is illustrated in the Figure 41.

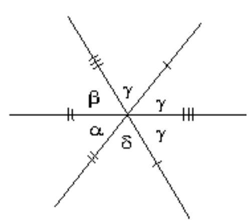

Figure 41: Angle arrangement.

As $2\gamma + \alpha < \pi$, then $3\gamma + \alpha = \pi$ or $2\gamma + \alpha + \delta = \pi$. If $3\gamma + \alpha = \pi$, having in account that $\alpha + \gamma + (r-1)\delta = \pi$ and $\beta + r\delta = \pi$, we conclude that $\beta + \delta + 2\gamma = \pi$, which is a contradiction.

Also, if $2\gamma + \delta + \alpha = \pi$, for the same reason $\beta + \gamma + 2\delta = \pi$, which is again an impossibility.

2.1.2 Suppose that $\lambda = m\gamma$, $m = 1, 2, 3$. If $m = 1$, then $3\gamma = \pi$ and the angle arrangement around vertex v_5 is illustrated in the Figure 42.

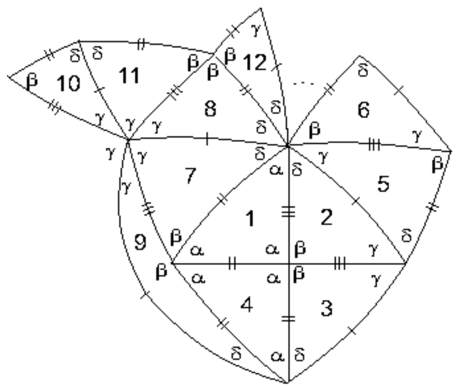

Figure 42: Local configuration.

Observing tiles labelled 8,11 and 12 and the position of the angle β, we can not expand the configuration, since $\beta > \frac{\pi}{2}$.

If $m = 2, 3$, then $4\gamma = \pi$ and $5\gamma = \pi$, but we are led to the same contradiction illustrated in Figure 42.

2.1.3 If $\lambda = \gamma + k\delta$, $k = 1, 2$, then $3\gamma + k\delta = \pi$ and the configuration ends up at a vertex surrounded by angles γ, δ and β, which is impossible since the sum of the alternate angles β and γ does not satisfy the angle folding relation (see Figure 43).

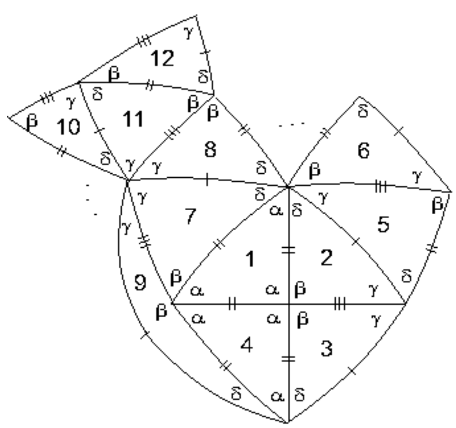

Figure 43: Local configuration.

2.1.4 Suppose that $\lambda = \delta$. If the sum $2\gamma+\delta$ satisfies the angle folding relation, then $\gamma > \dfrac{\pi}{3}$ and the local representation in Figure 39 extends to the one illustrated in Figure 44-I.

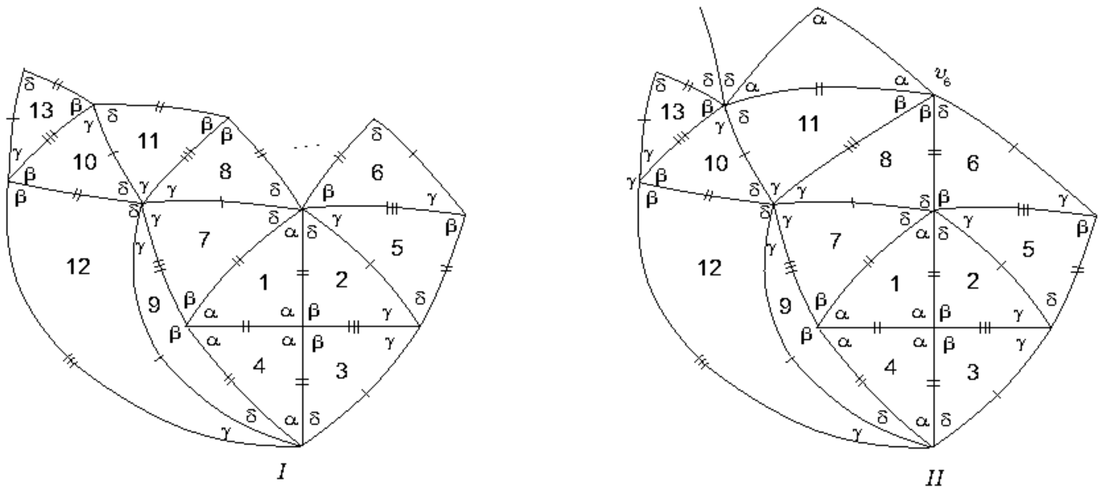

Figure 44: Local configuration.

The vertices surrounded by alternate angles β and γ must be of valency four, for which $\gamma = \alpha$ and from the assumption in **2.1**, $r = 2$, i.e. $\beta + 2\delta = \pi = \alpha + \gamma + \delta$. Figure 44-II illustrates the expanded configuration and looking at vertex v_6, we conclude that is of valency four, which is impossible since $\delta < \gamma = \alpha$. Therefore, $2\gamma + \delta < \pi$ and so, $2\gamma + k\delta = \pi$, for some $k \geq 2$. However, this case is similar to the case $2\gamma + \delta = \pi$.

2.2 Suppose that $\theta_7 = \gamma$ (see Figure 39). Consequently $\alpha + \gamma + \rho = \pi$, for some ρ different from β.

2.2.1 If $\rho = \alpha$, then we have $2\alpha + \gamma = \pi$, since $\gamma + \delta > \alpha > \dfrac{\pi}{3}$. However, due to the edge compatibility, it is impossible to arrange the angles in order to satisfy $2\alpha + \gamma = \pi$.

2.2.2 If $\rho = \gamma$, then $\alpha+2\gamma < \pi$, since $\alpha+2\gamma = \pi$ implies that the other sum of alternate angles is $\beta+\gamma+\delta = \pi$, which is an impossibility. Taking into account that, $\gamma+\delta > \alpha > \dfrac{\pi}{3}$ and $\gamma > \dfrac{\pi}{6}$, one has $\alpha + 3\gamma = \pi$ or $\alpha + 2\gamma + \delta = \pi$. Again, by the angle arrangement, the case $\alpha + 3\gamma = \pi$ leads us to the sum $\beta + \delta + 2\gamma = \pi$, which is impossible. Therefore, $\alpha+2\gamma+\delta = \pi$ and we may add some new cells to the configuration in Figure 39. Choosing for tile 7 one of its two possible positions, we end up at the configuration in Figure 45.

Looking at vertex labelled v_7, we conclude that it must be of valency four and therefore $\gamma = \alpha > \dfrac{\pi}{3}$, contradicting the sum $\alpha + 2\gamma + \delta = \pi$.

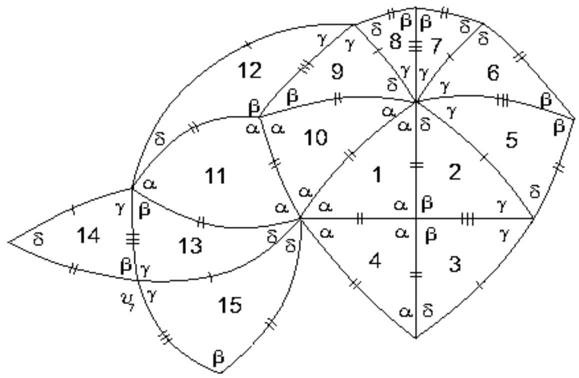

Figure 45: Local configuration.

The other position for tile numbered 7 leads us to a contradiction, Figure 46, since $\xi = \beta$ or $\zeta = \beta$.

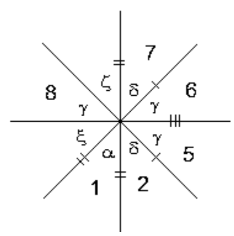

Figure 46: Angle arrangement.

2.2.3 If $\rho = \delta$, then $\alpha + \gamma + \delta \leq \pi$. We shall study the cases $\alpha + \gamma + \delta = \pi$ and $\alpha + \gamma + \delta < \pi$ separately.

2.2.3.1 Suppose, firstly, that $\alpha + \gamma + \delta = \pi$. Taking into account that $\gamma + \delta > \alpha$, one has $\gamma > \dfrac{\pi}{4}$. The local configuration in Figure 39 extends a bit more to the one in Figure 47.

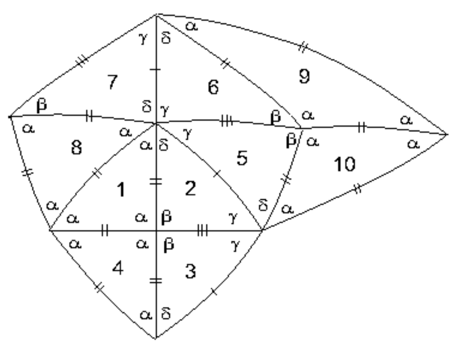

Figure 47: Local configuration.

In order to satisfy the angle folding relation, the sum of the alternate angles 2α is $2\alpha+\gamma=\pi$ or $2\alpha+p\delta=\pi$, for some $p\geq 1$. If $2\alpha+\gamma=\pi$, then $\delta=\alpha>\frac{\pi}{3}$ and consequently $\gamma>\frac{\pi}{3}$ not satisfying $\alpha+\gamma+\delta=\pi$. Therefore, $2\alpha+p\delta=\pi, p\geq 1$ and, by the assumption $\alpha+\delta+\gamma=\pi$, we get $\alpha+(p-1)\delta=\gamma$. Consequently, $\gamma\geq\alpha>\frac{\pi}{3}$, which implies that the sum of alternate angles at vertices surrounded by α and γ must satisfy $\alpha+\gamma+\delta=\pi$. Accordingly, $\alpha+\beta=\pi$, $2\alpha+p\delta=\pi, p\geq 1$ and $\alpha+\gamma+\delta=\pi$. Expanding the configuration and attending to the choice of the edge sides of tile 13, we get the one shown in Figure 48. Looking at vertex v_8, the configuration cannot be extended, since the sum 2β violates the angle folding relation.

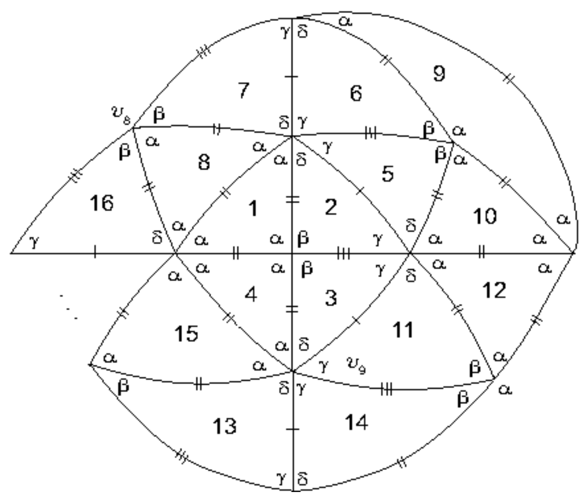

Figure 48: Local configuration.

The other position of tile numbered 13 implies that, at vertex v_9, the sequence of angles is $(\alpha,\delta,\gamma,\beta,\delta,\delta)$ (Figure 49).

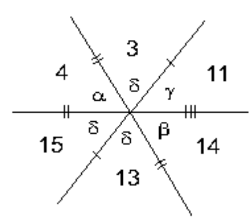

Figure 49: Angle arrangement.

Summarizing, $\alpha+\beta=\pi$, $\alpha+\gamma+\delta=\pi$, $\beta+2\delta=\pi$ and $2\alpha+p\delta=\pi, p\geq 1$, which implies that $\delta>\frac{\pi}{6}$ and $p=1$. Therefore, $\alpha=\gamma=\frac{2\pi}{5}, \delta=\frac{\pi}{5}$ and $\beta=\frac{3\pi}{5}$. The configuration extends to the following one and we are led to vertices surrounded by three angles β, whose sum 2β violates the angles folding relation.

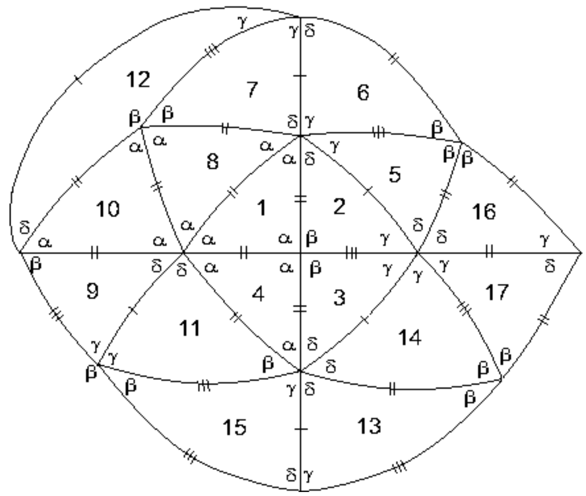

Figure 50: Local configuration.

2.2.3.2 Suppose now that $\alpha+\gamma+\delta<\pi$. Then, $\alpha+2\gamma+\delta=\pi$ or $\alpha+\gamma+r\delta=\pi$, for some $r\geq 1$. The first case is similar to the one studied in **2.2.2**.
If $\alpha+\gamma+r\delta=\pi, r\geq 1$, the configuration in Figure 39 can be extended and we get the one in Figure 51.

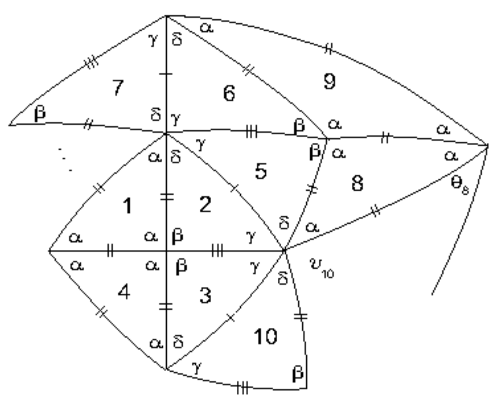

Figure 51: Local configuration.

A decision about the angle labelled $\theta_8\in\{\alpha,\beta\}$ must be taken.

2.2.3.2.1 Assuming that $\theta_8=\alpha$, the sum containing two alternate angles α must satisfy $2\alpha+p\delta=\pi$, for some $p\geq 1$, otherwise it would satisfy $2\alpha+\gamma=\pi$, and consequently, by the adjacency rules for the sides, the other sum would be $\beta+\alpha+\delta=\pi$, which is impossible. Therefore, the configuration extends a bit more to the one illustrated in Figure 52. Note that to avoid the appearance of one angle β at vertex v_{11}, the sides of tile 12 must be in the position illustrated. Looking at the angle $\omega=\beta$, we conclude that the configuration below can not be extended.

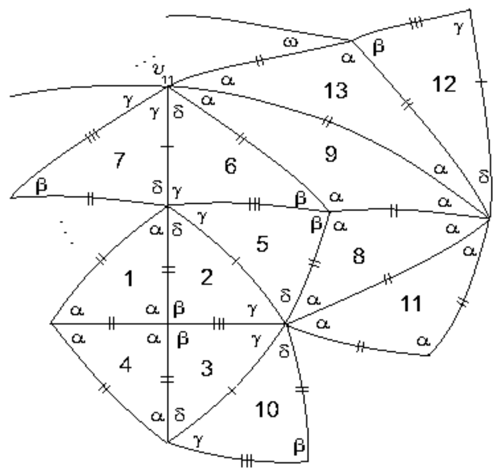

Figure 52: Local configuration.

2.2.3.2.2 If $\theta_8 = \beta$, the vertex labelled v_{10} is as illustrated in Figure 53 and we conclude that the other sum of alternate angles satisfies $\gamma + (r+1)\delta = \pi$.

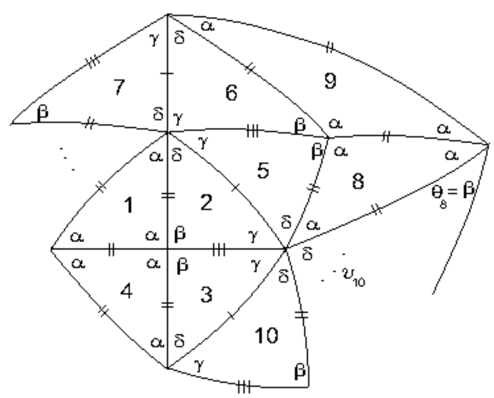

Figure 53: Local configuration.

Consequently $\delta = \alpha < \gamma$, which is impossible. ☐

Proposition 2.4. *If $x = \beta$ and $\alpha + x < \pi$, then $\Omega(T_1, T_2) = \varnothing$.*

Proof. Suppose that $x = \beta$ (see Figure 2) and $\alpha + x < \pi$. Then, $\gamma + \delta > \alpha > \dfrac{\pi}{3}$, $\gamma > \dfrac{\pi}{6}$ and consequently $\alpha + \beta + \gamma = \pi$ or $\alpha + \beta + t\delta = \pi$, for some $t \geq 1$.

1. Suppose that $\alpha + \beta + \gamma = \pi$. Then, the configuration in Figure 2 extends to the one illustrated in Figure 54 and tile 4 has two possible positions. Either way, the other sum of alternate angles satisfies $2\beta + \delta = \pi$, which is impossible since $\beta + \gamma + \delta > \pi$.

2. If $\alpha + \beta + t\delta = \pi$, for some $t \geq 1$ and since $\beta + \gamma + \delta > \pi$, one has $\gamma > \alpha > \dfrac{\pi}{3}$.

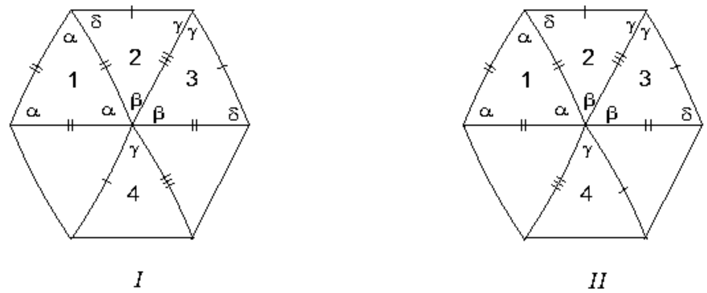

Figure 54: Local configuration.

Assuming that $\gamma \geq \frac{\pi}{2}$, then, $\beta > \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$, which contradicts the adjacency condition (1.1). Therefore, $\delta < \gamma < \frac{\pi}{2}, \alpha < \frac{\pi}{2}$ and consequently $\beta \geq \frac{\pi}{2}$.

2.1 Assume firstly, that $\beta = \frac{\pi}{2}$. The configuration in Figure 2 extends to the one below and a decision must be taken about the angle $\theta_1 \in \{\gamma, \delta\}$.

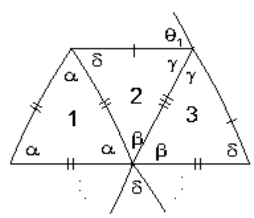

Figure 55: Local configuration.

2.1.1 If $\theta_1 = \gamma$, then the sum containing two angles γ is $2\gamma + \delta = \pi$, since $\gamma > \frac{\pi}{3}$ and $\gamma + \delta > \frac{\pi}{2}$. Extending the configuration above, tile 8 has two possible positions as is shown in Figure 56-I and II.

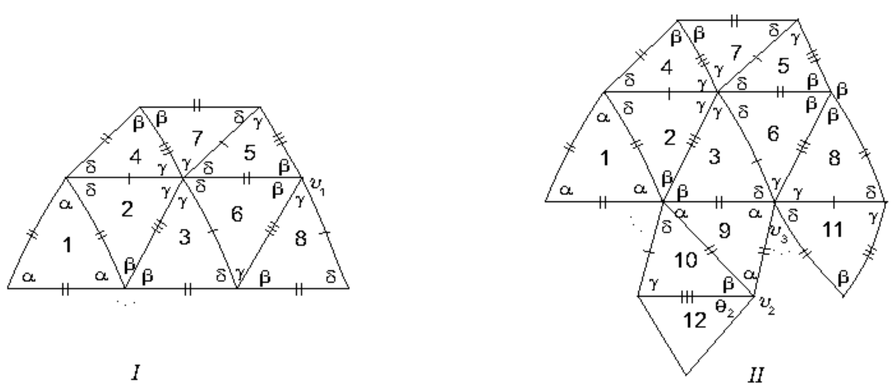

Figure 56: Local configuration.

In Figure 56-I, the sum of the alternate angles β and γ at vertex v_1 must satisfy $\beta + \gamma = \pi$, which is impossible since $\beta = \dfrac{\pi}{2}$ and $\gamma < \beta$.

In Figure 56-II, the angle θ_2 in tile 12 must be γ or β. If $\theta_2 = \gamma$, one of the sums of alternate angles at vertex v_2 satisfies $\alpha + \gamma + p\delta = \pi$ and the other $\beta + \delta + p\delta = \pi$, for some $p \geq 1$ (see Figure 57). From $\alpha + \gamma + p\delta = \beta + \delta + p\delta$, we conclude that $\alpha + \gamma = \beta + \delta$ and since $\gamma > \alpha$, then $\delta > \dfrac{\pi}{6}$, which contradicts the assumption $\alpha + \beta + t\delta = \pi,\ t \geq 1$. Therefore, $\theta_2 = \beta$.

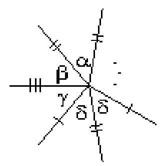

Figure 57: Angle arrangement at vertex v_2.

From the conditions, $\alpha + \beta + t\delta = \pi$ and $\alpha + \gamma + p\delta = \pi$, at vertex v_3, we get $\gamma + p\delta = \beta + t\delta$ and since $\gamma + \delta > \dfrac{\pi}{2}$, then $p < t + 1$. On the other hand, from the same assumptions and since $\gamma < \beta$, we get $t < p$. Therefore, $t < p < t + 1$, which is impossible.

2.1.2 If $\theta_1 = \delta$, the configuration in Figure 55 is now,

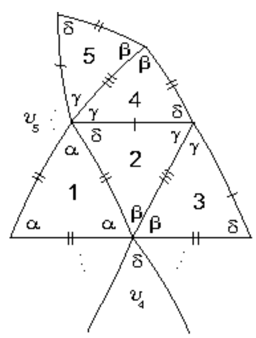

Figure 58: Local configuration.

Once again, by the conditions $\alpha + \beta + t\delta = \pi$, at vertex v_4 and $\alpha + \gamma + p\delta = \pi$, at vertex v_5, a contradiction is achieved.

2.2 Assuming that $\beta > \dfrac{\pi}{2}$ and since $\alpha + \delta < \alpha + \gamma < \alpha + \beta < \pi,\ 2\beta > \pi$ and $\delta < \gamma < \dfrac{\pi}{2}$, then vertices of valency four must be surrounded by alternate angles γ and β or δ and β.

2.2.1 Extending the same configuration in Figure 55 and if the angle labelled θ_1 is γ, then one has $2\gamma + q\delta = \pi$, for some $q \geq 1$ and we get the one illustrated in Figure 59.

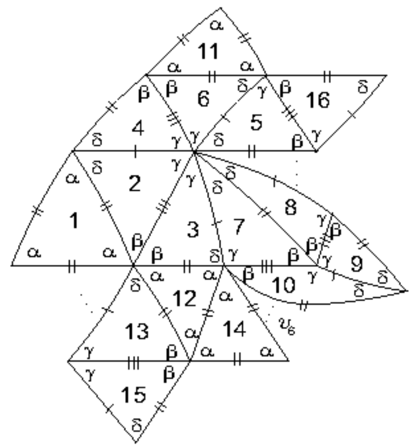

Figure 59: Local configuration.

Both sums of the alternate angles at vertex v_6 are $\alpha+\gamma+p\delta=\pi$ and $\alpha+\beta+p\delta=\pi$, which is a contradiction, since $\gamma<\beta$.

2.2.2 Suppose finally that $\theta_1=\delta$ (see Figure 55). The sum containing α and γ must satisfy $\alpha+\gamma+p\delta=\pi$, for some $p\geq 1$ and the configuration is the one illustrated in Figure 60.

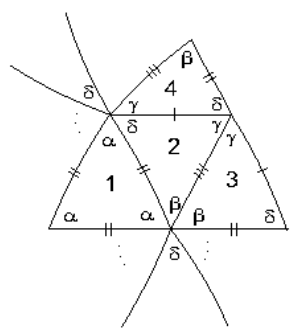

Figure 60: Local configuration.

From the sums $\alpha+\gamma+p\delta=\pi$ and $\alpha+\beta+t\delta=\pi$ assumed in **2.**, we get the same contradiction as in **2.1.1**.

$\square$

3 Symmetry Groups

Here we present the group of symmetries of the spherical f-tilings obtained $\mathcal{E}_\alpha, \mathcal{G}^k, k\geq 1, \mathcal{E}, \mathcal{F}, \mathcal{H}$ and $\mathcal{L}$. We also indicate the transitivity classes of isogonality and isohedrality.

In **Table 1** is shown a complete list of all spherical dihedral f-tilings, whose prototiles are an equilateral triangle T_1 of angle α and a scalene triangle T_2 of angles δ, γ, β, $(\delta<\gamma<\beta)$.

We have used the following notation:

- M and N are, respectively, the number of triangles congruent to T_1 and the number of triangles congruent to T_2 used in such dihedral f-tilings;
- $G(\tau)$ is the symmetry group of the f-tiling τ. The numbers of isohedrality-classes and isogonality-classes for the symmetry group are denoted, respectively, by # isoh. and # isog.;
- By C_n and D_n we denote, respectively, the cyclic group of order n and the dihedral group of order $2n$;
- $\beta = \beta_\alpha$, in f-tiling $\mathcal{E}_\alpha$, is given by

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\gamma + \cos\delta\cos\beta}{\sin\delta\sin\beta},$$

  with $\gamma = \pi - \alpha$ and $\delta = \pi - \beta$;
- $\delta = \delta_1^k$, in f-tiling $\mathcal{G}^k$, is the solution of

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\gamma + \cos\delta\cos\beta}{\sin\delta\sin\beta},$$

  with $\alpha = \pi - (2k+1)\delta, \gamma = (k+1)\delta$ and $\beta = \pi - (k+1)\delta$.

f-tiling	α	δ	γ	β	$G(\tau)$	# isoh.	# isog.
$\mathcal{E}_\alpha$	$]\frac{\pi}{2},\pi[$	$\pi-\beta$	$\pi-\alpha$	β_α	D_3	2	1
$\mathcal{G}^k, k\geq 1$	$\pi-(2k+1)\delta$	δ_1^k	$(k+1)\delta$	$\pi-(k+1)\delta$	D_3	$2k+2$	$k+1$
$\mathcal{E}$	72.75°	53.6°	$\frac{\pi}{3}$	107.25°	$C_2\times D_3$	2	3
$\mathcal{F}$	$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{\pi}{3}$	$\frac{3\pi}{5}$	D_6	3	3
$\mathcal{H}$	70.52°	24.74°	$\frac{\pi}{3}$	109.48°	D_6	4	3
$\mathcal{L}$	65.56°	34.72°	$\frac{\pi}{4}$	114.44°	D_8	4	3

Table 1: The Combinatorial Structure of the Dihedral f-Tilings of the Sphere by Equilateral and Scalene Triangles with adjacency of type III

In **Table 2** is shown a complete list of all spherical dihedral f-tilings, whose prototiles are an equilateral triangle T_1 of angle α and a scalene triangle T_2 of angles δ, γ, β, $(\delta < \gamma < \beta)$.

We have used the following notation.

- The angles δ and γ, in f-tiling $\mathcal{F}_1^\delta$, obey

$$\frac{-\cos(\gamma+\delta)}{1+\cos(\gamma+\delta)} = \frac{\cos\gamma\,\cos\delta}{\sin\gamma\,\sin\delta}.$$

  Besides, $\alpha = \pi - (\gamma + \delta)$;

- $\delta_0 \approx 54.74^\circ$ and $\gamma_0 \approx 70.53^\circ$;

- $\alpha = \alpha_1^p$, $p \geq 4$, in f-tiling $\mathcal{D}^p$, is the solution of

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\delta + \cos\gamma\cos\beta}{\sin\gamma\sin\beta},$$

  with $p\delta = \pi$, $\beta = \pi - \alpha$ and $\gamma = \pi - 2\alpha$;

- $\alpha = \alpha_2^p$, $p \geq 4$, in f-tiling $\mathcal{F}^p$, is the solution of

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\delta + \cos\gamma\cos\beta}{\sin\gamma\sin\beta},$$

  with $p\delta = \pi$, $\beta = \pi - \alpha$ and $\gamma = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$;

- $\alpha = \alpha_3^m$, $m \geq 5$, in f-tiling $\mathcal{E}^m$, is the solution of

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\delta + \cos\gamma\cos\beta}{\sin\gamma\sin\beta},$$

  with $m\delta = \pi$, $\beta = \pi - \alpha$ and $\gamma = \dfrac{\pi}{2} - \dfrac{\alpha}{2} - \dfrac{\pi}{2m}$;

- $\beta = \beta_\alpha$, in f-tiling $\mathcal{E}_\alpha$, is given by

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\gamma + \cos\delta\cos\beta}{\sin\delta\sin\beta},$$

  with $\gamma = \pi - \alpha$ and $\delta = \pi - \beta$;

- $\delta = \delta_1^k$, in f-tiling $\mathcal{G}^k$, is the solution of

$$\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{\cos\gamma + \cos\delta\cos\beta}{\sin\delta\sin\beta},$$

  with $\alpha = \pi - (2k+1)\delta, \gamma = (k+1)\delta$ and $\beta = \pi - (k+1)\delta$.

f-tiling	α	δ	γ	β	M	N
$\mathcal{F}_1^{\delta}$	$]\gamma_0, \frac{\pi}{2}[$	$]0, \delta_0[$	$]\delta_0, \frac{\pi}{2}[$	$\frac{\pi}{2}$	8	24
$\mathcal{F}_2^{\delta}$	$]\gamma_0, \frac{\pi}{2}[$	$]0, \delta_0[$	$]\delta_0, \frac{\pi}{2}[$	$\frac{\pi}{2}$	8	24
$\mathcal{D}^p, p \geq 4$	α_1^p	$\frac{\pi}{p}$	$\pi - 2\alpha$	$\pi - \alpha$	$4p$	$4p$
$\mathcal{F}^p, p \geq 4$	α_2^p	$\frac{\pi}{p}$	$\frac{\pi}{2} - \frac{\alpha}{2}$	$\pi - \alpha$	$2p$	$4p$
$\mathcal{E}^m, m \geq 5$	α_3^m	$\frac{\pi}{m}$	$\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2m}$	$\pi - \alpha$	$4m$	$8m$
$\mathcal{E}_\alpha$	$]\frac{\pi}{2}, \pi[$	$\pi - \beta$	$\pi - \alpha$	β_α	2	6
$\mathcal{G}^k, k \geq 1$	$\pi - (2k+1)\delta$	δ_1^k	$(k+1)\delta$	$\pi - (k+1)\delta$	2	$6(2k+1)$
$\mathcal{E}$	$72.75°$	$53.63°$	$\frac{\pi}{3}$	$107.25°$	6	12
$\mathcal{F}$	$\frac{2\pi}{5}$	$\frac{\pi}{5}$	$\frac{\pi}{3}$	$\frac{3\pi}{5}$	12	12
$\mathcal{H}$	$70.52°$	$24.74°$	$\frac{\pi}{3}$	$109.48°$	12	24
$\mathcal{L}$	$65.56°$	$34.72°$	$\frac{\pi}{4}$	$114.44°$	16	32

Table 2: Dihedral f-Tilings of the Sphere by Equilateral and Scalene Triangles

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