A Traceability Conjecture for Oriented Graphs

Marietjie Frick* and Susan A van Aardt†

Department of Mathematical Sciences,
University of South Africa, South Africa

{frickm, vaardsa}@unisa.ac.za

Jean E Dunbar

Converse College, South Carolina, USA

jean.dunbar@converse.edu

Morten H Nielsen and Ortrud R Oellermann‡

Department of Mathematics and Statistics,
University of Winnipeg, Canada

{m.nielsen, o.oellermann}@uwinnipeg.ca

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Abstract

A (di)graph G of order n is k-traceable (for some k, $1 \leq k \leq n$) if every induced sub(di)graph of G of order k is traceable. It follows from Dirac's degree condition for hamiltonicity that for $k \geq 2$ every k-traceable graph of order at least $2k-1$ is hamiltonian. The same is true for strong oriented graphs when $k = 2, 3, 4$, but not when $k \geq 5$. However, we conjecture that for $k \geq 2$ every k-traceable oriented graph of order at least $2k-1$ is traceable. The truth of this conjecture would imply the truth of an important special case of the Path Partition Conjecture for Oriented Graphs. In this paper we show the conjecture is true for $k \leq 5$ and for certain classes of graphs. In addition we show that every strong k-traceable oriented graph of order at least $6k-20$ is traceable. We also characterize those graphs for which all walkable orientations are k-traceable.

Keywords: Longest path, oriented graph, k-traceable, Path Partition Conjecture, Traceability Conjecture.

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1 Introduction

Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of G is called its *order* and the number of edges is called its *size* and are denoted by $n(G)$ and $m(G)$, respectively. Where no confusion arises we will suppress the G. For any nonempty set $X \subseteq V(G)$, $\langle X \rangle$ denotes the subgraph of G induced by X.

A graph G containing a cycle (path) through every vertex is said to be *hamiltonian* (*traceable*).

The *detour order* of G, denoted by $\lambda(G)$ (as in [3], [20] and [21]), is the order of a longest path in G. The *detour deficiency* of G is defined as $p(G) = n(G) - \lambda(G)$. A graph with detour deficiency p is called *p-deficient*. Thus a graph is traceable if and only if it is 0-deficient.

A graph G of order n is *k-traceable* (for some k, $1 \leq k \leq n$) if every induced subgraph of G of order k is traceable. Every graph is 1-traceable, but a graph is 2-traceable if and only if it is complete, and a graph of order n is n-traceable if and only if it is traceable.

The above concepts are defined analogously for digraphs. Often, a directed path (directed cycle, directed walk) will simply be called a path (cycle, walk).

We use $A(D)$ to denote the arc set of a digraph D. If v is a vertex in a digraph D, we denote the sets of *out-neighbours* and *in-neighbours* of v by $N^+(v)$ and $N^-(v)$ and the cardinalities of these sets by $d^+(v)$ and $d^-(v)$, respectively.

The *degree* of v in D is defined as $\deg(v) = d^+(v) + d^-(v)$ and the *minimum degree* of D is $\delta(D) = \min_{v\in V(D)} \deg(v)$. If H is a subdigraph of D, then $N^+(H) = \bigcup_{v\in V(H)} N^+(v)$. If S is a subdigraph of D or a set of vertices in D, we denote the out-neighbours of H that lie in S by $N_S^+(H)$. Similar notation is used with respect to in-neighbours.

A digraph is *traceable from* (*to*) $x \in V(D)$ if D has a hamiltonian path starting (ending) at x. A digraph D is *walkable* if it contains a walk that visits every vertex. A digraph D is (*dis*)*connected* if its underlying graph is (dis)connected and it is called *strong* (or *strongly connected*) if every vertex of D is reachable from every other vertex. Thus a nontrivial digraph D is strong if and only if it contains a closed walk that visits every vertex. A maximal strong subdigraph of a digraph D is called a *strong component* of D. The *components* of a digraph D are the components of its underlying graph G. The strong components of a digraph have an *acyclic ordering*, i.e. they may be labelled $D_1, \ldots, D_t$ such that if there is an arc from D_i to D_j, then $i \leq j$ (cf. [1], p. 17).

An oriented graph is a digraph that is obtained from a simple graph by assigning a direction to each edge. In this paper we concentrate on *oriented graphs*, though some of our results hold for digraphs in general. Section 3 gives a characterization of those graphs for which all walkable orientations are k-traceable.

Thomassen [24] showed that for every $k \geq 42$ there exists a k-traceable graph of order $k+1$ that is nontraceable. (Such graphs are called *hypotraceable.*) However, for $k \geq 2$ all k-traceable graphs of sufficiently large order are hamiltonian, as shown by the following result.

Proposition 1.1 *Let $k \geq 2$ and suppose G is a is k-traceable graph of order at least $2k-1$. Then G is hamiltonian.*

Proof. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian, by Dirac's degree condition for hamiltonicity (see [8]). Now suppose G has a vertex x such that $\deg(x) \leq \frac{n-1}{2}$. Then $|V(G) \setminus N(x)| \geq k$. Now let H be an induced subgraph of G such that $x \in V(H) \subseteq V(G) \setminus N(x)$ and $n(H) = k$ Then x is an isolated vertex in H, so H is not traceable and hence G is not k-traceable. ■

For digraphs, the situation is very different, even in the case of strong oriented graphs. In Section 4 we show that the analogue of Proposition 1.1 for strong oriented graphs is true when $k = 2, 3, 4$, but not when $k \geq 5$. In fact, we construct, for every $n \geq 5$, a nonhamiltonian strong oriented graph of order n that is k-traceable for every $k \in \{5, \ldots, n\}$.

For every $k \geq 6$ Grötschel et al. [17] constructed a k-traceable oriented graph of order $k+1$ that is nontraceable. However, we show that, for $k \geq 2$, every strong k-traceable oriented graph of order at least $6k - 20$ is traceable. It is therefore natural to ask: for given $k \geq 2$, what is the largest value of n such that there exists a k-traceable oriented graph of order n that is nontraceable? We formulate the following conjecture.

Conjecture 1 (The Traceability Conjecture (TC)) *For every integer $k \geq 2$, every k-traceable oriented graph D of order at least $2k - 1$ is traceable.*

This conjecture was motivated by the Path Partition Conjecture for 1-deficient oriented graphs, which is discussed in Section 2.

The TC asserts that every nontraceable oriented graph of order n has a nontraceable induced subdigraph of order k for each $k \in \{2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$.

In Section 5 we prove that the Traceabilty Conjecture holds for certain classes of oriented graphs and that it holds in general for $k = 2, 3, 4, 5$.

2 Background and motivation

Our interest in k-traceable graphs and digraphs arose from investigations into the Path Partition Conjecture (PPC) and its directed versions. We briefly sketch the background of these conjectures.

Throughout the paper a and b will denote positive integers. A vertex partition (A, B) of a graph G is an *(a, b)-partition* if $\lambda(\langle A \rangle) \leq a$ and $\lambda(\langle B \rangle) \leq b$. If G has an (a, b)-partition for every pair (a, b) such that $a + b = \lambda(G)$, then G is *λ-partitionable.* The PPC can be formulated as follows.

Conjecture 2 (Path Partition Conjecture (PPC)) *Every graph G is λ-partitionable.*

The PPC is a well-known, long-standing conjecture. It originated from a discussion between L. Lovász and P. Mihók in 1981 and was subsequently treated in the theses [18] and [25]. It first appeared in the literature in 1983, in a paper by Laborde, Payan and Xuong [21]. In [4] it is stated in the language of the theory of hereditary properties of graphs. It is also mentioned in [6].

The analogous conjecture for digraphs is called the DPPC and its restriction to oriented graphs is called the OPPC. In 1995 Bondy [3] stated a seemingly stronger version of the DPPC, requiring $\lambda(\langle A\rangle) = a$ and $\lambda(\langle B\rangle) = b$.

Results on the PPC and its relationship with other conjectures appear in [5], [9], [10], [11], [13] and [15]. Results on the DPPC appear in Laborde et al. [21], Havet [19], Frick et al. [14], and Bang-Jensen et al. [2].

Every graph may be regarded as a symmetric digraph with the same detour order, by replacing every edge with two oppositely directed arcs, and so the truth of the DPPC would imply the truth of the PPC and the OPPC. However, the relationship between the PPC and the OPPC is not clear.

The PPC has been proved for all graphs with detour deficiency $p \leq 3$. For $p \geq 4$ it has been proved for all p-deficient graphs of order at least $10p^2 - 3p$ (see [5],[15]). We present here an easy proof for the case $p = 1$, which relies on Proposition 1.1.

Proposition 2.1 *Every 1-deficient graph is λ-partitionable.*

Proof. Let G be a 1-deficient graph of order n and consider a pair of positive integers, a, b such that $a + b = \lambda(G) = n - 1$. We assume $a \leq b$. Then $a + 1 = n - b \leq \lceil \frac{n}{2} \rceil$. Since G is nonhamiltonian, it follows from Proposition 1.1 that G has a nontraceable induced subgraph H of order $a + 1$. But then $\lambda(H) \leq a$ and $|V(G) \setminus V(H)| = b$, so $(V(H), V(G) \setminus V(H))$ is an (a, b)-partition of G. ■

However, the restriction of the OPPC to 1-deficient oriented graphs has not yet been settled and it seems difficult and interesting enough to be formulated as a separate conjecture. We call it the OPPC1.

Conjecture 3 (OPPC1) *Every 1-deficient oriented graph is λ-partitionable.*

The OPPC1 may be formulated in terms of traceability, as follows.

Conjecture 4 (Alternative form of OPPC1) If D is a 1-deficient oriented graph of order $n = a + b + 1$, then D is not $(a + 1)$-traceable or D is not $(b + 1)$-traceable.

It is clear from the above formulation that the truth of the Traceabilty Conjecture will imply the truth of the OPPC1.

3 Graphs for which all walkable orientations are k-traceable

Grötschel and Harary [16] characterized those graphs for which all strong orientations are hamiltonian. Fink and Lesniak-Foster [12] studied the structure of graphs having the property that all walkable orientations are traceable. They showed that if G is obtained from a complete graph of order at least 4 by deleting the edges of a vertex-disjoint union of paths each of length 1 or 2, then every walkable orientation of D is traceable. Graphs for which all strong orientations are eulerian were characterized in [23]. In view of the next result we are mainly interested in walkable orientations of graphs.

Proposition 3.1 *Let D be an oriented graph of order n which is k-traceable for some $k \in \{2, \dots, n\}$. Then D is walkable.*

Proof. If D is a nonwalkable oriented graph, then D has two vertices x and y such that no path in D contains both x and y. But then every subdigraph of D containing both x and y is nontraceable. ■

We characterize here those graphs of order n for which all walkable orientations are k-traceable for some $k \in \{2, 3, \dots, n\}$.

Proposition 3.2 *The complete graph K_n is the only graph G of order n having the following properties:*

(1) *G has a walkable orientation and*

(2) *every walkable orientation of G is k-traceable for some $k \in \{2, 3, \dots, \lceil \frac{n}{2} \rceil\}$.*

Proof. Every orientation of K_n is a tournament, hence k-traceable for all $k \in \{2, 3, \dots, n\}$, so K_n certainly has the claimed properties.

Suppose $G \neq K_n$ is a graph of order n satisfying (1) and (2). Note that $n \geq 4$ and therefore $n - 2 \geq \lceil \frac{n}{2} \rceil$. Let x and y be two independent vertices of G.

Since G has a k-traceable orientation for some $k \in \{2, 3, \dots, \lceil \frac{n}{2} \rceil\}$, G itself is k-traceable, hence hamiltonian by Proposition 1.1. Therefore, G has a hamiltonian cycle C, and it is easy to construct an orientation D of G in which C is a (directed) hamiltonian cycle of D and with $d^-(x) = d^-(y) = 1$. Let x^- and y^- be the in-neighbours of x and y, respectively, in D. Then $D - \{x^-, y^-\}$ is not walkable, and hence not k-traceable for any $k \in \{2, 3, \dots, n-2\}$. Hence, D is not k-traceable for any k. This contradicts the assumption that G satisfies (2), so such a G cannot exist. ■

4 Hamiltonicity and traceability of strong, k-traceable oriented graphs

It is well-known that every tournament is traceable and every strong local tournament is hamiltonian [1]. (A digraph is a *local tournament* if, for every vertex v, each of $\langle N^-(v) \rangle$ and $\langle N^+(v) \rangle$ is a tournament.) Chen and Manalastas [7] also proved the following result for strong digraphs.

Theorem 4.1 *(Chen and Manalastas) If D is a strong digraph with $\alpha(D) \leq 2$, then D is traceable.*

Havet [19] strengthened this result to the following.

Theorem 4.2 *(Havet) If D is a strong digraph with $\alpha(D) = 2$ then D has two nonadjacent vertices that are end vertices of hamiltonian paths in D and two nonadjacent vertices that are initial vertices of hamiltonian paths in D.*

We shall often use the following result on the minimum degree of k-traceable oriented graphs.

Lemma 4.3 *Let D be an oriented graph of order n which is k-traceable for some $k \leq n$. Then $\delta(D) \geq n - k + 1$.*

Proof. Suppose, to the contrary, that D has a vertex x with $\deg(x) \leq n - k$. Then $|V(D) \setminus N(x)| \geq k$. Let H be an induced subdigraph of D such that $V(H)$ consist of x together with $k-1$ other vertices in $V(D) \setminus N(x)$. Then H has order k and H is nontraceable. Hence D is not k-traceable. ■

We now prove that for strong oriented graphs the analogue of Proposition 1.1 holds for $k = 2, 3, 4$.

Theorem 4.4 *For $k = 2, 3$ or 4, every strong k-traceable oriented graph of order at least $k+1$ is hamiltonian.*

Proof. If $k = 2$ or 3, then D is a strong local tournament and hence is hamiltonian.

Now let D be a strong 4-traceable oriented graph of order $n \geq 5$. Since a strong tournament is hamiltonian, we may assume $\delta(D) \leq n-2$ and hence, by Lemma 4.3, $\delta(D) = n-3$ or $n-2$.

Suppose first that $\delta(D) = n - 3$. Let x be a vertex of degree $n-3$ and let $\{y, z\} = V(D) \setminus N[x]$. If $v \in N^+(x)$, then $\langle \{x, v, y, z\} \rangle$ is traceable, so we may assume, without loss of generality, that $yz \in A(D)$. Then every vertex in $N^+(x)$ is adjacent to y. Furthermore, if v and w are two distinct vertices in $N^+(x)$, then $\langle \{x, v, w, y, \} \rangle$ is traceable, so v and w are adjacent. Thus $\langle N^+(x) \rangle$ is a tournament and hence has a hamiltonian path P. Similarly, every vertex in $N^-(x)$ is adjacent from z and $\langle N^-(x) \rangle$ has a hamiltonian path Q. Thus $xPyzQx$ is a hamiltonian cycle of D.

Now suppose $\delta(D) = n-2$. Let c be the circumference of D and let $C = v_1 v_2 \ldots v_c v_1$ be a longest cycle in D. Suppose $c \leq n-1$ and let $x \in V(G) \setminus C$. Suppose that no vertex of C is adjacent to x. Then all vertices of C (except possibly one) are adjacent from x. Since D is strongly connected there is some shortest path P from C to x. Suppose P is a $v_i - x$ path. Then x is adjacent to at least one of v_{i+1} and v_{i+2}, where the subscripts are modulo c. Let $j \in \{i+1, i+2\}$ be such that xv_j is an arc of D. Then, since P is a path of order at least 3, $Pv_j v_{j+1} \ldots v_i$ is a cycle longer than C. So at least one vertex of C is adjacent to x and, similarly, at least one vertex of C is adjacent from x.

We may also assume that at least one vertex of C is nonadjacent with x; otherwise, there will be a vertex v_j on C such that $v_j \in N^-(x)$ and $v_{j+1} \in N^+(x)$, producing a cycle longer than C. Hence, since $deg(x) \geq n-2$, exactly one vertex of C, say v_1, is nonadjacent with x and $N_C^+(x) = \{v_2, \ldots, v_\ell\}$ for some index ℓ, $2 \leq \ell < c$; otherwise the maximality of the order of C is contradicted. Further, since $deg(v_1) \leq n-2$, we note that v_1 is a universal vertex of $\langle V(C) \rangle$.

Next we note that v_1 and x have a common out-neighbour, v_2, and a common in-neighbour, v_c. If v_1 and x have a common out-neighbour, $v_k \neq v_2$, then $\langle \{v_1, x, v_2, v_k\} \rangle$ is nontraceable. Consequently, v_2 is the only common out-neighbour of x and v_1. Similarly,

v_c is the only common in-neighbour of x and v_1. Thus if $3 \le j \le c-1$, then $v_j \in N^+(x)$ if and only if $v_j \in N^-(v_1)$. Moreover, since $\langle\{x, v_1, v_2, v_c\}\rangle$ is traceable, $v_2v_c \in A(D)$.

Now, if $2 < \ell < c-1$ then, since l is the largest integer such that $v_\ell \in N^+(x)$, it follows that $v_\ell \in N^-(v_1)$ and $v_{\ell+1} \in N^+(v_1)$. But then the cycle $xv_2 \ldots v_\ell v_1 v_{\ell+1} v_{\ell+2} \ldots v_c x$ is longer than C.

If $\ell = 2$, then $v_3 \in N^+(v_1)$ and $xv_2v_cv_1v_3 \ldots v_{c-1}x$ is a cycle longer than C.

It is not difficult to show that $c > 3$ and so, if $\ell = c-1$, then $v_{c-1} \in N^-(v_1)$ and $v_1v_2v_cxv_3 \ldots v_{c-1}v_1$ is a cycle longer than C. ■

A strong 5-traceable oriented graph need not be hamiltonian. Fig. 1 depicts a strong 5-traceable oriented graph of order 6 that is nonhamiltonian.

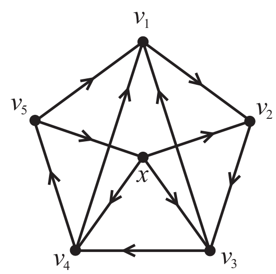

Figure 1: A strong 5-traceable oriented graph of order 6 that is nonhamiltonian.

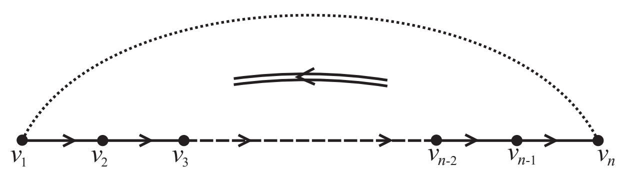

Figure 2: A strong k-traceable oriented graph that is nonhamiltonian.

Other nonhamiltonian strong 5-traceable oriented graphs are obtained from the graph in Fig. 1 by adding some or all of the arcs v_5v_2, v_5v_3 and v_4v_2. However, v_1 and x remain nonadjacent. Nielsen [22] generalized this construction to prove the following.

Theorem 4.5 *For every $n \ge 5$, there exists a strong nonhamiltonian oriented graph of order n that is k-traceable for every $k \in \{5, 6, \ldots, n\}$.*

Proof. Let T be a transitive tournament of order $n \ge 5$ with source vertex s and sink vertex t. Obtain D from T by removing the arc st and by reversing the arcs of the (unique) hamiltonian path of T. This strong oriented graph D is depicted in Fig. 2 with $s = v_n$ and $t = v_1$.

Suppose D has a hamiltonian cycle C. Then, since v_2 is the only out-neighbour of v_1, the cycle C contains the arc v_1v_2, and hence also the arcs $v_2v_3, v_3v_4, \ldots, v_{n-2}v_{n-1}$ and $v_{n-1}v_n$. But v_n is not adjacent to v_1. Thus D is nonhamiltonian.

Now let $k \in \{5, 6, \ldots, n\}$ and let H be a subdigraph of D of order k. If H does not contain both v_1 and v_n, then H is a tournament and hence is traceable. Now assume $v_1, v_n \in V(H)$ and let $P = u_1 \ldots u_{k-2}$ be a hamiltonian path of the tournament $H - \{v_1, v_n\}$. Since v_n has in-degree one in D and v_n is a universal vertex in $H - v_1$, either $v_n u_1 \ldots u_{k-2}$ or $u_1 v_n u_2 \ldots u_{k-2}$ is a hamiltonian path of $H - v_1$. Since $k \geq 5$, every subdigraph $H - v_1$ thus contains a hamiltonian path $P' = w_1 \ldots w_{k-1}$ with $v_n \neq w_{k-2}, w_{k-1}$. Similarly, v_1 has out-degree one, so either $w_1 \ldots w_{k-1} v_1$ or $w_1 \ldots w_{k-2} v_1 w_{k-1}$ is a hamiltonian path in H. ■

The graph D_0 constructed by Whitehead in [26], Theorem 2.3, also has the properties required to prove Theorem 4.5. Although that graph as well as the graphs constructed in Theorem 4.5 are nonhamiltonian, they are traceable. We shall prove that all strong k-traceable oriented graphs of sufficiently large order are traceable. The proof relies on the following result.

Theorem 4.6 *If D is a k-traceable oriented graph of order at least $6k - 20$, then D has independence number at most 2.*

Proof. Suppose $S = \{x_1, x_2, x_3\}$ is a set of three independent vertices in D and let $W = V(D) \backslash S$. Let $A_i = W \backslash N^-(x_i)$ and $B_i = W \backslash N^+(x_i)$; $i = 1, 2, 3$. Then $A_i \cup B_i = W$; $i = 1, 2, 3$. Now let i, j be any pair of distinct integers in $\{1, 2, 3\}$. If $|A_i \cap A_j| \geq k - 3$, let H be an induced subdigraph of D, whose vertex set consists of x_1, x_2, x_3 and $k - 3$ vertices of $A_i \cap A_j$. Then H has order k and is nontraceable, since both x_i and x_j have no in-neighbours in H. Hence $|A_i \cap A_j| \leq k - 4$. Similarly, $|B_i \cap B_j| \leq k - 4$. Now suppose $|A_1 \cap B_2| \geq 2k - 7$. Then, since $|A_1 \cap A_3| \leq k - 4$, at least $k - 3$ vertices of $A_1 \cap B_2$ are in B_3, but then $|B_2 \cap B_3| \geq k - 3$. This contradiction proves that $|A_1 \cap B_2| \leq 2k - 8$. But $A_1 = (A_1 \cap A_2) \cup (A_1 \cap B_2)$. Hence $|A_1| \leq (k - 4) + 2k - 8 = 3k - 12$. Similarly, $|B_1| \leq 3k - 12$. But $V(D) = A_1 \cup B_1 \cup \{x_1, x_2, x_3\}$, so $n(D) \leq (3k - 12) + (3k - 12) + 3 = 6k - 21$. ■

Theorems 4.1 and 4.6 imply the following:

Corollary 4.7 *If $k \geq 2$ and D is a strong k-traceable oriented graph of order at least $6k - 20$, then D is traceable.*

We now show that the case $k \leq 5$ of the TC holds for strong oriented graphs.

Corollary 4.8 *For each $k \in \{2, 3, 4, 5\}$, every strong k-traceable oriented graph of order at least $2k - 1$ is traceable.*

Proof. Theorem 4.4 proves the cases $k = 2, 3, 4$. Corollary 4.7 proves that every strong 5-traceable oriented graph of order at least 10 is traceable. Now let D be a strong 5-traceable oriented graph of order 9. Suppose D is not traceable. Then, by Theorem 4.1, D has an independent set of vertices $S = \{x_1, x_2, x_3\}$. Now define the sets A_i and B_i as in Theorem 4.6. Then it follows from the proof of Theorem 4.6 that $|A_1| \leq 3$ and $|B_1| \leq 3$, so $|A_1| = |B_1| = 3$ and $A_1 \cap B_1 = \emptyset$. Hence, since D is 5-traceable, $\{x_1, x_2\} \cup A_1$ has a hamiltonian path starting at x_1 and $\{x_1, x_3\} \cup B_1$ has a hamiltonian path ending at x_1. Thus D is traceable. ∎

Corollary 4.8 implies that the case $k \leq 5$ of the TC holds for strong oriented graphs. In Section 5 we shall show that the case $k \leq 5$ of the TC holds in general.

5 The Traceability Conjecture

In this section we deduce some properties of k-traceable oriented graphs and then use these to prove that the TC holds for certain classes of oriented graphs and also that the TC holds for $k \leq 5$. From these results we deduce new results concerning the OPPC1.

The following useful result follows immediately from the fact that the strong components of an oriented graph have an acyclic ordering.

Lemma 5.1 *If P is a path in a digraph D, then the intersection of P with any strong component of D is either empty or a path.*

In view of Proposition 3.1 we restrict our attention to *walkable* oriented graphs when investigating the TC. Suppose D is a walkable oriented graph with h strong components. Then the strong components have a unique acyclic ordering $D_1, \ldots, D_h$ such that if there is an arc from D_i to D_j then $i \leq j$ and there is at least one arc from D_i to D_{i+1} for $i = 1, \ldots, h-1$. Throughout the paper we shall label the strong components of a walkable oriented graph D in accordance with this unique acyclic ordering and denote the subdigraph of D induced by all the vertices in the strong components $D_r, D_{r+1}, \ldots, D_s$ by D_r^s, i.e.

$$D_r^s = \left\langle \bigcup_{i=r}^{s} V\left(D_i\right) \right\rangle .$$

Our next result gives a lower bound on the order of a strong component that is not a tournament in a k-traceable oriented graph.

Lemma 5.2 *Let $k \geq 2$ and let D be a k-traceable oriented graph of order $n \geq k$ with strong components $D_1, \ldots, D_h$. If D_i is not a tournament for some $i \in \{1, \ldots, h\}$, then $n(D_i) \geq n - k + 3$.*

Proof. Suppose $n(D_i) \leq n-k+2$. Then $n(D - V(D_i)) \geq n-(n-k+2) = k-2$. Now let H be an induced subdigraph of D such that H contains $k-2$ vertices of $D - V(D_i)$ and two nonadjacent vertices of D_i. Then it follows from Lemma 5.1 that H is nontraceable, contrary to our assumption that D is k-traceable. ∎

Next we consider the structure of k-traceable oriented graphs of sufficiently large order.

Lemma 5.3 *Let $k \geq 2$ and let D be a k-traceable oriented graph of order $n \geq 2k-5$ with strong components $D_1, \ldots, D_h$. Then for every positive integer $i \leq h-1$ at least one of the digraphs D_1^i and D_{i+1}^h is a tournament.*

Proof. Suppose, to the contrary, that for some $i \leq h-1$ neither D_1^i nor D_{i+1}^h is a tournament. Since $n \geq 2k-5$, one of D_1^i and D_{i+1}^h, say D_1^i, has at least $k-2$ vertices. Let H be an induced subdigraph of D such that H contains $k-2$ vertices of D_1^i together with any two nonadjacent vertices of D_{i+1}^h. Then it follows from Lemma 5.1 that H is nontraceable, contrary to the hypothesis. ■

For oriented graphs whose nontrivial strong components are all hamiltonian a slightly stronger result than the TC holds.

Theorem 5.4 *If $k \geq 2$ and D is a k-traceable oriented graph of order $n \geq 2k-3$ such that every nontrivial strong component of D is hamiltonian, then D is traceable.*

Proof. Let the strong components of D be $D_1, \ldots D_h$. Suppose D is nontraceable. Then, since each D_i is hamiltonian or a singleton, it is clear that $h \geq 3$. If D_1^{h-1} is a tournament, then every vertex in D_{h-1} is an end vertex of some hamiltonian path of D_1^{h-1}. Since D_h is hamiltonian or a singleton, this would imply that D is traceable. Thus D_1^{h-1} is not a tournament.

Let i be the smallest positive integer such that D_1^i is not a tournament. Then $i \leq h-1$ and it follows from Lemma 5.3 that D_{i+1}^h is a tournament. Hence $i > 1$ (otherwise D would be traceable) and D_1^{i-1} is a tournament by the minimality of i.

Since D is walkable, there exist vertices $y \in V(D_i)$ and $w \in V(D_{i+1})$ such that $yw \in A(D)$, and since D_i is either hamiltonian or a singleton, y is the endvertex of a hamiltonian path $P = x \ldots y$ in D_i. Suppose x has an in-neighbour $v \in V(D_{i-1})$. Since D_1^{i-1} and D_{i+1}^h are tournaments, v is the endvertex of a hamiltonian path P' in D_1^{i-1} and w is the initial vertex of a hamiltonian path P'' in D_{i+1}^h and so $P'PP''$ is a hamiltonian path of D, a contradiction. Therefore, some vertex of D_i has no in-neighbour in D_{i-1}, i.e. $N_{D_i}^+(D_{i-1}) \neq V(D_i)$ and D_i is not a singleton.

If $|N_{D_i}^+(D_{i-1})| \leq n-k$ then $n(D - N_{D_i}^+(D_{i-1})) \geq k$. Let H be an induced subdigraph of $D - N_{D_i}^+(D_{i-1})$ such that H has k vertices, of which at least one is in D_{i-1} and at least one in D_i. Since D_{i-1} has no out-neighbours in $V(H) \cap D_i$, it follows that H is nontraceable.

Hence $|N_{D_i}^+(D_{i-1})| \geq n-k+1 \geq (2k-3)-k+1 = k-2$. Now let $C : v_1 v_2 \ldots v_c v_1$ be a hamiltonian cycle of D_i. Then, for every $v_j \in N_C^+(D_{i-1})$, its predecessor, v_{j-1}, on C is not in $N_C^-(D_{i+1})$. Let H be a subdigraph of D induced by a set of $k-2$ of these predecessors, together with one vertex from D_{i-1} and one from D_{i+1}. Then H has order k but is nontraceable, since D_{i+1} has no in-neighbours in $V(H) \cap V(C)$. ■

We are now ready to prove the TC for $k \leq 5$.

Corollary 5.5 *If $k \in \{2,3,4,5\}$ and D is a k-traceable oriented graph of order at least $2k-1$, then D is traceable.*

Proof. Suppose, to the contrary, that D is nontraceable. By Theorem 5.4, D has a nontrivial strong component X that is nonhamiltonian. By Corollary 4.8 and Lemma 5.2, $n-k+3 \leq n(X) \leq n-1$. Hence $k \geq 4$ and $n(X) \geq n-k+3 \geq (2k-1)-k+3 \geq k+2 \geq 6$. It therefore follows from Theorem 4.4 that X is not 4-traceable, so $k=5$. Now choose any 4 vertices from X that induce a nontraceable subdigraph of X. Then these vertices together with any vertex from $V(D) - X$ induce a nontraceable subdigraph of D that is of order 5. ■

It follows from Theorem 5.4 that the TC holds for all oriented graphs with the property that every two cycles are vertex disjoint. In particular, it holds for unicyclic oriented graphs.

Lemma 4.3 implies that the TC holds (vacuously) for every oriented graph D satisfying $\delta(D) \leq \frac{n(D)}{2}$. Our next result shows that it also holds for oriented graphs with sufficiently large minimum degree.

Theorem 5.6 *If $k \geq 2$ and D is a k-traceable oriented graph of order $n \geq 2k - 3$ such that $\delta(D) \geq n - 2$, then D is traceable.*

Proof. Suppose D is nontraceable. Then D is not a tournament and therefore we may assume that $\delta(D) = n - 2$ and hence $\alpha(D) = 2$. Therefore, from Theorem 4.1, it follows that D is not strong. Let $D_1, \ldots, D_h$ be the strong components of D. By Theorem 5.4 there exists a nontrivial strong component D_i, $1 \leq i \leq h$, such that D_i is nonhamiltonian and by Theorem 4.2 there exist two nonadjacent vertices x and y in $V(D_i)$ such that both are end vertices of hamiltonian paths in D_i.

First we show that $2 \leq i \leq h - 1$. Suppose $i = 1$. Then Lemma 5.3 implies that D_2^h is a tournament. Hence, if $v \in V(D_2)$, then v is an initial vertex of a hamiltonian path of D_2^h. But then x and y are both nonadjacent with v, contradicting the fact that $\alpha(D) = 2$. Hence $i \neq 1$. Similarly we can show that $i \neq h$. It therefore follows from Lemma 5.3 that both D_1^{i-1} and D_{i+1}^h are tournaments. Hence every vertex in D_{i-1} is an end vertex of a hamiltonian path of D_1^{i-1} and every vertex in D_{i+1} is an initial vertex of a hamiltonian path of D_{i+1}^h.

Let $n(D_i) = n_i$ and let $P = v_1 v_2 \ldots v_{n_i - 1} v_{n_i}$ with $x = v_{n_i}$ be a hamiltonian path in D_i that ends in x.

Since $\delta(D) = n - 2$, x is adjacent to every vertex in D_{i+1}, and hence D_i^h has a hamiltonian path starting at v_1. This implies that no vertex in D_{i-1} is adjacent to v_1. Our assumption on $\delta(D)$ therefore implies that $n(D_{i-1}) = 1$ and any other hamiltonian path in D_i that ends in x also starts at v_1. Since D_i is strong, v_1 has an in-neighbour in D_i and since D_i is nonhamiltonian, v_{n_i} is an out-neighbour of v_1. But v_1 is adjacent to every vertex in D_i, so there exists an integer $r \in \{3, \ldots, n_i - 1\}$ such that $v_r \in N_{D_i}^-(v_1)$ and $v_{r+1} \in N_{D_i}^+(v_1)$. But then $v_2 \ldots v_r v_1 v_{r+1} \ldots v_{n_i}$ is a hamiltonian path of D_i starting at v_2 and ending in $x = v_{n_i}$. This contradiction proves the theorem. ■

The results in this section provide four new results concerning the OPPC1.

Corollary 5.7 *The OPPC1 holds for $a \leq 4$.*

Corollary 5.8 *Let D be a 1-deficient oriented graph. If every nontrivial strong component of D is hamiltonian, then D is λ-partitionable.*

Corollary 5.9 *Let D be a 1-deficient oriented graph of order n. If $\delta(D) \leq \lfloor \frac{n}{2} \rfloor$ or $\delta(D) \geq n - 2$, then D is λ-partitionable.*

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