The maximum piercing number for some classes of convex sets with the $(4, 3)$-property

Jan Kynčl Martin Tancer

Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI)*
Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
{kyncl,tancer}@kam.mff.cuni.cz

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Abstract

A finite collection $\mathcal{C}$ of closed convex sets in $\mathbb{R}^d$ is said to have a (p, q)*-property* if among any p members of $\mathcal{C}$ some q have a non-empty intersection, and $|\mathcal{C}| \geq p$. A *piercing number* of $\mathcal{C}$ is defined as the minimal number k such that there exists a k-element set which intersects every member of $\mathcal{C}$. We focus on the simplest non-trivial case in $\mathbb{R}^2$, i.e., $p = 4$ and $q = 3$. It is known that the maximum possible piercing number of a finite collection of closed convex sets in the plane with $(4, 3)$-property is at least 3 and at most 13. We consider the following three special types of collections of closed convex sets: segments in $\mathbb{R}^d$, unit discs in the plane and positively homothetic triangles in the plane, in each case only those satisfying $(4, 3)$-property. We prove that the maximum possible piercing number is 2 for the collections of segments and 3 for the collections of the other two types.

1 Introduction

A finite collection $\mathcal{C}$ of closed convex sets in d-dimensional Euclidean space is said to have a (p, q)*-property* if among any p members of $\mathcal{C}$ some q have a non-empty intersection, and $|\mathcal{C}| \geq p$. A *piercing number* of $\mathcal{C}$ is defined as the minimal number k such that there exists a k-element set which intersects every member of $\mathcal{C}$.

The well-known Helly Theorem [9] states that the piercing number of any finite collection of closed convex sets in $\mathbb{R}^d$ with $(d+1, d+1)$-property is equal to 1. By considering a collection of hyperplanes in $\mathbb{R}^d$ in general position, we get that for $q \leq d$ the piercing number is unbounded. Hadwiger and Debrunner [6] conjectured that for every $p \geq q \geq d + 1$

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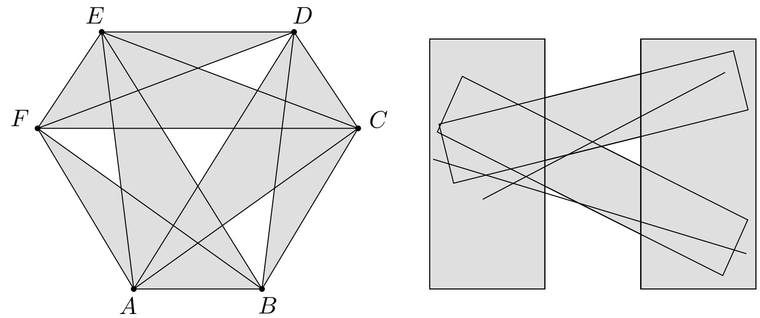

Figure 1: Collections of six sets with $(4,3)$-property and the piercing number 3. Left: six triangles ACD, BCD, CEF, DEF, EAB, FAB. Right: four rectangles and two segments, two of the rectangles are disjoint.

the piercing number of any finite collection of closed convex sets in $\mathbb{R}^d$ with (p,q)-property is bounded by a constant, which depends only on the values of p, q and d. This conjecture has been proved by Alon and Kleitman [1, 2].

Now, we can ask the following question: what is the exact value of $M(p,q;d)$, the maximum possible piercing number of a finite collection of closed convex sets in $\mathbb{R}^d$ with (p,q)-property? The general arguments [1, 2] do not give any reasonable upper bounds on $M(p,q;d)$. The simplest non-trivial case of this problem occurs for $p=4, q=3$ and $d=2$. It is easy to construct a collection of six triangles in the plane with $(4,3)$-property and the piercing number 3 (for an example, see Figure 1, left). There also exists such collection with two disjoint sets; see Figure 1, right. In this paper we construct two more examples with another restrictions on the involved sets. But a collection with the piercing number 4 has not been found yet. The best known upper bound, $M(4,3;2) \leq 13$, has been established by Kleitman, Gyárfás and Tóth [11].

It seems to be quite difficult to improve these bounds significantly. So, we approach the problem from a different direction: we try to find the exact value of the maximum possible piercing number for some restricted collections of closed convex sets which satisfy $(4,3)$-property. In particular, we consider collections of segments in $\mathbb{R}^d$, unit discs in the plane and positively homothetic triangles in the plane. We prove that the maximum possible piercing number is equal to 2 for segments and 3 for discs and triangles. Similar results were proved for d-dimensional boxes in $\mathbb{R}^d$ with edges parallel to the coordinate axes [7, 8]; it is known that for $2 \leq q \leq p \leq 2q-2$ the maximum piercing number is $p-q+1$. It is also known that the maximum piercing number is 6 for the collections of unit discs satisfying $(3,2)$-property [13], 4 for the collections of pairwise intersecting discs (i.e., with $(2,2)$-property) [3], and at most 3 for the collections of pairwise intersecting translates of a convex set [10]. See the survey by Eckhoff [4] for other related results. The survey [4] also refers to the paper of Wegner [13], who claims that the piercing number of the collection of unit discs with $(4,3)$-property is 3, but he does not give a proof of that. He only states that one can prove it similarly as the upper bound for the case of $(3,2)$-property, but it may be not so easy. So we believe it is worth giving the proof in this paper.

2 Preliminaries

In this section we introduce notation and some propositions used throughout the paper.

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. Let $x \in \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$. We will say that x *pierces* S if $x \in S$.

Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}^d$ and let X be a set of points in $\mathbb{R}^d$. We will say that X *pierces* $\mathcal{C}$ if every $C \in \mathcal{C}$ is intersected by X.

For a finite collection $\mathcal{S}$ of sets define $P(\mathcal{S})$ as the piercing number of S, i. e.,

$$P(\mathcal{S}) = \min\{|X|; X \text{ pierces } \mathcal{S}\}.$$

Now for a collection $\mathcal{C}$ define

$$M_{\mathcal{C}} = \max\left\{P(\mathcal{S}) | \mathcal{S} \text{ is a finite subcollection of } \mathcal{C} \text{ with the } (4,3)\text{-property}\right\}.$$

Our aim is to determine precise values of $M_{\mathcal{C}}$ for the three collections $\mathcal{C}$ mentioned in the introduction.

Now we formulate Helly's Theorem [9], and then introduce two lemmas widely used throughout the paper.

Theorem 1 (Helly). *Let $d \geq 1$ be an integer and let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, $k \geq d+1$, be a collection of convex subsets of $\mathbb{R}^d$. If the intersection of every $d+1$ elements of $\mathcal{C}$ is non-empty, then the intersection of all the elements of $\mathcal{C}$ is non-empty.*

Lemma 2. *Let $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^d$ satisfying $(4,3)$-property.*

1. *There are no three pairwise disjoint sets S_1, S_2, $S_3 \in \mathcal{S}$.*

2. *If $A, B \in \mathcal{S}$ are two disjoint sets, then one of these sets intersects all the sets from $\mathcal{S} \setminus \{A, B\}$.*

Proof.

1. For contradiction, suppose that there exist pairwise disjoint sets S_1, S_2, $S_3 \in \mathcal{S}$. According to $(4,3)$-property, $|\mathcal{S}| \geq 4$, so there exists $S \in \mathcal{S} \setminus \{S_1, S_2, S_3\}$. By using $(4,3)$-property for the quadruple S_1, S_2, S_3, S we obtain a contradiction.

2. For contradiction, suppose that there exist $C_1, C_2 \in \mathcal{S} \setminus \{A, B\}$ such that $C_1 \cap A = \emptyset$ and $C_2 \cap B = \emptyset$. According to the first part of this lemma, $C_1 \neq C_2$. But then $(4,3)$-property is not satisfied for A, B, C_1, C_2.

□

Let $S \subseteq \mathbb{R}^2$ and $v \in \mathbb{R}^2$. We will use the notation $-S = \{-s | s \in S\}$, and $S + v = \{s + v | s \in S\}$.

Lemma 3. *Let $S \subseteq \mathbb{R}^2$ be an arbitrary set and let $v_1, v_2, \ldots, v_k \in \mathbb{R}^2$. Then the sets $S + v_1, S + v_2, \ldots, S + v_k$ have a non-empty intersection if and only if there exists $v \in \mathbb{R}^2$ such that the set $-S + v$ contains all the points $v_1, v_2, \ldots, v_n$.*

Proof. The statement easily follows from the following equivalence: $v \in S + v_i \Leftrightarrow -v \in -S - v_i \Leftrightarrow v_i \in -S + v$. □

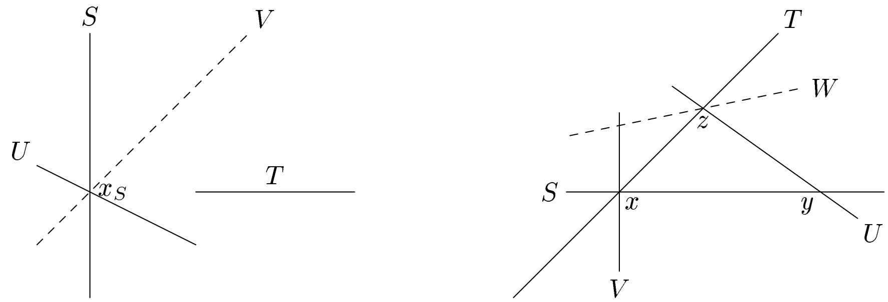

Figure 2: No two segments lie on a common line.

3 Segments

Let $\mathcal{G}_d$ be a collection of all segments in $\mathbb{R}^d$.

Theorem 4. $M_{\mathcal{G}_d} = 2$.

Proof. To show that $M_{\mathcal{G}_d} \geq 2$, consider a collection of four segments where three of them have a common point and are disjoint with the fourth segment.

For the second inequality, let $\mathcal{S}$ be a finite subcollection of $\mathcal{G}_d$ satisfying $(4,3)$-property. We will show that $P(\mathcal{S}) \leq 2$.

In this proof, *a segment* will always mean a segment which is an element of $\mathcal{S}$.

We distinguish several cases:

1. No two segments lie on a common line.

   See Figure 2, left, for the first of the following subcases, and right, for the second one.

   1.1. There exist two disjoint segments S, T.

   Let $U \in \mathcal{S}$ be any segment different from S and T. By Lemma 2 we observe that $S \cap U \neq \emptyset$ or $T \cap U \neq \emptyset$. Note that $|S \cap U| \leq 1$ and $|T \cap U| \leq 1$, since there are no two segments lying on a common line. If $S \cap U \neq \emptyset$, let $\{x_S\} = S \cap U$, otherwise choose x_S as an arbitrary point of S, similarly we choose a point x_T. According to the first observation, the segments S, T and U are pierced by the points x_S and x_T. Let V be a segment different from S, T and U. Then among the segments S, T, U, V there is a triple with non-empty intersection. Since $S \cap T = \emptyset$, we have $S \cap U \cap V = \emptyset$ or $T \cap U \cap V = \emptyset$, hence $x_S \in V$ or $x_T \in V$. Therefore, points x_S and x_T pierce $\mathcal{S}$.

   1.2. Every two segments intersect.

   If $S_1 \cap S_2 \cap S_3 \neq \emptyset$ for each triple of segments S_1, S_2, S_3, then by Helly's Theorem, one point is sufficient to pierce all the segments.

   So we can assume that there are three segments S, T, U such that $S \cap T \cap U = \emptyset$. According to the assumptions for this subcase, we have $|S \cap T| = |S \cap U| = |T \cap U| = 1$. Let $V \in \mathcal{S}$, $V \neq S, T, U$. Among the segments S, T, U, V there is a triple with non-empty intersection. This triple is different from the triple S, T, U. Without loss of generality, we can assume that from the remaining triples, S, T, V is the triple with non-empty intersection. Since $|S \cap T| = 1$, we have $|S \cap T \cap V| = 1$. Let $S \cap T \cap V = \{x\}$ and $S \cap U = \{y\}$. Then S, T, U and V are pierced by x, y. It remains to show that every other segment W from $\mathcal{S}$ contains x or y. Actually, we show that every such W contains x.

Let $T \cap U = \{z\}$. Note that $x \neq y \neq z \neq x$, because $S \cap T \cap U = \emptyset$. From the $(4, 3)$-property for the segments S, T, U, W we get that W contains at least one of the points x, y, z. Moreover, W does not contain two of them, according to the assumptions for this case. For contradiction, suppose that $x \notin W$. By symmetry, we can assume that $z \in W$ and $y \notin W$, so $W \cap U = \{z\}$. Then, among the segments S, V, U, W, no three have a common point. This violates the $(4, 3)$-property: $S \cap V = \{x\}$, but $x \notin W \cup U$; similarly $W \cap U = \{z\}$, but $z \notin S \cup V$. As a corollary we obtain that $\{x, y\}$ pierces $\mathcal{S}$.

2. Two of the segments lie on a common line.

   2.1. There exist two disjoint segments on a common line p.

   2.1.1. All the segments lie on p.
   Consider p being a horizontal line. Let $L \in M$ be a segment whose right endpoint x_l is the leftmost point among all right endpoints of the segments. Similarly, let R be a segment whose left endpoint x_r is the rightmost point among all left endpoints of the segments. Every other segment S intersects L or R. By the definition of x_l and x_r, S must contain at least one of these two points.

   2.1.2. There exists a segment $S \not\subseteq p$.
   Let T_1 and T_2 be two disjoint segments on p. By Lemma 2, S must intersect exactly one of these two segments, say T_1. Let $\{x\} = S \cap T_1$. By $(4, 3)$-property, every other segment T (different from S, T_1, T_2) must pass through x. Thus we can choose the second piercing point as an arbitrary point from T_2.

   2.2. Every two segments lying on a common line intersect.

   2.2.1. There are two disjoint segments S, T.
   S and T do not lie on a common line. Let s and t be the lines containing segments S and T. If every segment lies on s or t, then every two segments lying on s intersect. Hence, according to the one-dimensional Helly's Theorem, there exists a point $x_s \in s$ which is contained in all these segments. Similarly we can find a point $x_t \in t$ which is contained in all the remaining segments lying on t. In case there exists a segment which does not lie on any of the two lines s, t, we can choose the required two points the same way as in subcase 1.1.

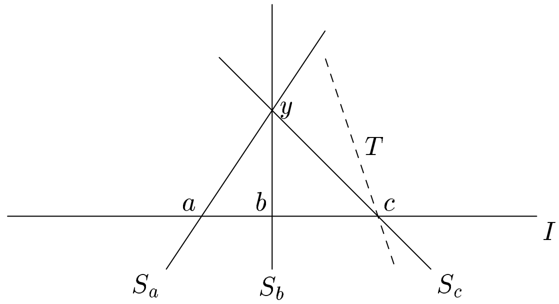

Figure 3: Sub-subcase 2.2.2.

2.2.2. Every two segments intersect.
See Figure 3. Let p be a line containing at least two segments. Considering p as a horizontal line, let L be the segment whose right endpoint is the leftmost point among all right endpoints of the segments lying on p. Similarly, let R be the segment whose left endpoint is the rightmost point among all left endpoints of the segments lying on p. Then $I = L \cap R$ is a non-empty interval (which may be degenerate), which is contained in all the segments lying on p. Let *oblique* segment be a segment not lying on p. According to the assumption of this sub-subcase, every oblique segment crosses p somewhere in the interval I. Let X be the set of all intersections of some oblique segment and I. If X is empty, then an arbitrary point from I pierces $\mathcal{S}$. If $|X| = 1$ or $|X| = 2$, then every segment contains at least one point from X. We are left with the case $|X| \geq 3$. Let a, b, c be three different points from X and let S_a, S_b, S_c be some oblique segments such that $a \in S_a, b \in S_b, c \in S_c$. Among the segments L, S_a, S_b, S_c, the triple S_a, S_b, S_c is the only one which can have a non-empty intersection. So there exists a point y not lying on p such that $\{y\} = S_a \cap S_b \cap S_c$. If every oblique segment contains y, we can choose y and an arbitrary point from I as the required two piercing points. In the other case there exists an oblique segment T not containing y. The segment T contains at most one point from $\{a, b, c\}$, so we can assume that $a, b \notin T$. But then, among the segments L, S_a, S_b, T no three have a point in common, which is a contradiction with $(4, 3)$-property.

$\square$

4 Discs

Let $\mathcal{D}$ be a set of all closed unit discs in $\mathbb{R}^2$.

Theorem 5. $M_{\mathcal{D}} = 3$.

Proof. First we establish the upper bound.

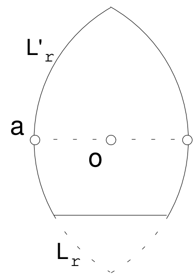

Figure 4: Case 1, regions L_r and L'_r.

Let $\mathcal{S}$ be a finite subcollection of $\mathcal{D}$ satisfying $(4,3)$-property. We will show that $P(\mathcal{S}) \leq 3$.

Let C be the set of all centers of discs in $\mathcal{S}$. According to Lemma 3 we can use the following dual form of $(4,3)$-property: among every four points in C some three can be covered by a closed unit disc. To prove that $P(\mathcal{S}) \leq 3$, it suffices to show that C can be covered by three unit discs.

Let r be the largest distance between two points of C. We will distinguish three cases:

1. $r \leq 2$ (every two discs from $\mathcal{S}$ have a non-empty intersection).

   Let $a, b \in C$ be the points whose distance is equal to r. Then all points from C lie in the lens-like region L_r, which is the intersection of two closed discs with the radius r centered at the points a, b; see Figure 4. Let x be the line determined by points a, b. Consider x as an x-axis of a Cartesian coordinate system with the origin o at the center of the segment ab, such that b has a positive x-coordinate. Let $(x_1, y_1) \in C$ be a point with the largest y-coordinate, and $(x_2, y_2) \in C$ a point with the least y-coordinate. Then $y_1 \geq 0$, $y_2 \leq 0$ and $y_1 - y_2 \leq r$. Without loss of generality, suppose that $|y_1| \geq |y_2|$. Then $y_2 \geq -\frac{r}{2}$, so all the points from C lie in the set $L'_r = \{(x,y) \in L_r; y \geq -\frac{r}{2}\}$. It now suffices to prove that L'_r can be covered by three closed unit discs. To finish the proof, we refer to case 3 ($2 < r \leq \sqrt{8}$), where a set larger then L'_r is covered by three unit discs.

2. $r > \sqrt{8}$.

   Let $A, B \in \mathcal{S}$ be the discs with the centers a, b such that $|a - b| = r$. The discs A and B are disjoint, so, according to Lemma 2 and by the symmetry, we can assume that all the discs from $\mathcal{S} \setminus \{A\}$ intersect B. By $(4,3)$-property and Helly's Theorem, all the discs from $\mathcal{S}$ disjoint with A have a common point $b_0 \in B$ (if there are just two such discs, use Lemma 2). Equivalently, all the points from C whose distance from a is larger than 2 can be covered by one closed unit disc centered at b_0. It remains to cover the set $C' = \{s \in C; |s - a| \leq 2\}$ by two closed unit discs. Except of a, every other point $s \in C'$ satisfies $|s - b| \leq 2$, so it lies in the region L, which is the intersection of two closed discs of radius 2 centered at points a, b; see Figure 5. Let $C'' = C' \setminus \{a\}$. If $|C''| \leq 1$, then C' contains at most two points and can be covered

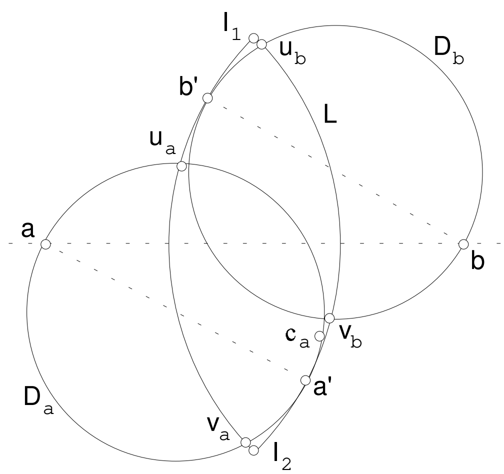

Figure 5: Case 2, discs D_a and D_b cover all the points from C'.

by at most two closed unit discs. For the rest of this case suppose that $|C''| \geq 2$. By $(4,3)$-property, for every two points $c_1, c_2 \in C''$ there exists either a closed unit disc containing points a, c_1, c_2 or a closed unit disc containing points b, c_1, c_2. Moreover, we can suppose that this disc has the point a (or b) on its boundary:

Since $r > \sqrt{8}$, for every point $c \in L$ the sizes of the angles bac and abc are less than $\frac{\pi}{4}$. Thus for every two points $c_1, c_2 \in L$ the angles c_1ac_2 and c_1bc_2 are acute. The rest follows from the following fact.

Lemma 6. *Let D be the smallest disc containing a triangle xyz. If the angle xyz is acute, then y lies on the boundary of D.*

Proof. Clearly, if $T = xyz$ is an obtuse or a right triangle with the longest side xy, then D has the segment xy as its diameter. If T is acute, then D is the disc whose boundary is the circumcircle of T. $\square$

Consider the same coordinate system as in case 1 and denote by l_1, l_2 the points of L with the largest and the least y-coordinate. We define two discs, D_a and D_b, which cover all the points from C'. The discs D_a and D_b will be determined by their diameters aa' and bb'.

Let D be a closed unit disc with diameter aa_1, such that D has a non-empty intersection with L. Then a_1 lies on the semicircle $S_a = \{z = (z_x, z_y) \in \mathbb{R}^2; |z - a| = 2, z_x > -\frac{r}{2}\}$. If $a_1 \in L \setminus \{l_1, l_2\}$, then the set $L \setminus D$ consists of two connected regions: the upper region $\overline{D}$ and the lower region $\underline{D}$.

In the other case $L \setminus D$ consists of a single connected region: either $\overline{D}$, if a_1 has a negative y-coordinate, or $\underline{D}$ otherwise. We define an *upper boundary* of D as an arc, which is a common boundary of D and $\overline{D}$ (in case that $a_1 = l_1$ we define the upper boundary as $\{l_1\}$). Similarly we define a *lower boundary* of D. For the points of the semicircle S_a we consider a "vertical" linear ordering: if $u = (u_x, u_y), v = (v_x, v_y) \in S_a$ and $u_y < v_y$, we write $u \prec v$ and say that u is *lower than* v, or equivalently, that v is *higher than* u. We make similar definitions also for the discs having point b on the boundary.

Suppose that no closed unit disc with a or b on its boundary covers all the points from C''. Consider the set M_a of all closed unit discs D with diameter aa_1 such that a_1 lies on the semicircle S_a and the boundary of D contains at least one point from C''. Symmetrically we define the set M_b. Let

$$M = \{(D'_a, D'_b) \in M_a \times M_b; \overline{D'_a} \cap (C'' \setminus D'_b) = \emptyset \wedge (C'' \setminus D'_a) \cap \underline{D'_b} = \emptyset\}\}.$$

Clearly, M is a non-empty finite set. We choose the pair $(D_a,\ D_b)$ as a maximal element of M according to the following partial order:

For the discs $D^1_a, D^1_b, D^2_a, D^2_b$ with the diameters aa_1, aa_2, bb_1, bb_2, we put $(D^1_a, D^1_b) \ll (D^2_a, D^2_b)$ if and only if

$$(a_1 \succeq a_2 \wedge b_1 \prec b_2) \vee (a_1 \succ a_2 \wedge b_1 \preceq b_2).$$

The discs D_a and D_b cover C'' if and only if $(\overline{D_a} \cup \underline{D_a}) \cap (\overline{D_b} \cup \underline{D_b}) \cap C'' = \emptyset$. By the definition of M, it suffices to prove that $\underline{D_a} \cap \overline{D_b} \cap C'' = \emptyset$.

From the maximality of (D_a, D_b) it follows that there exists a point $c_a \in C'' \setminus D_b$ on the upper boundary of D_a and a point $c_b \in C'' \setminus D_a$ on the lower boundary of D_b. It cannot happen both $c_a \in \overline{D_b}$ and $c_b \in \underline{D_a}$, since in this case there exists no closed unit disc with a or b on its boundary covering both points c_a and c_b. Thus, without loss of generality, we can suppose that $c_a \in \underline{D_b}$.

For contradiction, suppose that $\underline{D_a}$ and $\overline{D_b}$ have a non-empty intersection. Let u_a and v_a be the intersections of the boundary of D_a with the semicircle S_b, such that u_a is higher than v_a. Similarly we define u_b and v_b as the intersections of the boundary of D_b with the semicircle S_a, so that u_b is higher than v_b. The region $\underline{D_a}$ is bounded by three arcs $v_a a'$, $a' l_2$, $l_2 v_a$, and $\overline{D_b}$ is bounded by the arcs $u_b b'$, $b' l_1$, $l_1 u_b$. If b' were higher than v_a and u_b higher than a', the regions $\underline{D_a}$ and $\overline{D_b}$ would be disjoint (they would be separated by the line $v_a a'$, for example). Since there is a point $c_a \in \underline{D_b}$ on the upper boundary of D_a, either b' is higher than u_a or v_b is higher than a'. In the first of these two cases, a' must be higher than u_b, in the second case b' must be lower than v_a. Since these cases are symmetric, it suffices to consider only the first case.

We observe that u_a has larger y-coordinate than a': consider a horizontal line $x_{a'}$ going through the point a'. It crosses the boundary of D_a at the point t_a with the x-coordinate equal to $-\frac{r}{2}$ (the angle $a' t_a a$ is right). It implies that t_a lies outside the region L and that the line $x_{a'}$ crosses S_b (the left part of the boundary of L) at a point which lies inside D_a, thus lower than u_a. Similarly we get that u_b has larger y-coordinate than b', which contradicts the assumptions that b' is higher than u_a and a' higher than u_b.

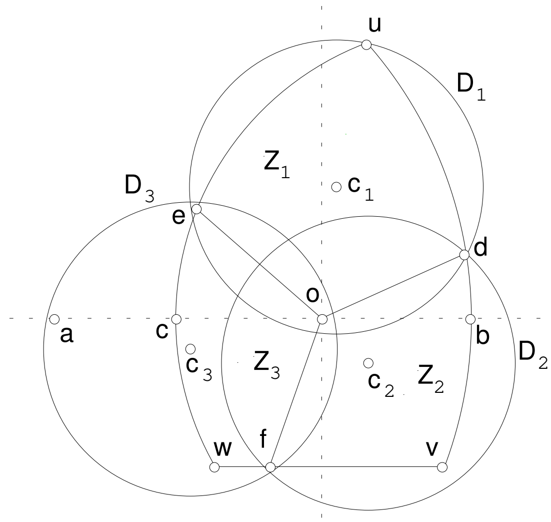

Figure 6: Case 3, discs D_1, D_2, D_3 cover the region L''_r.

3. $2 < r \leq \sqrt{8}$.

   See Figure 6.

   As in case 2, let $A, B \in \mathcal{S}$ be the discs with the centers a, b such that $|a - b| = r$ and all the discs from $\mathcal{S}$ except of A intersect B. Let $C' = C \setminus \{a\}$. All the points from C' lie in the area $L_r = \{z \in \mathbb{R}^2; |z - a| \leq r, |z - b| \leq 2\}$. Hence it is sufficient to cover the set $L_r \cup S_r$, where S_r is the segment ab, with three closed unit discs. Consider a Cartesian coordinate system such that $b = (1, 0)$, and $a = (1 - r, 0)$. By Lemma 2, every two discs from $\mathcal{S} \setminus \{A\}$ have a non-empty intersection, so every two points from C' have a vertical distance at most 2. Thus, without loss of generality, all the points from C' have y-coordinate at least -1. So it suffices to cover the set $L''_r = L'_r \cup S_r$, where $L'_r = \{z = (x_z, y_z); z \in L_r, y_z \geq -1\}$. Without loss of generality, we can assume $r = \sqrt{8}$. It is easy to see that $L''_{\sqrt{8}}$ covers every set L'_r defined in case 1 ($r \leq 2$).

   We will divide L''_r into three sets Z_1, Z_2, Z_3 and we cover each of them by one unit disc. Let $u = (\frac{2-\sqrt{2}}{2}, \frac{\sqrt{14}}{2})$, $v = (1 - \sqrt{8} + \sqrt{7}, -1)$, $w = (1 - \sqrt{3}, -1)$ (u, v and w are the boundary vertices of the region L'_r), $o = (0, 0)$, $c = (-1, 0)$, $d = (1 - \sqrt{8} + \sqrt{8 - 0.43^2}, 0.43)$, $e = (1 - \sqrt{4 - 0.75^2}, 0.75)$, and $f = (-0.35, -1)$. The points d, e, f lie on the boundary of L'_r and the segments od, oe, of divide L''_r into three parts $Z_1 = odue$, $Z_2 = ofvd$, and $Z_3 = oewf \cup ac$. For $i = 1, 2, 3$, part Z_i is covered by the closed unit disc D_i centered at c_i, where $c_1 = (0.1, 0.9)$, $c_2 = (0.3, -0.3)$, and $c_3 = (-0.9, -0.2)$: it is straightforward to check that $o, d, u, e \in D_1$, $o, f, v, d \in D_2$ and $o, e, w, f, a, c \in D_3$, which implies $Z_i \subseteq D_i$ for $i = 1, 2, 3$ (the boundary curves of the sets Z_i consist of segments or arcs which have less curvature than the unit circle).

The lower bound is established by the example of Grünbaum [5], who constructed nine

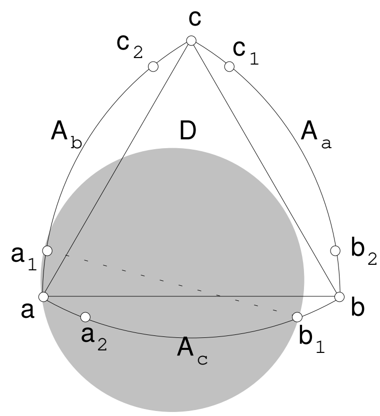

Figure 7: Nine points of S and the disc D covering points b_1, a_1, a.

pairwise intersecting unit discs that cannot be pierced by two points. For completeness, we describe the construction here (at least we need to show that the $(4,3)$-property is satisfied). We construct a nine-point set $S = \{a, b, c, a_1, a_2, b_1, b_2, c_1, c_2\}$ in the plane, which cannot be covered by two closed unit discs and satisfies the dual version of $(4,3)$-property. See Figure 7.

Let a, b, c be the vertices of an equilateral triangle with side of length 2. Let A_a be the arc of the circle with center a and radius 2, with endpoints b, c and central angle $\frac{\pi}{3}$. Similarly arcs A_b and A_c are defined. In other words, the points a, b, c and the arcs A_a, A_b, A_c form the vertices and edges of a Reuleaux triangle of width 2. Let $\varepsilon > 0$ be sufficiently small. Let $a_1, c_2 \in A_b$, $b_1, a_2 \in A_c$, $c_1, b_2 \in A_a$ be such points, that $|a_1 - a| = |a_2 - a| = |b_1 - b| = |b_2 - b| = |c_1 - c| = |c_2 - c| = \varepsilon$.

Now we verify that both required conditions are satisfied:

- Suppose that S can be covered by two closed unit discs. Then one of them, D_1, must cover two points from the set $\{a, b, c\}$, say a and b. The distance between a and b is equal to the length of the diameter of the unit disc. Hence, D_1 is centered at the center of the segment ab. Arcs A_a and A_b intersect D_1 only at points a and b, so D_1 covers only points a, b, a_2 and b_1. But points c, a_1 and b_2 obviously cannot be covered by a closed unit disc (the circle circumscribed to the acute triangle ca_1b_2 has radius almost $\frac{2\sqrt{3}}{3}$), which is a contradiction.

- Let $Q \subset S$ be a four-point set. Without loss of generality, suppose that Q contains at least two points from the set $\{a, a_1, a_2\}$. If Q contains all these three points, then $\{a, a_1, a_2\} \subset Q$ is a triple which is covered by a closed unit disc centered at a. Otherwise, Q contains exactly two of the points a, a_1, a_2. By symmetry, we can suppose that $a_1 \in Q$.

  Suppose that $a \in Q$. The quadruple Q contains at least one point from the set $R = \{c_2,\ c,\ c_1,\ b_2,\ b_1\}$. Hence it suffices to prove that every triple $\{a_1, a, x\}$, where $x \in R$, can be covered with a closed unit disc. If $x \in \{c_2,\ c,\ c_1,\ b_2\}$, then $\{a_1, a, x\}$ is covered by the disc with the diameter ax, since $|a - x| \leq 2$ and the angle

aa_1x is obtuse. The remaining triple $\{a_1, a, b_1\}$ is covered by the disc D with the diameter a_1b_1: let o be the circumcenter of the triangle abc. Then triangles oa_1b_1 and oab are similar and $|o - b_1| < |o - b|$, which implies $|a_1 - b_1| < 2$, so D has a radius less than 1. The disc D covers also the point a, since a_1ab_1 is a right triangle: $|\sphericalangle b_1aa_1| = |\sphericalangle baa_1| + |\sphericalangle b_1ab| = |\sphericalangle cbb_1| + |\sphericalangle b_1ab| = \frac{\pi}{3} + |\sphericalangle abb_1| + |\sphericalangle b_1ab| = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$.
Suppose that $a_2 \in Q$. Then all the triples $\{a_1, a_2, x\}$, where $x \in \{c_2, c, c_1, b_2, b, b_1\}$, can be covered by a closed unit disc: by symmetry, it suffices to prove that only for $x \in \{c_2, c, c_1\}$. In that case, $\{a_1, a_2, x\}$ is covered by the disc with the diameter a_2x, since $|a_2 - x| \leq 1$ and the angle a_2a_1x is obtuse.

□

5 Triangles

In this section we analyze the collections of positively homothetic triangles.

By applying an appropriate affine transformation, we can assume without loss of generality that all the considered triangles are equilateral. Let $T_\lambda \subset \mathbb{R}^2$ be an equilateral triangle defined as a convex hull of points $(0,0)$, $(\lambda, 0)$ and $\left(\frac{\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right)$. Let $\mathcal{T}$ be a collection of all triangles $T_\lambda + v$, where $\lambda > 0$ and $v \in \mathbb{R}^2$ is an arbitrary vector. Let $\mathcal{T}_1 \subset \mathcal{T}$ be a collection of all unit triangles from $\mathcal{T}$, i. e., the triangles with $\lambda = 1$. Our aim is to show that $M_{\mathcal{T}} = M_{\mathcal{T}_1} = 3$.

For $T = T_\lambda + v \in \mathcal{T}$, the line determined by points $(0,0) + v$ and $(\lambda, 0) + v$ is called the *bottom line* of T. Similarly, the *right line* is determined by points $(\lambda, 0) + v$ and $\left(\frac{\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right) + v$ and $(0,0) + v$, and the *left line* is determined by points $(\lambda, 0) + v$, $\left(\frac{\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right) + v$ and $(0,0) + v$, $\left(\frac{\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right) + v$. The point $\left(\frac{\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right) + v$ is called the *top* of the triangle $T_\lambda + v$.

Now we present the upper and the lower bound for the maximum piercing number of positively homothetic triangles.

Theorem 7. $M_{\mathcal{T}} \leq 3$.

Proof. As usual, let $\mathcal{S}$ be a finite subcollection of $\mathcal{T}$ satisfying $(4,3)$-property. We will show that $P(\mathcal{S}) \leq 3$.

Let $T \in \mathcal{S}$ be the triangle which has the largest y-coordinate of its bottom line. If T is not unique, pick one such triangle arbitrarily. Denote by a, b, c the vertices of T so that ab is the bottom line of T. Without loss of generality, we may assume that $T = T_1$, i. e., $a = (0,0)$ and $b = (1,0)$. If $S \in \mathcal{S}$ and $S \cap T$ is non-empty, then the intersection of the segment ab and T is a non-empty segment (which may be degenerate). Let l_S and r_S be the x-coordinates of the left and the right endpoint of this segment.

Consider several cases:

1. The triangle T does not intersect any other triangle.

   Let U, V, W be different elements of $\mathcal{S} \setminus \{T\}$. Then, according to the $(4,3)$-property for the quadruple T, U, V, W, the triple U, V, W has a non-empty intersection.

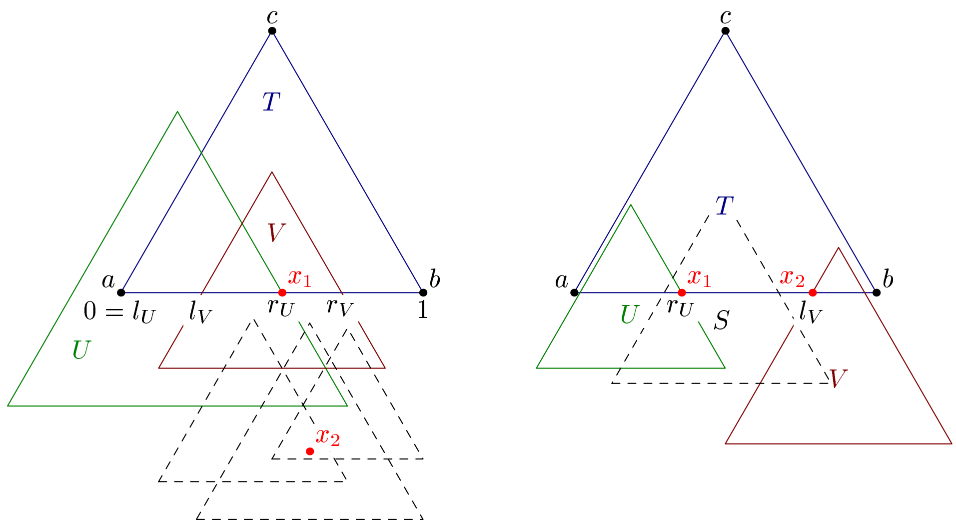

Figure 8: Subcases 2.1 and 2.2.

The assumption $|\mathcal{T}| \geq 4$ implies that $|\mathcal{S} \setminus \{T\}| \geq 3$. Hence, by Helly's Theorem, the intersection of $\mathcal{S} \setminus \{T\}$ is non-empty. Then arbitrary two points $x_1 \in \bigcap(\mathcal{S} \setminus \{T\})$ and $x_2 \in T$ pierce $\mathcal{S}$.

2. The triangle T intersects some other triangle.

   Let $U \in \mathcal{S}$ be the triangle such that r_U is the smallest possible. Similarly, let $V \in \mathcal{S}$ be the triangle such that l_V is the largest possible. We distinguish three subcases. See Figure 8 for the first two of these subcases.

   2.1. $r_U \geq l_V$.

   Let $x_1 = (r_U, 0)$. If $S \in \mathcal{S}$ is a triangle intersecting T, then $r_S \geq r_U$ and $l_S \leq l_V \leq r_U$, hence $x_1 \in S$. As in case 1, there exists a point x_2 which is in the intersection of triangles not intersecting T (if there are at most two such triangles, they can be pierced by two points x_2 and x_3).

   2.2. $r_U < l_V$ and $U \cap V$ is empty.

   Let $x_1 = (r_U, 0)$ and $x_2 = (l_V, 0)$. We will show that $\{x_1, x_2\}$ pierces $\mathcal{S}$. Clearly, T and U are pierced by x_1 and V is pierced by x_2. Let $S \in \mathcal{S} \setminus \{T, U, V\}$. We will use $(4,3)$-property for the triangles S, T, U, V. Since $U \cap V = \emptyset$, we have $S \cap T \cap U \neq \emptyset$ or $S \cap T \cap V \neq \emptyset$. It means that $S \cap T \neq \emptyset$ and $l_S \leq r_U \leq r_S$ or $r_S \geq l_V \geq l_S$, which implies that $x_1 \in S$ or $x_2 \in S$.

   2.3. $r_U < l_V$ and $U \cap V$ is non-empty.

   Let $x_1 = (r_U, 0)$ and $x_2 = (l_V, 0)$. We will find a point $x_3 \in U \cap V$ such that $\{x_1,\, x_2,\, x_3\}$ pierces $\mathcal{S}$. Let $\mathcal{M} = \{S \in \mathcal{S} | S \cap T = \emptyset\}$ and $\mathcal{Z} = \mathcal{M} \cup \{U, V\}$. First, we establish that $\bigcap \mathcal{Z} \neq \emptyset$. If $\mathcal{Z} = \{U, V\}$, then $\bigcap \mathcal{Z} \neq \emptyset$ from the assumption of this subcase. If $|\mathcal{Z}| \geq 3$, then it is sufficient to show that Helly's property from Theorem 1 is satisfied. Let X, Y, $Z \in \mathcal{Z}$. According to the

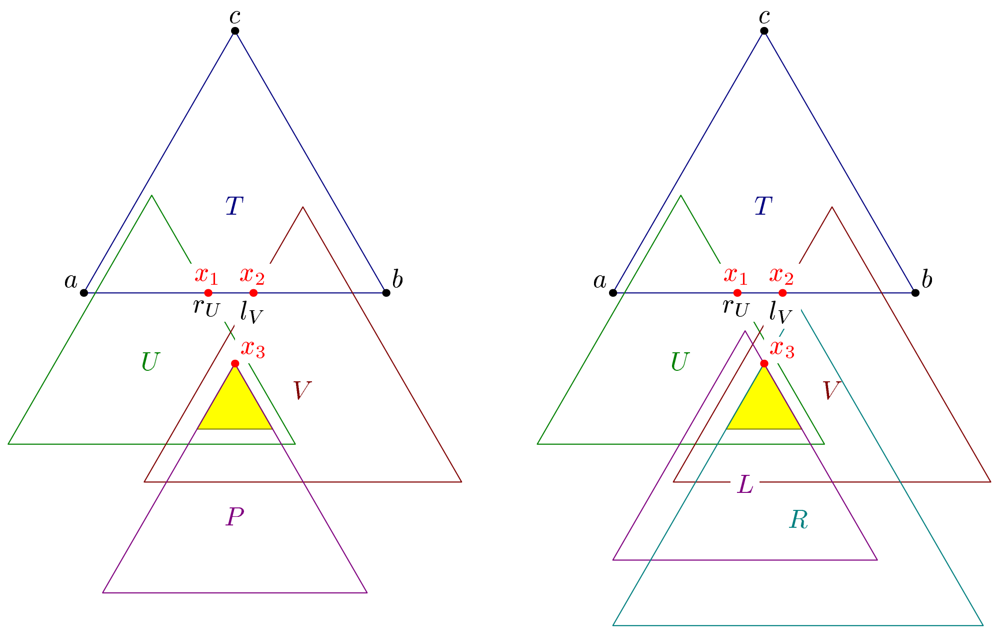

Figure 9: Sub-subcases 2.3.1 and 2.3.2.

$(4,3)$-property for the quadruple X, Y, Z, T we obtain that $X \cap Y \cap Z$ is non-empty (by an easy discussion of three cases according to the cardinality of $\{X,Y,Z\} \cap \{U,V\}$). Hence Helly's property is satisfied. It is easy to see that $\bigcap \mathcal{Z}$ is a triangle or a point. In the first case pick x_3 as the top of that triangle, in the second case pick x_3 such that $\{x_3\} = \bigcap \mathcal{Z}$. It remains to show that every triangle $S \in \mathcal{S}$ is pierced by some of the points x_1, x_2, x_3. We know this for $S \in \{T,U,V\}$ and for every triangle $S \in \mathcal{S}$ disjoint with T. Now, let $S \neq T,U,V$ be a triangle from $\mathcal{S}$ intersecting T. If $l_S \leq r_U$, then $x_1 \in S$. If $r_S \geq l_V$, then $x_2 \in S$.

Suppose that $r_U < l_S \leq r_S < l_V$. We will show that $x_3 \in S$. We have $T \cap U \cap S = \emptyset$ and $T \cap V \cap S = \emptyset$. Now, distinguish two sub-subcases.

2.3.1. There exists $P \in \mathcal{Z}$ such that x_3 is a top of P.
Then $P \neq U,V$, since the tops of U and V are above the bottom line of T. Use the $(4,3)$-property for the quadruple S, T, U, P. We know that $T \cap P = \emptyset$ and $S \cap T \cap U = \emptyset$, hence $S \cap U \cap P \neq \emptyset$. In particular, $S \cap P \neq \emptyset$. On the other hand, x_3 is below the left line of V, which is below the left line of S ($l_S < l_V$). Similarly, x_3 is below the right line of S. Thus $S \cap P \neq \emptyset$ implies that $x_3 \in S$.

2.3.2. There are two distinct triangles $L,R \in \mathcal{Z}$ such that x_3 is on the left line of R and on the right line of L.
Then the statement easily follows from the $(4,3)$-property for the triangles S, T, L and R.

We have discussed all the cases so the proof is finished. $\square$

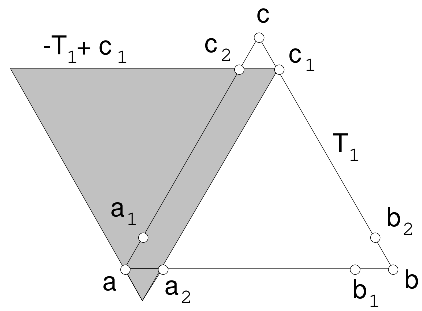

Figure 10: Nine points of S and the triangle $-T_1+c_1$.

Theorem 8. $M_{\mathcal{T}_1} \geq 3$.

Proof. Similarly as in the case of unit discs, we construct a nine-point set $S=\{a,b,c,a_1,a_2,b_1,b_2,c_1,c_2\}$ which cannot be covered by two translated copies of $-T_1$ and satisfies a dual version of the $(4,3)$-property: among every four points from S some three can be covered by a translated copy of $-T_1$. See Figure 10.

Let $a=(0,0), b=(1,0), c=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ be the vertices of the equilateral triangle T_1. We put the remaining six points on the sides ab, bc, ca of this triangle: let $a_1, c_2 \in ca$, $b_1, a_2 \in ab$, $c_1, b_2 \in bc$ be such points, that $|a_1-a|=|a_2-a|=|b_1-b|=|b_2-b|=|c_1-c|=|c_2-c|=\varepsilon$, where $\varepsilon>0$ is sufficiently small.

Suppose that S can be covered by two translated copies of $-T_1$. Then one of these two copies, T', covers two of the points a,b,c. By symmetry, we can assume that $a,b\in T'$. Then a and b are two vertices of T' and $T'\cap T=ab$, so the triangle T' covers only those four points from S lying on ab. The remaining points have to be covered by the second translated copy of $-T_1$, but it is obviously impossible even for the three points c,a_1,b_2.

It remains to verify that S satisfies the dual $(4,3)$-property. Let Q be a four-point subset of S. Without loss of generality, we can assume that Q contains exactly two points from the set $\{c,c_1,c_2\}$ and that $c_2\in Q$. If $a_1\in Q$ or $a_2\in Q$, then the triangle $-T_1+c+a_2$ (a translated copy of $-T_1$ with the bottom vertex at a_2) contains at least three points from Q, since it covers points c,c_1,c_2,a_1 and a_2. The case when $b_1\in Q$ or $b_2\in Q$ is symmetric. We are left with the case when Q contains points a,b,c_2 and one of the points c,c_1. If $Q=\{a,b,c_2,c\}$, then points a,c_2,c are covered by the triangle $-T_1+c$ with the bottom vertex at a. If $Q=\{a,b,c_2,c_1\}$, then the points a,c_2,c_1 are covered by the triangle $-T_1+c_1$ with the upper right vertex at c_1. ◻

According to the inequality $M_{\mathcal{T}} \geq M_{\mathcal{T}_1}$, the previous two results imply the following corollary.

Theorem 9. $M_{\mathcal{T}} = M_{\mathcal{T}_1} = 3$.

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