# Nonexistence of triples of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs *

Xueliang Li and Yan Liu

Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
lxl@nankai.edu.cn, liuyanday@yahoo.com.cn

Submitted: Aug 21, 2006; Accepted: Feb 3, 2008; Published: Feb 11, 2008
Mathematics Subject Classifications: 05C60, 05C75

**Abstract**

In the paper "Broersma and Hoede, *Path graphs*, J. Graph Theory **13** (1989) 427-444", the authors asked a problem whether there is a triple of mutually nonisomorphic connected graphs which have an isomorphic connected $P_3$-graph. In this paper, we show that there is no such triple, and thus completely solve this problem.

Keywords: path graph, connected, isomorphism

## 1 Introduction

Broersma and Hoede [3] generalized the concept of line graphs to that of path graphs by defining adjacency as follows. Let $k$ be a positive integer, and $P_k$ and $C_k$ denote a path and a cycle with $k$ vertices, respectively. Let $\Pi_k(G)$ be the set of all $P_k$'s in $G$. The *path graph* $P_k(G)$ of $G$ is a graph with vertex set $\Pi_k(G)$ in which two $P_k$'s are adjacent whenever their union is a path $P_{k+1}$ or a cycle $C_k$. Broersma and Hoede got many results on $P_3$-graphs and, in particular, described two infinite classes of pairs of nonisomorphic connected graphs which have isomorphic connected $P_3$-graphs. They also raised a number of unsolved problems or questions, most of which have been solved in the intervening years. Only the following one remains unanswered.

**Problem**. Does there exist a triple of mutually nonisomorphic connected graphs which have an isomorphic connected $P_3$-graph ?

*Research supported by PCSIRT, NSFC and the "973" program.

For $k = 2$, i.e., line graphs, from Whitney's result (see [4]) it is not difficult to see that the problem has a negative answer. In [5] the authors showed that for $k \geq 4$ there are not only triples of but also arbitrarily many mutually nonisomorphic connected graphs with isomorphic connected $P_k$-graphs. Interestingly, however, we will show in this paper that for $k = 3$ there does not exist any triple of mutually nonisomorphic connected graphs with an isomorphic connected $P_3$-graph; just like the case for $k = 2$ but very different from the case for $k \geq 4$. Note that if one drops the connectedness of the original graph or its $P_3$-graph, then it is easy to find arbitrarily many mutually nonisomorphic graphs with an isomorphic $P_3$-graph.

# 2 Preliminaries

All graphs in this paper are undirected, finite and simple. We follow the terminology and notations used in [1, 2]. If $\sigma$ is an isomorphism from $G$ to $H$, then $\sigma$ induces a $P_k$-isomorphism $\sigma^*$ from $G$ to $H$, where $\sigma^*(a_1 a_2 \cdots a_k) = \sigma(a_1)\sigma(a_2)\cdots\sigma(a_k)$ for all $a_1 a_2 \cdots a_k \in \Pi_k(G)$. A $P_k$-isomorphism $\tau$ is *induced* if $\tau = \sigma^*$ for some isomorphism $\sigma$. If $\tau_i$ is a $P_k$-isomorphism from $G_i$ to $H_i$ for $i = 1$ and 2, then we say that $\tau_1$ and $\tau_2$ are *equivalent* if there are isomorphisms $\sigma$ and $\rho$ from $G_1$ to $G_2$ and $H_1$ to $H_2$, respectively, such that $\tau_1 = (\rho^*)^{-1} \circ \tau_2 \circ \sigma^*$.

A vertex of degree 1 is called *terminal*, and an edge is *terminal* if it has a terminal end. Define an *$i$-thorn* to be a $P_3$ with exactly $i$ ($i = 1$ or 2) terminal ends in $G$. Let $T_i(G)$ be the set of $i$-thorns in $G$. A $P_3$ in $G$ is called *terminal* if it has degree 1 in $P_3(G)$.

We say that two $P_3$-isomorphisms $\tau_i$ from $G_i$ to $H_i$ for $i = 1$ and 2, are *$T$-related* if $(i)$ $G_1$ and $G_2$ differ only in their star components, so do $H_1$ and $H_2$; $(ii)$ $|T_2(G_1)| = |T_2(G_2)|$; and $(iii)$ $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \Pi_3(G_1) - T_2(G_1) = \Pi_3(G_2) - T_2(G_2)$.

Consider two 1-thorns $abc$ and $abd$ where $deg(a) \geq 2$ and $deg(c) = deg(d) = 1$, then swapping $abc$ and $abd$ gives a $P_3$-isomorphism, which we call a *$B$-swap*.

Suppose $abcde$ is a $P_5$ in $G$ such that both $abc$ and $cde$ are terminal 1-thorns, i.e., $deg(a) = deg(e) = 1$ and $deg(c) = 2$, then swapping $abc$ and $cde$ gives a $P_3$-isomorphism, which we call an *$S$-swap*.

For distinct $a, b \in V(G)$, let $D_{a,b}$ denote the subgraph of $G$ consisting of the union of all $P_3$'s with ends $a$ and $b$ and with middle vertex of degree 2 in $G$. If $D_{a,b}$ is nonempty we call it a *diamond* with ends $a$ and $b$. We usually write $V(D_{a,b}) - \{a, b\}$ as $\{c_1, c_2, \cdots, c_k\}$ and call $k$ the *width* of $D_{a,b}$, and refer to $D_{a,b}$ as a $k$-diamond. Note that if $a$ and $b$ are adjacent, the edge $ab$ is not included in $D_{a,b}$. To distinguish the two possibilities, we say that the diamond $D_{a,b}$ is *braced* if $a$ and $b$ are adjacent in $G$ and *unbraced* otherwise. For $1 \leq i < j \leq k$, the $P_3$'s $ac_ib$ are called *diamond paths* while the pair of $P_3$'s $c_iac_j$ and $c_ibc_j$ is called a *diamond pair*. Then swapping $c_iac_j$ and $c_ibc_j$ gives a $P_3$-isomorphism, which we call a *$D$-swap*.

Suppose $\tau_1$ and $\tau_2$ are $P_3$-isomorphisms from $G$ to $H$. We say that $\tau_1$ and $\tau_2$ are *$B$-related* if $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of $B$-swaps. The *$S$-related* and *$D$-related* are defined similarly. We use joins of these four equivalence relations: for example, two

$P_3$-isomorphisms are *TBSD-related* if we can get from one to the other by a chain of zero or more $T$-, $B$-, $S$- and/or $D$-relations.

The following is the main result of [1], based on which we shall solve our problem by case analysis.

**Theorem 2.1** *Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$ such that at least one of $G$ or $H$ is connected. Then $\tau$ is one of the following:*

*(i) $T$-related to a $P_3$-isomorphism of generalized $K_{3,3}$ type;*
*(ii) of special Whitney type;*
*(iii) $D$-related to a $P_3$-isomorphism of Whitney type 3, 4, 5 or 6;*
*(iv) $D$-related to a $P_3$-isomorphism of bipartite type; or*
*(v) $TBSD$-related to an induced $P_3$-isomorphism.*

*The definition for each of the above types will be given in the successive subsections.*

For solving our problem, in Theorem 2.1 we only need to consider that the original graphs $G$ and $H$ are nonisomorphic connected graphs with $T_2(G) = T_2(H) = \emptyset$. Below, we will analyze the types in Theorem 2.1 case by case in detail.

## 2.1 Generalized $K_{3,3}$ type

First, we introduce the following notation which is used in the definition of generalized $K_{3,3}$ type. We write $(c,d)ab(e,f) \mapsto uvwxu$ if $G$ contains the edges $ab$, $ac$, $ad$, $be$, $bf$, $H$ contains the $C_4$ $uvwxu$, and $\tau$ maps $cab \mapsto xuv$, $dab \mapsto vwx$, $abe \mapsto uvw$ and $abf \mapsto wxu$. We also write $abc(d,e) \mapsto uvwxy$ if $G$ contains the edges $ab, bc, cd, ce$, $H$ contains the $P_5$ $uvwxy$, and $\tau$ maps $abc \mapsto vwx$, $bcd \mapsto uvw$ and $bce \mapsto wxy$. This notation will be reversed (e.g., $abcda \mapsto (w,x)uv(y,z)$) as needed. Then, define the generalized $K_{3,3}$ type as follows:

Either $\tau$ or $\tau^{-1}$ as in the following cases (i) through (vii), or any equivalent $P_3$-isomorphism, is said to be of *generalized $K_{3,3}$ type*.

(i) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, and $cad$ and $ebf$ map to $P_3$ components of $H$.
(ii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $kebfh \mapsto yv_3u_1(v_1,v_2)$, and $cad$ maps to a $P_3$ component.
(iii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(k,l)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, and $cad$, $kel$ and $hfi$ map to $P_3$ components.
(iv) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ecadg \mapsto xu_3v_1(u_1,u_2)$, and $cebfh \mapsto yv_3u_1(v_1,v_2)$. Note that $G$ and $H$ are connected and isomorphic.
(v) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ebfhe \mapsto (v_1,v_2)u_1v_3(y,z)$, and $cad$ maps to $yv_3z$. Again $G$ and $H$ are connected and isomorphic.
(vi) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(c,d)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, $aceda \mapsto (w,x)u_3v_1(u_1,u_2)$, and $hfi$ maps to $wu_3x$. Again $G$ and $H$ are connected and isomorphic.

(vii) The construction on $K_{3,3}$; $G \cong H \cong K_{3,3}$.

**Remark 1**. For generalized $K_{3,3}$ type, it is easy to get the following results:

1. For cases (i), (ii) and (iii), $G$ and $H$ are nonisomorphic, but $H$ is not connected and there are isolated vertices in $P_3(G)$ and $P_3(H)$.
2. For cases (iv) and (vii), $G$ and $H$ are connected with $T_2(G) = T_2(H) = \emptyset$, but $G$ and $H$ are isomorphic.
3. For cases (v) and (vi), $G$ and $H$ are connected, but are isomorphic and there are isolated vertices in $P_3(G)$ and $P_3(H)$.

Thus there is no pair of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs in generalized $K_{3,3}$ type.

## 2.2 Special Whitney type

Let $SW$ be the graph obtained by subdividing each edge of $K_{1,3}$ exactly once, then $P_3(SW) \cong C_6$. Rotation of this $C_6$ by one step is a noninduced $P_3$-isomorphism from $SW$ to itself, then we say this or any equivalent $P_3$-isomorphism is of *special Whitney type*.

Hence there is also no pair of nonisomorphic connected graphs with isomorphic connected $P_3$-graphs by the definition of special Whitney type.

## 2.3 Whitney type 3, 4, 5 or 6

In this subsection, we begin with a general idea which will be used here and in the next subsection. Suppose $F$ is a graph. A *diamond inflation* of $F$ is a graph obtained by replacing each edge $ab \in E(F)$ by an unbraced $s_{ab}$-diamond $D_{a,b}$ ($s_{ab} \geq 1$), and adding $t_a$ terminal edges incident with each $a \in V(F)$ ($t_a \geq 0$). Suppose $\varphi$ is an edge-isomorphism between graphs $F$ and $F^{'}$, and suppose $I$ and $I^{'}$ are diamond inflations of $F$ and $F^{'}$, respectively, with the following property: for every $ab \in E(F)$, if $\varphi(ab) = uv$ then (i) $s_{uv} = s_{ab}$ and (ii) $t_u + t_v = t_a + t_b$. Obtain $G$ and $H$ from $I$ and $I^{'}$, respectively, by adding star components to one of them (if necessary) to make the numbers of 2-thorns equal. Then we can define a $P_3$-isomorphism $\tau$ from $G$ to $H$ and say that $\tau$ is a *diamond inflation* of $\varphi$.

**Remark 2**. If $D_{a,b}$ is a nontrivial diamond (i.e., $s_{ab} > 1$) in $G$, then there exists a unique and nontrivial diamond $D_{u,v}$ in $H$ (see the proof in [1]).

The type in this subsection is related to Whitney's exceptional edge-isomorphisms which is stated as follows:

**Theorem 2.2 (Whitney [6])** *Suppose that $\varphi$ is an edge-isomorphism from $G$ to $H$ where $G$ and $H$ are both connected. If $\varphi$ is not induced, then $i = |E(G)| = |E(H)| \in \{3,4,5,6\}$, $G$ and $H$ are isomorphic to $W_i$ and $W_i^{'}$ in some order, and $\varphi$ is equivalent to $\varphi_i$ or $\varphi_i^{-1}$, where* *(i)* $W_6 \cong W_6' \cong K_4$, *with* $V(W_6) = \{a,b,c,d\}$, $V(W_6') = \{u,v,w,x\}$, *and* $\varphi_6$ *maps* $ab \mapsto uv$, $ac \mapsto uw$, $ad \mapsto vw$, $bc \mapsto ux$, $bd \mapsto vx$ *and* $cd \mapsto wx$*;*

*(ii)* $W_5 = W_6 - cd$, $W_5' = W_6' - wx$ *and* $\varphi_5 = \varphi_6|E(W_5)$*;*

*(iii)* $W_4 = W_6 - \{bd, cd\}$, $W_4' = W_6' - \{vx, wx\}$ *and* $\varphi_4 = \varphi_6|E(W_4)$*; and*

*(iv)* $W_3 = W_6 - \{bc, bd, cd\} \cong K_{1,3}$, $W_3' = W_6' - x \cong K_3$, *and* $\varphi_3 = \varphi_6|E(W_3)$*.*

Then a $P_3$-isomorphism $\tau$ is said to be of *Whitney type* $i$ if $\tau$ or $\tau^{-1}$ is equivalent to a diamond inflation of $\varphi_i$ as above for $i = 3,4,5,6$.

Denote by $t_z$ the number of terminal edges incident with $z$ for $z$ in $\{a,b,c,d\}$ or $\{u,v,w,x\}$. Whitney type $P_3$-isomorphisms, according to condition (ii) of Diamond Inflation, give one equation from each pair of corresponding edges of the original Whitney graphs. In all four types, the corresponding edges $ab$ and $uv$, $ac$ and $uw$, $ad$ and $vw$ give that $t_a + t_b = t_u + t_v$, $t_a + t_c = t_u + t_w$, $t_a + t_d = t_v + t_w$, respectively. Solving for $t_u$, $t_v$, $t_w$ in terms of $t_a$, $t_b$, $t_c$, $t_d$ we get

$$\begin{cases} t_u = \frac{1}{2}(t_a + t_b + t_c - t_d) \\ t_v = \frac{1}{2}(t_a + t_b - t_c + t_d) \\ t_w = \frac{1}{2}(t_a - t_b + t_c + t_d) \end{cases} \qquad (1)$$

In Whitney type 4, 5 or 6 the corresponding edges $bc$ and $ux$ give that $t_b + t_c = t_u + t_x$. Combining Equ.(1), we obtain

$$t_x = \frac{1}{2}(-t_a + t_b + t_c + t_d) \qquad (2)$$

We assume that $t_x = 0$ in Whitney type 3 as $x \notin V(W_3')$. Because we require connected $P_3$-graphs, in the above Equ.(1) and Equ.(2) we must have $t_z = 0$ or 1 for every $z \in \{a,b,c,d\} \cup \{u,v,w,x\}$. We write $(t_a, t_b, t_c, t_d) \mapsto (t_u, t_v, t_w, t_x)$ if $t_a$, $t_b$, $t_c$, $t_d$ = 0 or 1, the corresponding solutions for $t_u$, $t_v$, $t_w$, $t_x$ by Equ.(1) and Equ.(2). For example: $(1,0,0,1) \mapsto (0,1,1,0)$ denotes that $t_a = 1$, $t_b = t_c = 0$ and $t_d = 1$ correspond to solutions $t_u = 0$ and $t_v = t_w = 1$ by Equ.(1) and $t_x = 0$ by Equ.(2). So it is easy to check that there are only the following eight cases satisfying $t_z = 0$ or 1 for every $z \in \{a,b,c,d\} \cup \{u,v,w,x\}$:

(i) $(0,0,0,0) \mapsto (0,0,0,0)$ for all four types.

(ii) $(1,1,0,0) \mapsto (1,1,0,0)$ for all four types.

(iii) $(1,0,1,0) \mapsto (1,0,1,0)$ for all four types.

(iv) $(1,0,0,1) \mapsto (0,1,1,0)$ for all four types.

(v) $(0,1,1,0) \mapsto (1,0,0,0)$ for type 3 and $(0,1,1,0) \mapsto (1,0,0,1)$ for type 4, 5 or 6.

(vi) $(0,1,0,1) \mapsto (0,1,0,0)$ for type 3 and $(0,1,0,1) \mapsto (0,1,0,1)$ for type 4, 5 or 6.

(vii) $(0,0,1,1) \mapsto (0,0,1,0)$ for type 3 and $(0,0,1,1) \mapsto (0,0,1,1)$ for type 4, 5 or 6.

(viii) $(1,1,1,1) \mapsto (1,1,1,0)$ for type 3 and $(1,1,1,1) \mapsto (1,1,1,1)$ for type 4, 5 or 6.

If a $P_3$-isomorphism $\tau$ or $\tau^{-1}$ is equivalent to a diamond inflation of $\varphi_i$ as above, and falls into one of the above cases (i) through (viii), then $\tau$ is said to be of *special Whitney type* $i$ for $i = 3$, 4, 5 or 6.

Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$. It is well-known that $K_{1,3}$ and $K_3$ are the only pair of nonisomorphic connected graphs with the same line graph. So, if $\tau$ is of special Whitney type 3, then $G \ncong H$. If $\tau$ is of special Whitney type $i$ ($i = 4$, 5 or 6), in order to find pairs of nonisomorphic connected graphs with isomorphic $P_3$-graphs, we should choose suitable widths of each corresponding diamonds. Otherwise, for example, let $G$ and $H$ be diamond inflations of $W_4$ and $W_4^{'}$, respectively, with $s_e = 2$ and $t_z = 1$ for every $e \in E(W_4) \cup E(W_4^{'})$ and every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$. Obviously, $G \cong H$.

## 2.4 Bipartite type

First, we also introduce the definition of bipartite type. Start with a positive integer $k$ and an arbitrary bipartite graph $F$ with at least one edge and with a bipartition $(A, B)$. Let $I$ and $I^{'}$ be different diamond inflations of $F$, where each edge $e$ is inflated to a diamond of the same width $s_e$ both times, but in producing $I$ each vertex $v$ has $t_v$ terminal edges added, while in producing $I^{'}$ it has $t_v^{'}$ terminal edges added. where

$$t_v^{'} = \begin{cases} t_v - k & \text{if } v \in A \\ \\ t_v + k & \text{if } v \in B \end{cases} \qquad (3)$$

Thus, we need $t_v \geq k$ for all $v \in A$. Let $\varphi$ be the identity edge-isomorphism from $F$ to itself. Clearly $\varphi$, $I$ and $I^{'}$ satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of $F$ has the form $ab$ with $a \in A$ and $b \in B$, so that $t_a^{'} + t_b^{'} = (t_a - k) + (t_b + k) = t_a + t_b$. We can therefore obtain a $P_3$-isomorphism $\tau$ by diamond inflation; $\tau$ is in general not induced. We say $\tau$ and $\tau^{-1}$, or any equivalent $P_3$-isomorphisms, are of *bipartite type*.

This case is similar to the above Whitney type. Because we require that the $P_3$-graphs of $I$ and $I^{'}$ are connected, we must have $t_v$, $t_v^{'} = 0$ or 1 for every $v \in A \cup B$. Since $k \leq t_v (v \in A)$, we have $k = 0$ or 1. If $k = 0$, then $I \cong I^{'}$. If $k = 1$, then $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$. Otherwise, if there is a vertex $u_0 \in A$ with $t_{u_0} = 0$ or a vertex $v_0 \in B$ with $t_{v_0} = 1$, then $t_{u_0}^{'} = -1$ or $t_{v_0}^{'} = 2$ by Equ.(3). Therefore we have a $P_3$-isomorphism $\tau_0$ from $I$ to $I^{'}$, where $t_u = 1$ and $t_u^{'} = 0$ for all $u \in A$, $t_v = 0$ and $t_v^{'} = 1$ for all $v \in B$, respectively. Then we say that $\tau_0$ and $\tau_0^{-1}$, or any equivalent $P_3$-isomorphism, are of *special bipartite type*. Therefore, this is the only case to find pairs of nonisomorphic connected graphs which have isomorphic connected $P_3$-graphs in the bipartite type.

## 2.5 $TBSD$-related to an induced $P_3$-isomorphism

In this subsection, we require that there are no isolated vertices in $P_3$-graphs. Then all $P_3$-isomorphisms are $BSD$-related to an induced one. It is clear that if two original graphs

$G$ and $H$ are connected with an isomorphic $P_3$-graph, then $G \cong H$ by the definition of $BSD$-related. Thus in this type, if we require connected $P_3$-graphs, then the original graph and its $P_3$-graph are one to one.

From the arguments in the above five subsections, we get the following corollary which is essential to the solution of our problem.

**Corollary 2.3** *Let $\tau$ be a $P_3$-isomorphism from $G$ to $H$, where $G$ and $H$ are nonisomorphic connected graphs with an isomorphic connected $P_3$-graph. Then $\tau$ is one of the following:*

*(i) $D$-related to a $P_3$-isomorphism of special Whitney type 3, 4, 5 or 6; or*

*(ii) $D$-related to a $P_3$-isomorphism of special bipartite type.*

# 3 Main result

Now we can state and show the main result of this paper.

**Theorem 3.1** *There is no triple of mutually nonisomorphic connected graphs with an isomorphic connected $P_3$-graph.*

**Proof.** Assume, to the contrary, that there exists a triple of mutually nonisomorphic connected graphs $G_1$, $G_2$ and $G_3$ which have an isomorphic connected $P_3$-graph. Let $\tau_i$ be a $P_3$-isomorphism from $G_i$ to $G_{i+1}$, then $\tau_i$ will be one of two types in Corollary 2.3 for $i = 1, 2$.

**Case 1.** $\tau_1$ and $\tau_2$ are of the same type.

**Subcase 1.1** $\tau_1$ and $\tau_2$ are both of $D$-related to a $P_3$-isomorphism of special Whitney type $i$ for $i = 3$, 4, 5 or 6.

By the definition of $\tau_1$, let $G_1$ and $G_2$ be diamond inflations of $W_i$ and $W_i^{'}$, respectively, with one of the eight cases in subsection 2.3. For $i = 3$, without loss of generality, suppose that $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$, $t_u = 0$, $t_v = t_w = 1$ and $t_x = 0$. Since $\tau_1$ and $\tau_2$ are of the same type, $G_3$ is also a diamond inflation of $W_3$, and $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$ by Equ.(1). Hence $G_1 \cong G_3$, a contradiction. For $i = 4$, 5 or 6, by a similar argument as the case for $i = 3$, we also get $G_1 \cong G_3$, a contradiction.

**Subcase 1.2** $\tau_1$ and $\tau_2$ are both of $D$-related to a $P_3$-isomorphism of special bipartite type.

This subcase is similar to Subcase 1.1. Denote by $F$ an arbitrary bipartite graph with a bipartition $(A, B)$. Then assume that $G_1$ and $G_2$ are different diamond inflations of $F$, respectively, where $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in $G_1$; $t_u = 0$ for all $u \in A$ and $t_v = 1$ for all $v \in B$ in $G_2$. Thus we can easily obtain that $G_3$ is also a diamond inflation of $F$ with $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in $G_3$ by the definition of $\tau_2$. Then $G_1 \cong G_3$, contrary to the assumption.

**Case 2.** $\tau_1$ and $\tau_2$ are of different types.

Let $\tau_1$ be $D$-related to a $P_3$-isomorphism of special Whitney type $i$ for $i = 3$, 4, 5 or 6 and $\tau_2$ be $D$-related to a $P_3$-isomorphism of special bipartite type.

For $i = 4, 5$ or 6, by the definition of $\tau_1$, $G_1$ and $G_2$ are diamond inflations of $W_i$ and $W_i^{'}$ which have odd cycles; and also by $\tau_2$, $G_2$ and $G_3$ are different diamond inflations of some bipartite graph. Then $\tau_1$ must be $D$-related to a $P_3$-isomorphism of special Whitney type 3. Thus there is only one possibility: $G_2$ is a diamond inflation of $K_{1,3}$, where $K_{1,3}$ has a bipartition $A = \{a\}$, $B = \{b, c, d\}$. By the definition of special Whitney type 3, $\tau_1$ falls into one of the following eight cases: $(0, 0, 0, 0) \mapsto (0, 0, 0, 0)$, $(1, 1, 0, 0) \mapsto (1, 1, 0, 0)$, $(1, 0, 1, 0) \mapsto (1, 0, 1, 0)$, $(1, 0, 0, 1) \mapsto (0, 1, 1, 0)$, $(0, 1, 1, 0) \mapsto (1, 0, 0, 0)$, $(0, 1, 0, 1) \mapsto (0, 1, 0, 0)$, $(0, 0, 1, 1) \mapsto (0, 0, 1, 0)$, or $(1, 1, 1, 1) \mapsto (1, 1, 1, 0)$. However, by the definition of special bipartite type, there are only two choices: either $t_a = 0$, $t_b = t_c = t_d = 1$, or $t_a = 1$, $t_b = t_c = t_d = 0$. Finally, there does not exist any graph $G_2$ that has common property of two different types at the same time. So $\tau_1$ and $\tau_2$ must be of the same type, a contradiction. The proof is thus complete. ∎

**Acknowledgement.** The authors are very grateful to the referees for their valuable suggestions and comments, which helped to improve the presentation of the paper.

# References


[1] R.E.L. Aldred, M.N. Ellingham, R.L. Hemminger and P. Jipsen, *$P_3$-isomorphisms for graphs*, J. Graph Theory **26** (1997) 35-51.

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, MacMillan/Elesevier, London/New York, 1976.

[3] H.J. Broersma and C. Hoede, *Path graphs*, J. Graph Theory **13** (1989) 427-444.

[4] R.L. Hemminger and L.W. Beineke, *Line graphs and line digraphs*, in: L.W. Beineke and R.J. Wilson, eds, Selected Topics in Graph Theory, Academic Press, London, New York, San Francisco, 1978.

[5] X. Li and B. Zhao, Isomorphisms of $P_k$-graphs for $k \geq 4$, Discrete Math. **289** (2004) 71-80.

[6] H. Whitney, *Congruent graphs and connectivity of graphs*, Amer. J. Math. **54** (1932) 150-168.