Generating function identities for $\zeta(2n+2), \zeta(2n+3)$ via the WZ method

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Abstract

Using WZ-pairs we present simpler proofs of Koecher, Leshchiner and Bailey-Borwein-Bradley's identities for generating functions of the sequences $\{\zeta(2n+2)\}_{n\geq 0}$ and $\{\zeta(2n+3)\}_{n\geq 0}$. By the same method, we give several new representations for these generating functions yielding faster convergent series for values of the Riemann zeta function.

1 Introduction

The Riemann zeta function is defined by the series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad \text{for} \quad \mathrm{Re}(s)>1.$$

Apéry's irrationality proof of $\zeta(3)$ and series acceleration formulae for the first values of the Riemann zeta function going back to Markov's work [8]

$$\zeta(2)=3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}, \qquad \zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}, \qquad \zeta(4)=\frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}$$

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stimulated intensive search of similar formulas for other values $\zeta(n), n \geq 5$. Many Apéry-like formulae have been proved with the help of generating function identities (see [6, 1, 5, 11, 4]). M. Koecher [6] (and independently Leshchiner [7]) proved that

$$\sum_{k=0}^{\infty} \zeta(2k+3)a^{2k} = \sum_{n=1}^{\infty} \frac{1}{n(n^2-a^2)} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \, \frac{5k^2-a^2}{k^2-a^2} \prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right), \tag{1}$$

for any $a \in \mathbb{C}$, with $|a| < 1$. For even zeta values, Leshchiner [7] (in an expanded form) showed that (see [4, (31)])

$$\sum_{k=0}^{\infty}\left(1-\frac{1}{2^{k+1}}\right)\zeta(2k+2)a^{2k} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} \, \frac{3k^2+a^2}{k^2-a^2}\prod_{m=1}^{k-1}\left(1-\frac{a^2}{m^2}\right), \tag{2}$$

for any complex a, with $|a| < 1$. Recently, D. Bailey, J. Borwein and D. Bradley [4] proved another formula

$$\sum_{k=0}^{\infty}\zeta(2k+2)a^{2k} = \sum_{n=1}^{\infty}\frac{1}{n^2-a^2} = 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-a^2)}\prod_{m=1}^{k-1}\left(\frac{m^2-4a^2}{m^2-a^2}\right), \tag{3}$$

for any $a \in \mathbb{C}, |a| < 1$.

In this paper, we present simpler proofs of identities (1)–(3) using WZ-pairs. By the same method, we give some new representations for the generating functions (1), (3) yielding faster convergent series for values of the Riemann zeta function.

We recall [12] that a discrete function $A(n,k)$ is called hypergeometric or closed form (CF) if the quotients

$$\frac{A(n+1,k)}{A(n,k)} \qquad \text{and} \qquad \frac{A(n,k+1)}{A(n,k)}$$

are both rational functions of n and k. A pair of CF functions $F(n,k)$ and $G(n,k)$ is called a WZ-pair if

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{4}$$

First application of the WZ-pairs to obtain convergence acceleration formulae for certain slowly convergent numerical series of hypergeometric type (in particular, for $\zeta(3)$) refers to Markov's work [8] in 1890. Markov starts with a proper hypergeometric kernel $H(n,k)$ and then tries to determine two functions $P(n,k)$, and $Q(n,k)$, which are polynomials in k with coefficients depending on n, in such a way that $F(n,k) = H(n,k)P(n,k)$ and $G(n,k) = H(n,k)Q(n,k)$ form a WZ-pair.

Recently, M. Mohammed and D. Zeilberger [10] turned out that Markov's method can be combined with the parametric Gosper algorithm to produce an algorithm which, for a given $H(n,k)$, outputs the desired $P(n,k) = \sum_{i=0}^{d} a_i(n)k^i$ and $Q(n,k)$, where $Q(n,k)$ is a rational function of k and the sequences $a_i(n)$ satisfy the initial conditions

$$a_0(0) = 1, \qquad a_i(0) = 0, \qquad 1 \leq i \leq d.$$

Paper [10] is accompanied by the Maple package MarkovWZ together with examples of accelerating formulae available from the second author's website. Many other new representations for $\log 2, \zeta(2), \zeta(3)$ were found in [9].

In all the proofs considered below, we start with a simple kernel $H(n,k)$, apply the Maple package MarkovWZ and find that $d=0$ implying

$$F(n,k)=H(n,k)a_0(n), \quad G(n,k)=H(n,k)Q(n,k), \quad F(0,k)=H(0,k).$$

We need the following summation formulas.

Proposition 1. ([3, Formula 2]) *For any WZ-pair* (F,G)

$$\sum_{k=0}^{\infty} F(0,k)-\lim_{n\to\infty}\sum_{k=0}^{n} F(n,k)=\sum_{n=0}^{\infty} G(n,0)-\lim_{k\to\infty}\sum_{n=0}^{k} G(n,k),$$

whenever both sides converge.

Proposition 2. ([3, Formula 3]) *For any WZ-pair* (F,G) *we have*

$$\sum_{n=0}^{\infty} G(n,0)=\sum_{n=0}^{\infty}(F(n+1,n)+G(n,n))-\lim_{n\to\infty}\sum_{k=0}^{n-1} F(n,k),$$

whenever both sides converge.

As usual, let $(\lambda)_\nu$ be the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1, & \nu=0;\\ \lambda(\lambda+1)\dots(\lambda+\nu-1), & \nu\in\mathbb{N}.\end{cases}$$

2 Proof of Koecher's identity

Consider

$$H(n,k)=\frac{k!}{(2n+k+1)!((n+k+1)^2-a^2)}.$$

Then we have

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$$

with

$$F(n,k)=\frac{(-1)^n k!(1+a)_n(1-a)_n}{(2n+k+1)!((n+k+1)^2-a^2)},$$

$$G(n,k)=\frac{(-1)^n k!(1+a)_n(1-a)_n(5(n+1)^2-a^2+k^2+4k(n+1))}{(2n+k+2)!((n+k+1)^2-a^2)(2n+2)}.$$

Hence (F,G) is a WZ-pair and by Proposition 1, we get

$$\sum_{k=0}^{\infty} H(0,k)=\sum_{k=0}^{\infty} F(0,k)=\sum_{n=0}^{\infty} G(n,0),$$

or

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2-a^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n (1+a)_n (1-a)_n (5(n+1)^2-a^2)}{(2n+2)!(2n+2)((n+1)^2-a^2)}$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(5n^2-a^2)}{n^3 \binom{2n}{n}(n^2-a^2)} \prod_{m=1}^{n-1} \left(1 - \frac{a^2}{m^2}\right).$$

3 Proof of Leshchiner's identity

Consider

$$H(n,k) = \frac{(-1)^k k!(n+k+1)}{(2n+k+1)!((n+k+1)^2-a^2)}.$$

Then we have

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

with

$$F(n,k) = \frac{(-1)^k k!(1+a)_n(1-a)_n(n+k+1)}{(2n+k+1)!((n+k+1)^2-a^2)},$$

$$G(n,k) = \frac{(-1)^k k!(1+a)_n(1-a)_n(3(n+1)^2+a^2+k^2+4k(n+1))}{2(2n+k+2)!((n+k+1)^2-a^2)},$$

and by Proposition 1, we get

$$\sum_{k=0}^{\infty} H(0,k) = \sum_{n=0}^{\infty} G(n,0),$$

or

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2-a^2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1+a)_n(1-a)_n(3(n+1)^2+a^2)}{(2n+2)!((n+1)^2-a^2)}$$
$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{3n^2+a^2}{n^2\binom{2n}{n}(n^2-a^2)} \prod_{m=1}^{n-1} \left(1 - \frac{a^2}{m^2}\right).$$

4 Proof of the Bailey-Borwein-Bradley identity

Consider

$$H(n,k) = \frac{(1+a)_k(1-a)_k}{(1+a)_{n+k+1}(1-a)_{n+k+1}}.$$

Then we have

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

with

$$F(n,k) = \frac{n!^2(1+a)_k(1-a)_k(1+2a)_n(1-2a)_n}{(2n)!(1+a)_{n+k+1}(1-a)_{n+k+1}},$$

$$G(n,k)=\frac{(1+a)_k(1-a)_k(1+2a)_n(1-2a)_n n!(n+1)!(3n+3+2k)}{(1+a)_{n+k+1}(1-a)_{n+k+1}(2n+2)!},$$

and (F,G) is a WZ-pair. Then

$$\sum_{k=0}^{\infty}H(0,k)=\sum_{n=0}^{\infty}G(n,0),$$

and therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(k^2-a^2)}&=3\sum_{n=0}^{\infty}\frac{(1+2a)_n(1-2a)_n(n+1)!^2}{(1+a)_{n+1}(1-a)_{n+1}(2n+2)!}\\&=3\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}(n^2-a^2)}\prod_{m=1}^{n-1}\left(\frac{m^2-4a^2}{m^2-a^2}\right),\end{aligned}$$

as required.

5 New generating function identities for $\zeta(2n+2)$ and $\zeta(2n+3)$

Theorem 1 *Let a be a complex number not equal to a non-zero integer. Then*

$$\sum_{k=1}^{\infty}\frac{1}{k(k^2-a^2)}=\sum_{n=1}^{\infty}\frac{a^4-a^2(32n^2-10n+1)+2n^2(56n^2-32n+5)}{2n^3\binom{2n}{n}\binom{3n}{n}((2n-1)^2-a^2)(4n^2-a^2)}\prod_{m=1}^{n-1}\left(\frac{a^2}{m^2}-1\right). \tag{5}$$

Expanding both sides of (5) in powers of a^2 and comparing coefficients of a^{2n} gives Apéry-like series for $\zeta(2n+3)$ for every non-negative integer n convergent at the geometric rate with ratio $1/27$. In particular, comparing constant terms recovers Amdeberhan's formula [2] for $\zeta(3)$

$$\zeta(3)=\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{56n^2-32n+5}{n^3(2n-1)^2\binom{2n}{n}\binom{3n}{n}}.$$

Similarly, comparing coefficients of a^2 gives

$$\zeta(5)=\frac{3}{16}\sum_{n=1}^{\infty}\frac{(4n-1)(16n^3-8n^2+4n-1)}{(-1)^{n-1}n^5(2n-1)^4\binom{2n}{n}\binom{3n}{n}}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n(56n^2-32n+5)}{n^3(2n-1)^2\binom{2n}{n}\binom{3n}{n}}\sum_{k=1}^{n-1}\frac{1}{k^2}.$$

Proof. Consider

$$H(n,k)=\frac{k!(1+a)_k(1-a)_k}{(2n+k+1)!(1+a)_{2n+k+1}(1-a)_{2n+k+1}}.$$

Then application of the Markov-WZ algorithm produces

$$F(n,k)=\frac{(-1)^n n!(2n)!k!(1+a)_k(1-a)_k(1+a)_n(1-a)_n(1+a)_{2n}(1-a)_{2n}}{(3n)!(2n+k+1)!(1+a)_{2n+k+1}(1-a)_{2n+k+1}},$$

$$G(n,k)=\frac{(-1)^n k!n!(2n)!(1+a)_k(1-a)_k(1+a)_n(1-a)_n(1+a)_{2n}(1-a)_{2n}}{6(3n+2)!(2n+k+2)!(1+a)_{2n+k+2}(1-a)_{2n+k+2}}q(n,k)$$

satisfying (4), with

$$\begin{aligned}q(n,k)&=2(2n+1)(a^4-a^2(32n^2+54n+23)+2(n+1)^2(56n^2+80n+29))\\&+k^4(9n+6)+k^3(90n^2+132n+48)+k^2(348n^3+792n^2-15a^2n+594n+147\\&-9a^2)+k(624n^4+1932n^3+2214n^2-84a^2n^2-117a^2n+1113n+207-39a^2).\end{aligned}$$

By Proposition 1, we have

$$\sum_{k=0}^{\infty} H(0,k)=\sum_{n=0}^{\infty} G(n,0)$$

or equivalently,

$$\sum_{k=1}^{\infty}\frac{1}{k(k^2-a^2)}=$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n n!(1-a)_n(1+a)_n(a^4-a^2(32n^2+54n+23)+2(n+1)^2(56n^2+80n+29))}{2(3n+3)!((2n+1)^2-a^2)((2n+2)^2-a^2)},$$

and the theorem follows.

Theorem 2 *Let a be a complex number not equal to a non-zero integer. Then*

$$\sum_{k=1}^{\infty}\frac{1}{k^2-a^2}=\sum_{n=1}^{\infty}\frac{n^2(21n-8)-a^2(9n-2)}{\binom{2n}{n}n(n^2-a^2)(4n^2-a^2)}\prod_{k=1}^{n-1}\left(\frac{k^2-4a^2}{(k+n)^2-a^2}\right). \qquad (6)$$

Formula (6) generates Apéry-like series for $\zeta(2n+2)$ for every non-negative integer n convergent at the geometric rate with ratio $1/64$. In particular, it follows that

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3} \qquad (7)$$

and

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{69n-32}{4n^5\binom{2n}{n}^3}-\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3}\sum_{k=1}^{n-1}\left(\frac{4}{k^2}-\frac{1}{(k+n)^2}\right).$$

Another proof of formula (7) can be found in [12, §12].

Proof. Consider

$$H(n,k)=\frac{(1+a)_{n+k}(1-a)_{n+k}}{(1+a)_{2n+k+1}(1-a)_{2n+k+1}}.$$

Application of the Markov-WZ algorithm produces

$$F(n,k) = \frac{n!^2(1+2a)_n(1-2a)_n(1+a)_{n+k}(1-a)_{n+k}}{(2n)!(1+a)_{2n+k+1}(1-a)_{2n+k+1}},$$

$$G(n,k) = \frac{n!^2(1+a)_{n+k}(1-a)_{n+k}(1+2a)_n(1-2a)_n}{2(2n+1)!(1+a)_{2n+k+2}(1-a)_{2n+k+2}} q(n,k)$$

satisfying (4), with

$$q(n,k) = (n+1)^2(21n+13) - a^2(9n+7) + 2k^3 + k^2(13n+11) + k(28n^2+48n+20-2a^2).$$

By Proposition 1,

$$\sum_{k=0}^{\infty} H(0,k) = \sum_{n=0}^{\infty} G(n,0),$$

which implies (6).

Theorem 3 *Let a be a complex number not equal to a non-zero integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2-a^2)} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(1+a)_n^2(1-a)_n^2((n+1)^2(30n+19)-a^2(12n+7))}{(1+a)_{2n+2}(1-a)_{2n+2}(n+1)(2n+1)}.$$

Proof. Consider

$$H(n,k) = \frac{(1+a)_k(1-a)_k}{(1+a)_{2n+k+1}(1-a)_{2n+k+1}(n+k+1)}.$$

Then application of the Markov-WZ algorithm produces

$$F(n,k) = \frac{(1+a)_k(1-a)_k(1+a)_n^2(1-a)_n^2}{(1+a)_{2n+k+1}(1-a)_{2n+k+1}(n+k+1)},$$

$$G(n,k) = \frac{(1+a)_k(1-a)_k(1+a)_n^2(1-a)_n^2 q(n,k)}{4(1+a)_{2n+k+2}(1-a)_{2n+k+2}(n+k+1)(n+1)(2n+1)},$$

with

$$\begin{aligned} q(n,k) = {} & (n+1)^3(30n+19) - a^2(n+1)(12n+7) + 2k^3(n+1) + 2k^2(7n^2+13n+6) \\ & + k(34n^3+93n^2+84n-4a^2n+25-3a^2). \end{aligned}$$

Now by Proposition 1, the theorem follows.

Theorem 4 *Let a be a complex number not equal to a non-zero integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2-a^2)} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}^5} \frac{p(n,a)}{(n^2-a^2)(4n^2-a^2)} \prod_{m=1}^{n-1}\left(\frac{(1-a^2/m^2)^2}{1-a^2/(n+m)^2}\right), \quad (8)$$

where

$$p(n,a) = a^4 - a^2(62n^2 - 40n + 8) + n^2(205n^2 - 160n + 32).$$

Formula (8) generates Apéry-like series for $\zeta(2n+3)$, $n \geq 0$, convergent at the geometric rate with ratio 2^{-10}. In particular, if $a = 0$ we get the formula of Amdeberhan and Zeilberger [3]

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5}.$$

Comparing coefficients of a^2 leads to

$$\begin{aligned} \zeta(5) = & \sum_{n=1}^{\infty} \frac{(-1)^n (31n^2 - 20n + 4)}{n^7 \binom{2n}{n}^5} \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n (205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5} \left(\sum_{m=1}^{n-1} \frac{1}{m^2} - \sum_{m=0}^{n} \frac{1}{2(m+n)^2} \right). \end{aligned}$$

Proof. Consider

$$F(n,k) = \frac{(-1)^k (1+a)_k (1-a)_k (1+a)_n^2 (1-a)_n^2 (2n-k-1)! k! n!^2}{2(n+k+1)!^2 (2n)! (1+a)_{2n} (1-a)_{2n}}.$$

Then

$$G(n,k) = \frac{(-1)^k (1+a)_k (1-a)_k (1+a)_n^2 (1-a)_n^2 (2n-k)! k! n!^2 q(n,k)}{4(2n+1)!(n+k+1)!^2 (1+a)_{2n+2} (1-a)_{2n+2}},$$

with

$$q(n,k) = (n+1)^3(30n+19) - a^2(n+1)(12n+7) + k(21n^3 + 55n^2 + 47n + 13 - 3a^2 n - a^2),$$

is a WZ mate such that

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} \frac{(1+a)_n^2 (1-a)_n^2 ((n+1)^2(30n+19) - a^2(12n+7))}{4(n+1)(2n+1)(1+a)_{2n+2}(1-a)_{2n+2}} = \sum_{k=1}^{\infty} \frac{1}{k(k^2-a^2)},$$

by Theorem 3. Now by Proposition 2, the theorem follows.

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