An elementary proof of the hook formula

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Submitted: Nov 29, 2007; Accepted: Mar 4, 2008; Published: Mar 12, 2008
Mathematics Subject Classification: 05E10

Abstract

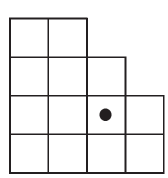
The hook-length formula is a well known result expressing the number of standard tableaux of shape λ in terms of the lengths of the hooks in the diagram of λ. Many proofs of this fact have been given, of varying complexity. We present here an elementary new proof which uses nothing more than the fundamental theorem of algebra. This proof was suggested by a q, t-analog of the hook formula given by Garsia and Tesler, and is roughly based on the inductive approach of Greene, Nijenhuis and Wilf. We also prove the hook formula in the case of shifted Young tableaux using the same technique.

1 Introduction

For a natural number n, we say λ is a *partition* of n, and write $\lambda \vdash n$, if λ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers satisfying

1. $\sum_{i=1}^{k} \lambda_i = n$ and
2. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

The *Young diagram* of a partition is an array of boxes, or cells, in the plane, left-justified, with λ_i cells in the i^{th} row from the bottom. We label these cells (i, j), with i denoting the row and j the column. For example, in the following Young diagram of $(4, 4, 3, 2)$, the cell $(2, 3)$ is marked: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*Research supported in part by NSF grant DMS-0500557

We identify a partition with its Young diagram throughout. The *hook length* of a cell $c \in \lambda$ is the number of cells weakly above and strictly to the right of c. We denote this by $h_\lambda(c)$. For example, in the diagram above, $h_\lambda((2,3)) = 3$.

A *standard tableau* of shape λ is a labelling of the cells of the Young diagram of λ with the numbers 1 to n so that the labels are strictly increasing from bottom to top along columns and from left to right along rows. For example, there are 5 standard tableaux of shape $(3, 2)$:

$$\begin{array}{ccc} 4 & 5 & \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 3 & 5 & \\ 1 & 2 & 4 \end{array} \qquad \begin{array}{ccc} 3 & 4 & \\ 1 & 2 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 5 & \\ 1 & 3 & 4 \end{array} \qquad \begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}.$$

We denote the number of standard tableaux of shape λ by f_λ. This number has implications beyond combinatorics; The work of Alfred Young [You01, You02] shows that f_λ gives the dimension of the irreducible representation indexed by λ.

The following is the celebrated hook length formula.

Theorem 1 ([FRT54]).

$$f_\lambda = \frac{n!}{\prod_{s\in\lambda} h_\lambda(s)} \tag{1}$$

Since this was first proved by Frame, Robinson and Thrall, many different proofs have been given ([GNW79], [Kra95], [NPS97] for just some examples). These proofs are quite useful; [GNW79], for example, provides an "intuitive" reason to believe the formula, while [Kra95] and [NPS97] provide bijective proofs. However, these have the disadvantage of appearing somewhat complicated to those readers unfamiliar with probability theory or the combinatorics of Young diagrams. Other proofs have the disadvantage of not being particularly combinatorial. We offer the proof below as a simple combinatorial approach for the non-specialist. In addition, we hope that more experienced combinatorialists will be interested in the connections which the proof reveals. In the section 4, we apply the proof to the shifted tableaux case.

2 Acknowledgements

The author is grateful to Adriano Garsia for pointing out the q, t-analog of the hook formula in [GT96]. This suggested that the recursion for the hook formula should have an expression in terms of a rational function involving content numbers, which led to the proof given here. Thanks also is due to the National Science Foundation for support, and to an anonymous reviewer for many useful suggestions.

3 A proof of the hook formula

Given partitions $\lambda \vdash n$, $\mu \vdash (n-1)$, we say that μ precedes λ (denoted by $\mu \to \lambda$) if the Young diagram of μ is contained in the Young diagram of λ. Given a standard tableau of shape λ, it is immediate that removing the cell containing n gives a standard tableau of shape $\mu \to \lambda$. It is not hard to see that all standard tableaux of a shape preceding λ can be obtained in such a manner. Thus we see that the number of standard tableaux satisfies the recursion

$$f_\lambda = \sum_{\mu \to \lambda} f_\mu.$$

Our goal is to show the right side of (1) satisfies the same recursion. That is, we wish to show

$$\frac{n!}{\prod_{s \in \lambda} h_\lambda(s)} = \sum_{\mu \to \lambda} \frac{(n-1)!}{\prod_{s \in \mu} h_\mu(s)}$$

or, more simply,

$$\sum_{\mu \to \lambda} \frac{\prod_{s \in \lambda} h_\lambda(s)}{\prod_{s \in \mu} h_\mu(s)} = n. \tag{2}$$

The proof will proceed in the following three steps. We let m be the number of corners of λ, and define certain numbers $x_i, y_i, 0 \leq i \leq m$, depending on λ. We then prove

$$\begin{aligned} \sum_{\mu \to \lambda} \frac{\prod_{s \in \lambda} h_\lambda(s)}{\prod_{s \in \mu} h_\mu(s)} &= -\sum_{i=1}^{m} \frac{\prod_{j=0}^{m} (x_i - y_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{m} (x_i - x_j)} && (3) \\ &= -\frac{1}{2} \sum_{i=0}^{m} \left(x_i^2 - y_i^2 \right) && (4) \\ &= n. && (5) \end{aligned}$$

The proof of (3) is completely combinatorial, the proof of (4) uses nothing more than the fundamental theorem of algebra, and the proof of (5) is a simple geometric argument.

Proof of (3). The *outer corners* of the Young diagram for λ are those cells which can be removed to give the diagram of a partition $\mu \to \lambda$. For a fixed λ we label the outer corners from top to bottom as $X_i = (\alpha_i, \beta_i)$, for $1 \leq i \leq m$. We then set $Y_i = (\alpha_{i+1}, \beta_i)$, for $0 \leq i \leq m$, where we set $\beta_0 = 0 = \alpha_{m+1}$; we call these cells the *inner corners* of the diagram. We also define the cell $X_0 = (\alpha_0, \beta_0) = (0, 0)$. Note that the cells X_0, Y_0, Y_m are outside of the diagram of the partition. For $1 \leq i \leq m$, we denote by $\mu^{(i)}$ the partition given by λ with the cell X_i removed. An example of a partition with labelled corners is shown in Figure 1.

The *content* of a cell $c = (i, j)$ is defined to be $j - i$, and is denoted by $ct(c)$. For example, the diagram of the partition $(4, 3, 2, 2)$ with the content of every cell labelled is

$$D = \begin{array}{|c|c|cc}
\cline{1-2}
-3 & -2 & & \\ \cline{1-2}
-2 & -1 & & \\ \cline{1-3}
-1 & 0 & \multicolumn{1}{c|}{1} & \\ \hline
0 & 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline
\end{array}\ .$$

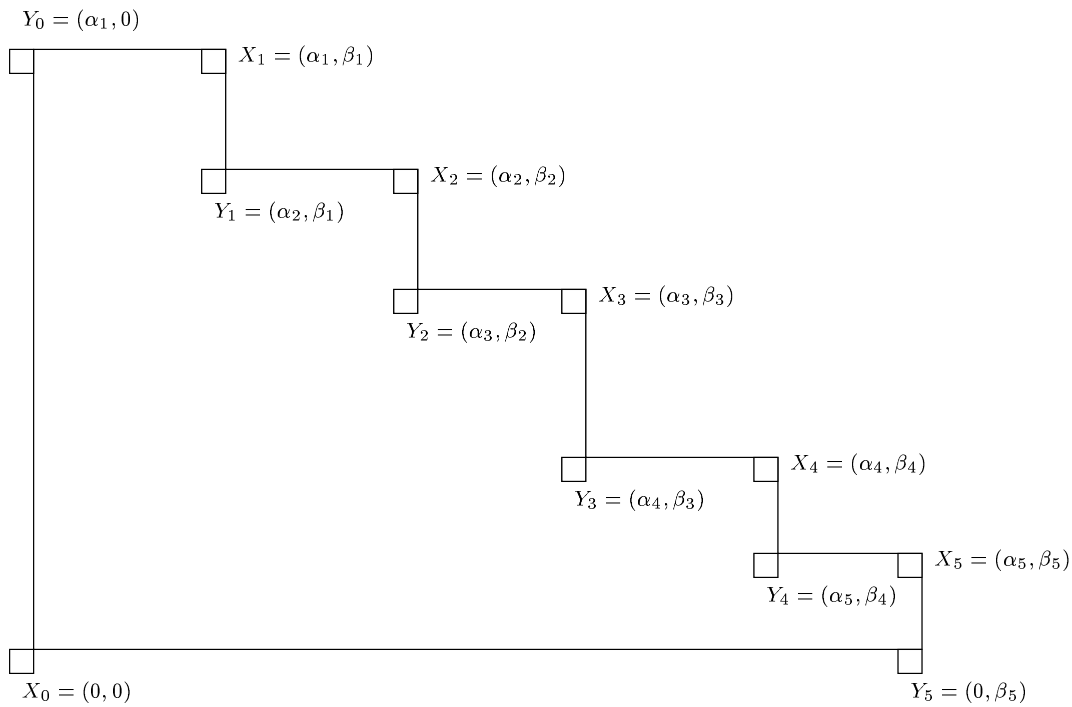


Figure 1: Partition with labelled corners

Content is a well-known statistic on the cells of a Young diagram, with many applications. For us, the primary use of content will be to express hook-lengths. If we set $E(c)$ to be the cell at the East end of the row containing c, and $N(c)$ to be the cell at the North end of the column containing c, we have:

$$h_\lambda(c) = ct(E(c)) - ct(N(c)) + 1. \tag{6}$$

This is because the content changes by one with each step along the hook, as we traverse from East to North. For example, it is easy to see from the diagram D that for $\lambda = (4, 3, 2, 2)$, we have $h_\lambda(2, 2) = (1) - (-2) + 1 = 4$. For $0 \leq i \leq m$, we set $x_i = ct(X_i)$ and $y_i = ct(Y_i)$.

We note for reference here a relation that follows from this labelling:

$$\sum_{i=0}^{m} x_i = \sum_{i=0}^{m} y_i. \tag{7}$$

This is due to the fact that in every row or column of the diagram in which a labelled cell appears, we have exactly one cell labelled with an X and exactly one labelled with a Y. Thus every row-coordinate and every column-coordinate will cancel in the sum

$\sum_{i=0}^{m}(x_i - y_i)$. In more detail we have

$$\sum_{i=0}^{m}(x_i - y_i) = \sum_{i=0}^{m}(\beta_i - \alpha_i) - (\beta_i - \alpha_{i+1})$$
$$= -\alpha_0 + \alpha_{m+1} = 0.$$

We now express the left side of (2) in terms of the x_i and y_i. For fixed i, there will be massive cancellation in the quotient

$$\frac{\prod_{c\in\lambda} h_\lambda(c)}{\prod_{c\in\mu^{(i)}} h_{\mu^{(i)}}(c)}.$$

This cancellation is illustrated in Figure 2, and described below.

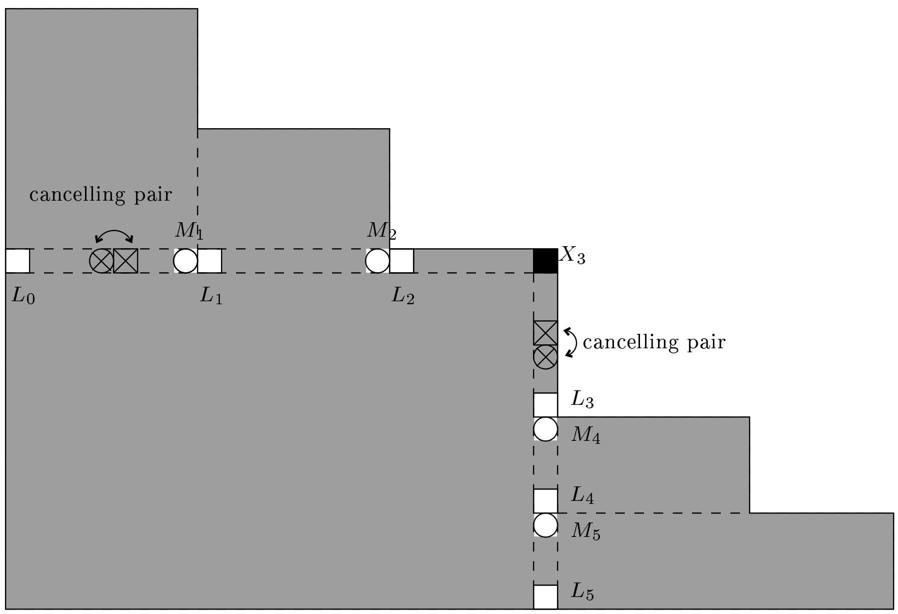


Figure 2: Non-cancelling cells. Squares should be viewed as cells in λ, circles as cells in $\mu^{(3)}$.

We first note that every cell not in the row or column of X_i will have the same hook length whether considered as a cell in λ or a cell in $\mu^{(i)}$. Thus, the factors corresponding to these cells will all cancel in the quotient. In fact, there will be even more cancellation. A "generic" cell of λ in the row of X_i will have the same hook length as the cell immediately to its left, considered as a cell of $\mu^{(i)}$. In symbols, we can say most pairs of cells of the form

$$(\alpha_i, b) \in \lambda \quad \text{and} \quad (\alpha_i, b-1) \in \mu^{(i)}$$

will have equal (and thus cancelling) hook-lengths. The cells for which this won't work will be the those beneath corner cells. To be precise, we label the cells of λ in the row of X_i which do not cancel as

$$L_j = (\alpha_i, \beta_j + 1) \qquad \text{for } 0 \le j < i.$$

We label the corresponding non-cancelling cells of $\mu^{(i)}$ in the row of X_i as

$$M_j = (\alpha_i, \beta_j) \qquad \text{for } 1 \leq j < i.$$

Note that we do not need to worry about the cell X_i itself, as it has a hook-length of 1 in λ and does not exist in $\mu^{(i)}$.

The cells in the column of X_i are similarly described. We label the non-cancelling cells in λ as

$$L_j = (\alpha_{j+1} + 1, \beta_i) \qquad \text{for } i \leq j \leq m$$

and the corresponding cells in $\mu^{(i)}$ as

$$M_j = (\alpha_j, \beta_i) \qquad \text{for } i < j \leq m.$$

Thus the left hand side of (3) reduces to

$$\sum_{i=1}^{m} \frac{\prod_{j=0}^{m} h_\lambda(L_j)}{\prod_{\substack{j=1\\ j\neq i}}^{m} h_{\mu^{(i)}}(M_j)} \ .$$

We now compute these hook lengths using equation (6). For $0 < j < i$, we have $y_j = ct(N(L_j)) - 1$, since Y_j is one unit to the left of $N(L_j)$. Thus

$$\begin{aligned} h_\lambda(L_j) &= ct(E(L_j)) - ct(N(L_j)) + 1 \\ &= x_i - y_j. \end{aligned}$$

Similarly, for $1 < j < i$, the cell X_i is one unit to the right of $E(M_j)$ in $\mu^{(i)}$. Therefore

$$\begin{aligned} h_{\mu^{(i)}}(M_j) &= ct(E(M_j)) + 1 - ct(N(M_j)) \\ &= x_i - x_j. \end{aligned}$$

An analogous computation for $i \leq j \leq m$ gives

$$\begin{aligned} h_\lambda(L_j) &= ct(E(L_j)) - x_i + 1 \\ &= y_j - x_i \end{aligned}$$

and for $i < j \leq m$,

$$\begin{aligned} h_{\mu^{(i)}}(M_j) &= x_j - ct(N(M_j)) + 1 \\ &= x_j - x_i. \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{\mu\to\lambda} \frac{\prod_{s\in\lambda} h_\lambda(s)}{\prod_{s\in\mu} h_\mu(s)} &= \sum_{i=1}^{m} \frac{\prod_{j=0}^{m} h_\lambda(L_j)}{\prod_{\substack{j=1\\ j\neq i}}^{m} h_{\mu^{(i)}}(M_j)} \\ &= \sum_{i=1}^{m} \frac{\prod_{j=0}^{i-1}(x_i - y_j)\prod_{j=i}^{m} -(x_i - y_j)}{\prod_{j=1}^{i-1}(x_i - x_j)\prod_{j=i+1}^{m} -(x_i - x_j)} \\ &= -\sum_{i=1}^{m} \frac{\prod_{j=0}^{m}(x_i - y_j)}{\prod_{\substack{j=1\\ j\neq i}}^{m}(x_i - x_j)} \ . \end{aligned}$$

$\square$

Proof of (4). We wish to show

$$-\sum_{i=1}^{m} \frac{\prod_{j=0}^{m}(x_i - y_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}(x_i - x_j)} = -\frac{1}{2}\sum_{i=0}^{m}\left(x_i^2 - y_i^2\right).$$

Those readers familiar with Lagrange interpolation may find the left hand side of this equation suggestive. In this vein, we consider the polynomial (in the single variable t)

$$P(t) = -\sum_{i=1}^{m} \frac{\prod_{j=0}^{m}(x_i - y_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}(x_i - x_j)} \prod_{\substack{j=1 \\ j\neq i}}^{m}(t - x_j).$$

One quickly verifies that this polynomial has the following properties:

1. $P(x_s) = -\prod_{j=0}^{m}(x_s - y_j)$ for $1 \leq s \leq m$ and
2. $P(t)$ has degree $m-1$, with leading coefficient

$$-\sum_{i=1}^{m} \frac{\prod_{j=0}^{m}(x_i - y_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}(x_i - x_j)}.$$

Since this quantity is the left hand side of (4), we can complete the proof by evaluating the coefficient of t^{m-1} in $P(t)$ in a different manner. Consider the polynomial

$$Q(t) = \prod_{j=0}^{m}(t - y_j)$$

and note that $Q(t)$ satisfies

1. $Q(x_s) = \prod_{j=0}^{m}(x_s - y_j)$ for $1 \leq s \leq m$ and
2. the leading term of $Q(t)$ is t^{m+1}.

Thus the polynomial $Q(t) + P(t)$ has leading term t^{m+1} and has a zero at $t = x_s$ for $1 \leq s \leq m$. Hence, for some α,

$$\begin{aligned}
Q(t) + P(t) =& (t-\alpha)\prod_{s=1}^{m}(t - x_s) \\
\implies P(t) =& (t-\alpha)\prod_{s=1}^{m}(t - x_s) - \prod_{j=0}^{m}(t - y_j) \\
=& \left(-\alpha - \sum_{i=1}^{m} x_i + \sum_{i=0}^{m} y_i\right) t^m + \\
& \left(\alpha\sum_{i=1}^{m} x_i + \sum_{1\leq i<j\leq m} x_i x_j - \sum_{0\leq i<j\leq m} y_i y_j\right) t^{m-1} + \ldots
\end{aligned}$$

Since $P(t)$ has degree $m-1$, the coefficient of t^m must be 0. Since $x_0 = 0$, (7) implies that $\alpha = 0$.

Using again that $x_0 = 0$, the coefficient of t^{m-1} in $P(t)$ can be written as

$$\sum_{0 \le i < j \le m} (x_i x_j - y_i y_j) \, .$$

Finally, we note that

$$\begin{aligned} \sum_{0 \le i < j \le m} (x_i x_j - y_i y_j) &= \frac{1}{2} \left(\left(\sum_{i=0}^m x_i \right)^2 - \left(\sum_{i=0}^m y_i \right)^2 - \sum_{i=0}^m \left(x_i^2 - y_i^2 \right) \right) \\ &= -\frac{1}{2} \sum_{i=0}^m \left(x_i^2 - y_i^2 \right) \end{aligned}$$

where the second equality follows from another application of (7). □

Proof of (5). Expanding the x_i and y_i in terms of the coordinates gives

$$\begin{aligned} -\frac{1}{2} \sum_{i=0}^m (x_i^2 - y_i^2) &= -\frac{1}{2} \sum_{i=0}^m (\beta_i - \alpha_i)^2 - (\beta_i - \alpha_{i+1})^2 \\ &= \alpha_0^2 - \alpha_{m+1}^2 - \frac{1}{2} \sum_{i=0}^m -2\beta_i \alpha_i + 2\beta_i \alpha_{i+1} \\ &= \sum_{i=1}^m \beta_i (\alpha_i - \alpha_{i+1}). \end{aligned}$$

By considering the diagram of λ as the disjoint union of rectangles of width β_i and height $(\alpha_i - \alpha_{i+1})$ (see Figure 3), we see that this sum is equal to n. □

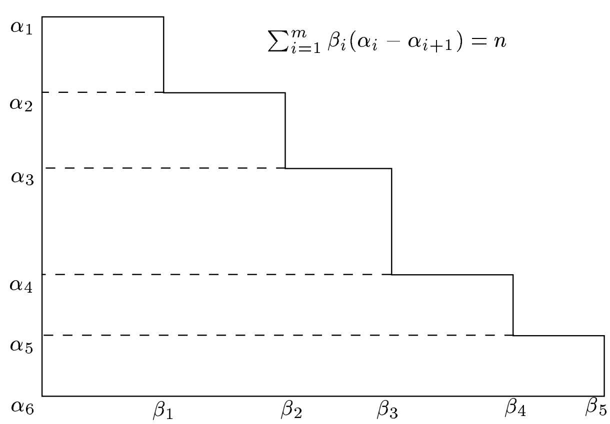
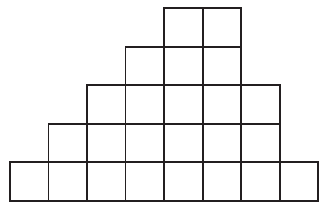


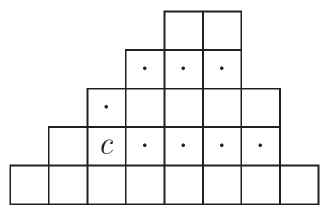
Figure 3: The diagram of λ as disjoint rectangles.

4 The Shifted Tableaux Formula

A *strict* partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ in one for which $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. Every strict partition has a *shifted* Young diagram associated with it; we take the usual Young diagram and shift the i^{th} row above the first to the right by i units. For example, the shifted diagram for the strict partition $(8, 6, 5, 3, 2)$ is shown below:



The definition of a standard shifted tableau is analogous to that of a shifted tableau. Content is also defined analogously. However, for cells in "the staircase" (precisely, those cells for which the column index is less than the number of parts of λ), we have a different definition of hook length. For such a cell, we add to the usual hook length the number of cells in the row one unit North of the cell $N(c)$. For example, the shifted hook length of the cell $c = (2, 3)$ in the diagram below is 9:



From here on, $h_\lambda(c)$ will denote the shifted hook length of the cell c.

Let g_λ denote the number of standard shifted tableaux of shape λ. The shifted hook length formula is as follows:

Theorem 2 ([Thr52]). *The number of standard shifted tableaux is given by*

$$g_\lambda = \frac{n!}{\prod_{c\in\lambda} h_\lambda(c)}.$$

The proof is very similar to the unshifted case. Once again we use the recursion

$$g_\lambda = \sum_{\mu\to\lambda} g_\mu.$$

to reduce to showing that

$$\frac{n!}{\prod_{c\in\lambda} h_\lambda(c)} = \sum_{\mu\to\lambda} \frac{(n-1)!}{\prod_{c\in\mu} h_\mu(c)}$$

or equivalently

$$n = \sum_{\mu \to \lambda} \frac{\prod_{c \in \lambda} h_\lambda(c)}{\prod_{c \in \mu} h_\mu(c)}.$$

Again, similarly to the unshifted case, the proof consists of showing three equalities.

$$\sum_{\mu \to \lambda} \frac{\prod_{c \in \lambda} h_\lambda(c)}{\prod_{c \in \mu} h_\mu(c)} = -\frac{1}{2} \sum_{i=1}^{m} \frac{\prod_{j=1}^{m} x_i(x_i+1) - y_j(y_j+1)}{\prod_{\substack{j=1 \\ j \neq i}} x_i(x_i+1) - x_j(x_j+1)} \tag{8}$$

$$= -\frac{1}{2} \left(\sum_{i=1}^{m} x_i(x_i+1) - y_i(y_i+1) \right) \tag{9}$$

$$= n. \tag{10}$$

Once again, each step is completely elementary.

Proof of (8). We begin by defining the cells $X_1, \ldots, X_m$, $Y_0, \ldots, Y_m$, in an analogous manner to the unshifted case and setting $x_1, \ldots, x_m$ and $y_0, \ldots, y_m$ to be the corresponding contents. Note that, as in the unshifted case, the cells Y_0 and Y_m lie outside of the diagram of λ. In the shifted case, we will not need the cell X_0. An example of a shifted tableau with these cells labelled is given in Figure 4.

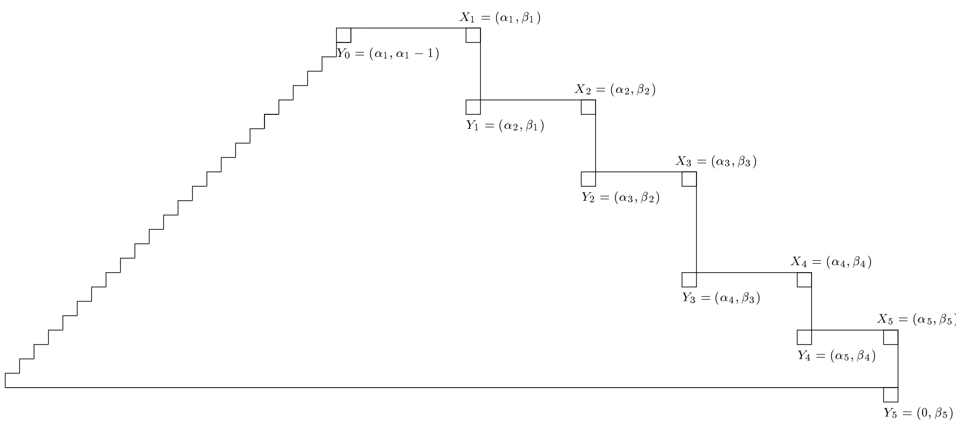


Figure 4: Labelled cells in a shifted tableau.

In analogy to equation (7), we have the following:

$$\sum_{i=1}^{m} (x_i - y_i) = \sum_{i=1}^{m} (\beta_i - \alpha_i) - (\beta_i - \alpha_{i+1})$$

$$= \alpha_{m+1} - \alpha_1 = -\alpha_1. \tag{11}$$

We now must express the quotient

$$\frac{\prod_{c\in\lambda} h_\lambda(c)}{\prod_{c\in\mu^{(i)}} h_{\mu^{(i)}}(c)}$$

in terms of the variables x_i and y_i, again taking advantage of the cancellations that occur. Figure 5 illustrates the location of the non-cancelling cells in the diagram of $\lambda \setminus X_i$.

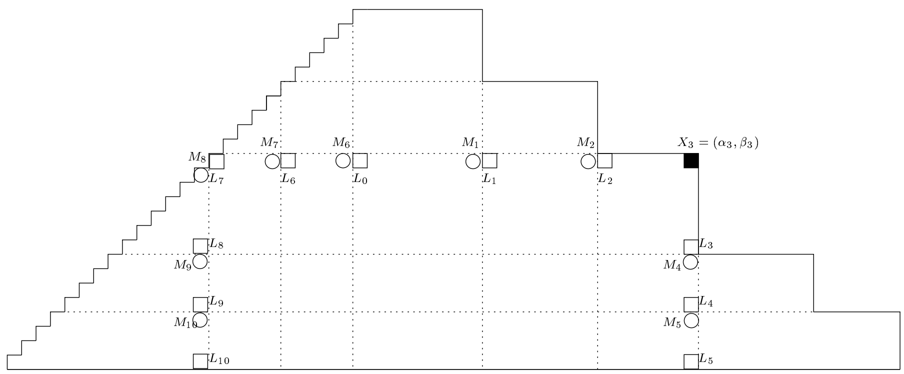


Figure 5: Non-cancelling cells. Squares should be viewed as cells in λ, circles as cells in $\lambda \setminus X_3$.

We now describe the contribution of the non-cancelling cells. Note first that, exactly as in the non-shifted tableaux case, we have the following non-cancelling cells in λ:

$$L_j = (\alpha_i, \beta_j + 1) \qquad \text{for } 0 \le j < i$$

and

$$L_j = (\alpha_{j+1} + 1, \beta_i) \qquad \text{for } i \le j \le m.$$

Similarly, the following cells in $\mu^{(i)}$ will not cancel:

$$M_j = (\alpha_i, \beta_j) \qquad \text{for } 1 \le j < i$$

and

$$M_j = (\alpha_{j+1}, \beta_i) \qquad \text{for } i < j \le m.$$

However, there are more non-cancelling cells, further to the left, in the shifted tableaux case. Precisely, we have in λ, cells

$$L_{m+j} = (\alpha_i, \alpha_{j+1}) \qquad \text{for } 1 \le j < i$$

and

$$L_{m+j} = (\alpha_{j+1} + 1, \alpha_i - 1) \qquad \text{for } i \le j \le k.$$

Similarly, the non-cancelling cells in $\mu^{(i)}$ are:

$$M_{m+j} = (\alpha_i, \alpha_j - 1) \qquad \text{for } 1 \le j < i$$

and

$$M_{m+j} = (\alpha_j - 1, \alpha_i - 1) \qquad \text{for } i \le j \le m.$$

We can compute the hooks of these cells in terms of the x_i and y_i. This gives

$$\begin{aligned} h_\lambda(L_j) &= x_i - y_j && \text{for } 0 \le j < i \\ h_\lambda(L_j) &= y_j - x_i && \text{for } i \le j \le m \\ h_\lambda(L_{m+j}) &= x_i + y_j + 1 && \text{for } 1 \le j \le m. \end{aligned}$$

To see this last equality, consider first the case where L_{m+j} is not in the same row as X_i. Note that the number of cells in the row of X_i is $x_i + 1$. Similarly, the number of cells strictly west of Y_j is equal to y_j which is equal to the number of cells weakly north or east of L_{m+j}. In the case where L_{m+j} is in the same row as X_i, we have $x_i + 1$ equal to the number of cells weakly north or east of X_i, and y_j equal to the length of the remaining row in the hook of L_{m+j}.

Similar computations give

$$\begin{aligned} h_{\mu^{(i)}}(M_j) &= x_i - x_j && \text{for } 1 \le j < i \\ h_{\mu^{(i)}}(M_j) &= x_j - x_i && \text{for } i \le j < m \\ h_{\mu^{(i)}}(M_{m+i}) &= 2(x_i + 1) && \\ h_{\mu^{(i)}}(M_{m+j}) &= x_i + x_j + 1 && \text{for } 1 \le j \le m,\ j \ne i. \end{aligned}$$

We now give an expression for the sum of the quotient of the hooks. This is the rational function

$$\sum_{i=1}^{m} \frac{h_\lambda(L_0)\prod_{j=1}^{m} h_\lambda(L_j) h_\lambda(L_{m+j})}{h_{\mu^{(i)}}(M_i)\prod_{\substack{j=1\\ j\ne i}}^{m} h_{\mu^{(i)}}(M_j) h_{\mu^{(i)}}(M_{m+j})} = -\sum_{i=1}^{m} \frac{(x_i - y_0)\prod_{j=1}^{m}(x_i - y_j)(x_i + y_j + 1)}{2(x_i+1)\prod_{\substack{j=1\\ j\ne i}}^{m}(x_i - x_j)(x_i + x_j + 1)}$$

However, since $y_0 = -1$, we can simplify this to

$$-\frac{1}{2}\sum_{i=1}^{m} \frac{\prod_{j=1}^{m} x_i(x_i+1) - y_j(y_j+1)}{\prod_{\substack{j=1\\ j\ne i}}^{m} x_i(x_i+1) - x_j(x_j+1)}.$$

□

Proof of 9. We begin by making the substitutions $\tilde{x}_i = x_i(x_i+1)$ and $\tilde{y}_i = y_i(y_i+1)$. We set

$$P(t) = -\frac{1}{2}\sum_{i=1}^{m} \frac{\prod_{j=1}^{m} \tilde{x}_i - \tilde{y}_j}{\prod_{\substack{j=1\\ j\ne i}}^{m} \tilde{x}_i - \tilde{x}_j} \prod_{\substack{j=1\\ j\ne i}}^{m} (t - \tilde{x}_j).$$

This has the properties

1. For $1 \le s \le n$,

$$P(\tilde{x}_s) = -\frac{1}{2}\prod_{j=1}^{m}(\tilde{x}_s - \tilde{y}_j)$$

   and

2. the degree of P is $m-1$, and the coefficient of t^{m-1} is

$$-\frac{1}{2}\sum_{i=1}^{m}\frac{\prod_{j=1}^{m}\tilde{x}_i - \tilde{y}_j}{\prod_{\substack{j=1\\ j\neq i}}^{m}\tilde{x}_i - \tilde{x}_j} .$$

As in the unshifted case, we complete the proof by finding another expression for the coefficient of t^{m-1}. We begin by defining

$$Q(t) = \frac{1}{2}\prod_{j=1}^{m}(t - \tilde{y}_j).$$

Thus Q has leading term $(1/2)t^m$, and we have $Q(\tilde{x}_s) + P(\tilde{x}_s) = 0$ for $1 \le s \le m$. Adding Q and P gives

$$\begin{aligned} Q(t) + P(t) &= \frac{1}{2}\prod_{j=1}^{m}(t - \tilde{x}_j) \\ \implies P(t) &= \frac{1}{2}\left(\prod_{j=1}^{m}(t - \tilde{x}_j) - \prod_{j=1}^{m}(t - \tilde{y}_j)\right) \\ &= -\frac{1}{2}\left(\sum_{j=1}^{m}\tilde{x}_j - \tilde{y}_j\right)t^{m-1} + \ldots \end{aligned}$$

since the degree of P is $m-1$. □

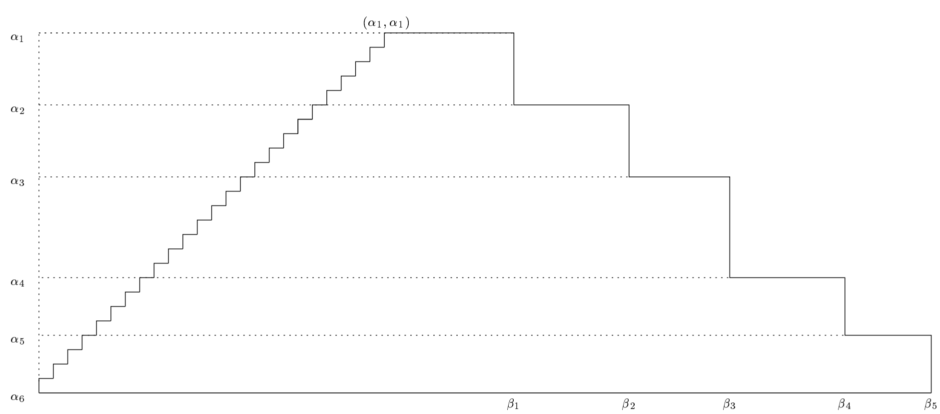


Figure 6: The final summation. Note that the area above the staircase is exactly $\binom{\alpha_1}{2}$.

Proof of (10). We rewrite the previous expression in terms of the coordinates and simplify:

$$
\begin{aligned}
-\frac{1}{2}\left(\sum_{i=1}^{m} \tilde{x}_i - \tilde{y}_i\right) &= -\frac{1}{2}\left(\sum_{i=1}^{m} x_i(x_i+1) - y_i(y_i+1)\right) \\
&= -\frac{1}{2}\left(\sum_{i=1}^{m}(x_i^2 - y_i^2) + (x_i - y_i)\right) \\
&= -\frac{1}{2}\left(\sum_{i=1}^{m}\left((\beta_i - \alpha_i)^2 - (\beta_i - \alpha_{i+1})^2\right) - \alpha_1\right) \qquad \text{(by (11))} \\
&= -\frac{1}{2}\left((\alpha_1^2 - \alpha_1) + 2\left(\sum_{i=1}^{m} \beta_i(\alpha_{i+1} - \alpha_i)\right)\right) \\
&= \left(\sum_{i=1}^{m} \beta_i(\alpha_i - \alpha_{i+1}) - \frac{\alpha_1(\alpha_1 - 1)}{2}\right) \\
&= n
\end{aligned}
$$

where the last equality can be seen by considering Figure 6. □

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