A quantified version of Bourgain's sum-product estimate in $\mathbb{F}_p$ for subsets of incomparable sizes

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Abstract

Let $\mathbb{F}_p$ be the field of residue classes modulo a prime number p. In this paper we prove that if $A, B \subset \mathbb{F}_p^*$, then for any fixed $\varepsilon > 0$,

$$|A+A| + |AB| \gg \Big(\min\Big\{|B|, \frac{p}{|A|}\Big\}\Big)^{1/25-\varepsilon}|A|.$$

This quantifies Bourgain's recent sum-product estimate.

1 Introduction

Let $\mathbb{F}_p$ be the field of residue classes modulo a prime number p and let A be a non-empty subset of $\mathbb{F}_p$. It is known from [4, 5] that if $|A| < p^{1-\delta}$, where $\delta > 0$, then one has the sum-product estimate

$$|A+A| + |AA| \gg |A|^{1+\varepsilon} \tag{1}$$

for some $\varepsilon = \varepsilon(\delta) > 0$. This estimate and its proof has been quantified and simplified in [3], [6]–[11]. Improving upon our earlier estimate from [6], Katz and Shen [11] have shown that in the most nontrivial range $1 < |A| < p^{1/2}$ one has

$$|A+A| + |AA| \gg |A|^{14/13}(\log|A|)^{O(1)}.$$

A version of sum-product estimates with subsequent application to exponential sum bounds is given in [3]. In particular, from [3] it follows that if $1 < |A| < p^{12/23}$, then

$$|A-A| + |AA| \gg |A|^{13/12}(\log|A|)^{O(1)}.$$

We also mention that in the case $|A| > p^{2/3}$ one has

$$\max\{|A+A|, |AA|\} \gg p^{1/2}|A|^{1/2},$$

which is optimal in general settings bound, apart from the value of the implied constant; for the details, see [7].

Sum-product estimates in $\mathbb{F}_p$ for different subsets of incomparable sizes have been obtained by Bourgain [1]. More recently, he has shown in [2] that if $A, B \subset \mathbb{F}_p^*$, then

$$|A+A| + |AB| \gg \left(\min\left\{|B|, \frac{p}{|A|}\right\}\right)^c |A| \tag{2}$$

for some absolute positive constant c. In the present paper we prove the following explicit version of this result.

Theorem 1. *For any non-empty subsets $A, B \subset \mathbb{F}_p^*$ and any $\varepsilon > 0$ we have*

$$|A+A| + |AB| \gg \left(\min\left\{|B|, \frac{p}{|A|}\right\}\right)^{1/25-\varepsilon} |A|,$$

where the implied constant may depend only on ε.

Remark. One can expect that appropriate adaptation of techniques of [3] and [11] may lead to quantitative improvement of the exponent $1/25$.

2 Lemmas

Below in statements of lemmas all the subsets are assumed to be non-empty. The first two lemmas are due to Ruzsa [12, 13]. They hold for subsets of any abelian group, but here we state them only for subsets of $\mathbb{F}_p$.

Lemma 1. *For any subsets X, Y, Z of $\mathbb{F}_p$ we have*

$$|X - Z| \le \frac{|X-Y||Y-Z|}{|Y|}.$$

Lemma 2. *For any subsets $X, B_1, \ldots, B_k$ of $\mathbb{F}_p$ we have*

$$|B_1 + \ldots + B_k| \le \frac{|X+B_1| \ldots |X+B_k|}{|X|^{k-1}}.$$

In the proof of estimate (2) (as well as in the proofs of exponential sum bounds) Bourgain used his result

$$|8XY - 8XY| \ge 0.5\{|X||Y|, p\}$$

valid for any non-empty subsets $X, Y \subset \mathbb{F}_p^*$, see [2, Lemma 2]. In the proof of our Theorem 1 we shall use the following lemma instead.

Lemma 3. *Let $X, Y \subset \mathbb{F}_p^*$, $|Y| \geq 2$. Then there are elements $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that either*

$$\Big|(x_1 - x_2)Y + (y_1 - y_2)X + (y_1 - y_2)X\Big| \geq \frac{0.5|X|^2|Y|}{|XY|}$$

or

$$\Big|(x_1 - x_2)Y + (y_1 - y_2)X\Big| \geq 0.5p.$$

Thus, at the cost of a slight worsening of the right hand side, we simplify the expression on the left hand side.

Proof. If $|XY| = |X||Y|$ then we are done. Let $|XY| < |X||Y|$. Since

$$\sum_{x \in X} \sum_{y \in Y} |xY \cap yX| \geq \frac{|X|^2|Y|^2}{|XY|},$$

there are elements $x_0 \in X$, $y_0 \in Y$ such that

$$|x_0 Y \cap y_0 X| \geq \frac{|X||Y|}{|XY|}.$$

Let $x_0 Y_1 = x_0 Y \cap y_0 X$. Then,

$$Y_1 \subset Y, \qquad \frac{x_0}{y_0} Y_1 \subset X, \qquad |Y_1| \geq \frac{|X||Y|}{|XY|} > 1.$$

If

$$\frac{X - X}{Y_1 - Y_1} \neq \mathbb{F}_p,$$

then

$$\frac{X - X}{Y_1 - Y_1} + \frac{x_0}{y_0} \neq \frac{X - X}{Y_1 - Y_1}.$$

Thus, for some $(x_1, x_2, y_1, y_2) \in X^2 \times Y_1^2$,

$$\frac{x_1 - x_2}{y_1 - y_2} + \frac{x_0}{y_0} \notin \frac{X - X}{Y_1 - Y_1}.$$

Hence,

$$\Big|\Big(\frac{x_1 - x_2}{y_1 - y_2} + \frac{x_0}{y_0}\Big)Y_1 + X\Big| = |X||Y_1|.$$

Since

$$\frac{x_0}{y_0} Y_1 \subset X,$$

we conclude that

$$\Big|(x_1 - x_2)Y_1 + (y_1 - y_2)X + (y_1 - y_2)X\Big| \geq |X||Y_1| \geq \frac{|X|^2|Y|}{|XY|}.$$

If

$$\frac{X - X}{Y_1 - Y_1} = \mathbb{F}_p,$$

then we use the well-known fact that for some $z \in \mathbb{F}_p$ we have

$$|X + zY_1| \geq 0.5 \min\{|X||Y_1|, p\}.$$

This implies that for some $(x_1, x_2, y_1, y_2) \in X^2 \times Y_1^2$,

$$|(x_1 - x_2)Y_1 + (y_1 - y_2)X| \geq 0.5 \min\{|X||Y_1|,\ p\}.$$

$\square$

The following statement follows from the aforementioned work [7]. We shall only use it in order to avoid a minor inconvenience that may arise when $p/|A|$ is as small as a fixed power of $\log |B|$.

Lemma 4. *Let $A, B, C \subset \mathbb{F}_p^*$. Then*

$$|A + C||AB| \gg \min\Big\{p|A|, \frac{|A|^2|B||C|}{p}\Big\}.$$

3 Proof of Theorem 1

If $G \subset X \times Y$ then for a given $x \in X$ we denote by $G(x)$ the set of all elements $y \in Y$ for which $(x, y) \in G$. The notation $E_+(X, Y)$ is used to denote the additive energy between X and Y, that is the number of solutions of the equation

$$x_1 + y_1 = x_2 + y_2, \qquad (x_1, x_2, y_1, y_2) \in X^2 \times Y^2.$$

We can assume that $|A| > 10$, $|B| > 10$. In view of Lemma 4, we can also assume that $p/|A| > (\log |B|)^{100}$.

Let

$$|A + A| + |AB| = |A|\Delta.$$

Then,

$$\sum_{b \in B} \sum_{b' \in B} |bA \cap b'A| \geq \frac{|A|^2|B|^2}{|AB|} \geq \frac{|A||B|^2}{\Delta}.$$

Hence, for some fixed $b_0 \in B$,

$$\sum_{b \in B} |bA \cap b_0 A| \geq \frac{|A||B|}{\Delta}. \tag{3}$$

Define

$$B_1 = \Big\{b \in B :\ |bA \cap b_0 A| \geq \frac{|A|}{2\Delta}\Big\}. \tag{4}$$

From Ruzsa's triangle inequalities (Lemma 1 and Lemma 2 with $k = 2$),

$$|bA \pm b_0 A| \le \frac{|bA + (bA \cap b_0 A)| \cdot |(bA \cap b_0 A) + b_0 A|}{|bA \cap b_0 A|} \le \frac{|A + A|^2}{|bA \cap b_0 A|},$$

which, in view of (4), implies that

$$|bA \pm b_0 A| \le \frac{2|A + A|^2 \Delta}{|A|} \le 2|A|\Delta^3 \quad \text{for} \quad \text{any} \quad b \in B_1. \tag{5}$$

For a given $a \in A$ let $aB_1(a) = aB_1 \cap b_0 A$. From (3) and (4) it follows that

$$\sum_{a \in A} |B_1(a)| = \sum_{a \in A} |aB_1 \cap b_0 A| = \sum_{b \in B_1} |bA \cap b_0 A| \ge \frac{|A||B|}{2\Delta}.$$

Obviously, we can assume that $|B_1| \ge 2$, since otherwise the statement is trivial from $2|B_1|\Delta \ge |B|$. We allot the values of $|B_1(a)|$ into duadic intervals and derive that for some subset $A_0 \subset A$ and for some number $N \ge 1$,

$$N|A_0| \ge \frac{|A||B|}{8\Delta \log |B|} \tag{6}$$

and

$$N \le |B_1(a)| \le 2N \quad \text{for} \quad \text{any} \quad a \in A_0. \tag{7}$$

In what follows, up to the inequality (10), is based on Bourgain's idea from [2]. We have

$$\sum_{(a,a') \in A_0^2} |B_1(a) \cap B_1(a')| \ge \frac{1}{|B_1|} \Big(\sum_{a \in A_0} |B_1(a)| \Big)^2 \ge \frac{N^2 |A_0|^2}{|B_1|}.$$

We allot the values of $|B_1(a) \cap B_1(a')|$ into duadic intervals and get that for some $G \subset A_0 \times A_0$ and some number $M \ge 1$,

$$M \le |B_1(a) \cap B_1(a')| \le 2M \quad \text{for} \quad \text{any} \quad (a, a') \in G$$

and

$$M|G| \ge \frac{N^2 |A_0|^2}{10|B_1| \cdot \log |B|}.$$

In particular,

$$M \ge \frac{N^2}{10|B_1| \cdot \log |B|}. \tag{8}$$

Let

$$A_1 = \Big\{ a \in A_0 : \ |G(a)| \ge \frac{N^2 |A_0|}{20M|B_1| \cdot \log |B|} \Big\}.$$

From

$$\sum_{a \in A_0} |G(a)| = |G| \ge \frac{N^2 |A_0|^2}{10M|B_1| \cdot \log |B|}$$

it follows

$$|A_1| \geq \frac{N^2|A_0|}{20M|B_1| \cdot \log|B|}. \tag{9}$$

For a given $a_1 \in A_1$ we shall estimate $|a_1B_1 \pm b_0G(a_1)|$ for any choice of the symbol "$\pm$". Let $\delta \in \{-1, 1\}$. To each element $x \in a_1B_1 + \delta b_0 G(a_1)$ we assign one representation

$$x = a_1 b + \delta b_0 a_1', \qquad b \in B_1, \quad a_1' \in G(a_1)$$

and define $B_{11}(x) = B_1(a_1) \cap B_1(a_1')$. Then

$$\delta b_0^2 A + xB_{11}(x) \subset \delta b_0^2 A + ba_1B_1(a_1) + \delta b_0 a_1' B_1(a') \subset b_0(bA + \delta b_0 A + \delta b_0 A),$$

whence, by Lemma 2 with $k = 3$ and estimate (5),

$$|\delta b_0^2 A + xB_{11}(x)| \leq \frac{|bA + \delta b_0 A| \cdot |A + A|^2}{|A|^2} \leq 2|A|\Delta^5.$$

Hence, for a given $x \in a_1B_1 + \delta b_0 G(a_1)$, we have

$$E_+(b_0^2 A, xB_1(a_1)) \geq E_+(b_0^2 A, xB_{11}(x)) \geq \frac{|A|^2 M^2}{2|A|\Delta^5} = \frac{|A|M^2}{2\Delta^5}.$$

Summing up this inequality over $x \in a_1B_1 + \delta b_0 G(a_1)$ and observing that the number of solutions of the equation

$$b_0^2 a' + xb' = b_0^2 a'' + xb'', \quad a', a'' \in A;\ b', b'' \in B_1(a_1);\ x \in a_1B_1 + \delta b_0 G(a_1)$$

is not greater than $2N|A| \cdot |a_1B_1 + \delta b_0 G(a_1)| + 4N^2|A|^2$, we get

$$\frac{|A|M^2}{2\Delta^5} |a_1B_1 + \delta b_0 G(a_1)| \leq 2N|A| \cdot |a_1B_1 + \delta b_0 G(a_1)| + 4N^2|A|^2.$$

If $|A|M^2 \leq 10|A|N\Delta^5$, then we are done in view of (8) and (6). Therefore, we can assume that

$$|a_1B_1 \pm b_0G(a_1)| \ll \frac{|A|N^2\Delta^5}{M^2} \quad \text{for} \quad \text{any} \quad a_1 \in A_1. \tag{10}$$

By Lemma 3, for some $a_1, a_{11} \in A_1$ and $b_1, b_{11} \in B_1$, either

$$\Big|(a_1 - a_{11})B_1 + (b_1 - b_{11})A + (b_1 - b_{11})A\Big| \gg \frac{|A_1|^2|B_1|}{|A_1B_1|} \gg \frac{|A_1|^2|B_1|}{\Delta|A|}$$

or

$$\Big|(a_1 - a_{11})B_1 + (b_1 - b_{11})A\Big| \gg p.$$

In the first case, by Lemma 2 with $k = 3$ and $X = (b_1 - b_{11})A$,

$$\Big|(a_1 - a_{11})B_1 + (b_1 - b_{11})A\Big| |A|\Delta^3 \gg |A_1|^2|B_1|.$$

Again by Lemma 2 with $k = 4$ and $X = b_0A$, and by (5),

$$|a_1B_1 + b_0A| \cdot |a_{11}B_1 - b_0A|\Delta^9 \gg |A_1|^2|B_1|.$$

To each of the cardinalities on the left hand side we again apply Lemma 2, with $k = 2$ and $X = b_0G(a_1)$ or $X = -b_0G(a_{11})$, and recalling the lower bound for $|G(a)|$ when $a \in A_1$, we deduce

$$|a_1B_1 + b_0G(a_1)| \cdot |a_{11}B_1 - b_0G(a_{11})| \cdot |A|^2\Delta^{11} \gg |A_1|^2|B_1|\Big(\frac{N^2|A_0|}{M|B_1| \cdot \log|B|}\Big)^2.$$

Combining this with (10), we get

$$|A|^4\Delta^{21} \gg \frac{M^2|A_1|^2|A_0|^2}{|B_1| \cdot \log^2|B|}.$$

Using (9) to substitute $M|A_1|$, and then (6) to substitute $N|A_0|$, we obtain

$$|A|^4\Delta^{21} \gg \frac{|A|^4|B|^4}{\Delta^4|B_1|^3\log^8|B|} \gg \frac{|A|^4|B|}{\Delta^4\log^8|B|}.$$

This proves our assertion in the first case.

In the second case we have

$$\Big|(a_1 - a_{11})B_1 + (b_1 - b_{11})A\Big| \gg p,$$

which implies

$$|a_1B_1 + b_0A| \cdot |a_{11}B_1 - b_0A|\Delta^6 \gg p|A|.$$

Then as in the first case,

$$|A|^2\Delta^{18} \gg \frac{p|A_0|^2M^2}{|A||B_1|^2\log^2|B|}.$$

Using (8) and then (6), we get

$$\Delta^{22} \gg \frac{p}{|A|\log^8|B|}$$

and the result follows in view of the assumption $p/|A| > (\log|B|)^{100}$.

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