The MacNeille Completion of the Poset of Partial Injective Functions

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Abstract. Renner has defined an order on the set of partial injective functions from $[n] = \{1, \ldots, n\}$ to $[n]$. This order extends the Bruhat order on the symmetric group. The poset P_n obtained is isomorphic to a set of square matrices of size n with its natural order. We give the smallest lattice that contains P_n. This lattice is in bijection with the set of alternating matrices. These matrices generalize the classical alternating sign matrices. The set of join-irreducible elements of P_n are increasing functions for which the domain and the image are intervals.

Keywords: alternating matrix, Bruhat, dissective, distributive lattice, join-irreducible, Key, MacNeille completion.

1 Introduction

The symmetric group S_n, the set of bijective functions from $[n]$ into itself, with the Bruhat order is a poset; it is not a lattice. In [5], Lascoux and Schützenberger show that the *smallest* lattice that contains S_n as a subposet is the lattice of *triangles*; this lattice is in bijection with the set of alternating sign matrices. The main objective of this paper is to construct the smallest lattice that contains the poset P_n of the partial injective functions, partial meaning that the domain is a subset of $\{1, \ldots, n\}$.

In section 2, we give the theory on the construction for a finite poset P of the smallest lattice, noted $L(P)$, which contains P as a subposet. We give also results [9] on join-irreducible and upper-dissector elements of a poset : $L(P)$ is distributive iff a join-irreducible element of P is exactly an upper-dissector element of P. We will show in section 4.4 that $L(P_n)$ is distributive.

In section 3.1, we give the definition of the set P_n with its order, due to Renner. This order extends the Bruhat order on S_n. In section 3.2, we associate to $f \in P_n$ a matrix over $\{0, \ldots, n\}$. In section 3.3, we give two posets of matrices RG_n and R_n, the elements

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of $R_n \subseteq RG_n$ being the matrices defined in section 3.2, for which the order is the natural order. We show that P_n and R_n are in bijection. In section 3.4, we show that P_n and R_n are isomorphic posets : it is one of the main results of this article. Thus $L(P_n)$ and $L(R_n)$ are isomorphic lattices.

In section 4.1, after having observed that RG_n is a lattice, see [3], we show that R_n is not a lattice and we see that $L(R_2) = RG_2$. In sections 4.2 and 4.3, we define the matrices $B_{r,s,a,n}$ and the matrices $C_{r,s,a,n}$ which are $\in R_n$; we show that all matrices of RG_n are the *sup* of matrices $B_{r,s,a,n}$ and the *inf* of matrices $C_{r,s,a,n}$; thus $L(R_n) = RG_n$: it is another one of the main results of this article. In sections 4.4, we show that the matrices $B_{r,s,a,n}$ are the join-elements and the upper-elements of R_n : thus RG_n is distributive; we show also that the matrices $C_{r,s,a,n}$ are the meet-elements of RG_n. In section 4.5, we obtain the the join-elements and the meet-elements of P_n. In section 4.6, we give a morphism of poset of P_n to S_{2n} : we may see P_n as a subposet of S_{2n}.

In section 5.1, we define the notion of a rectrice (and corectrice) which has been introduced by Lascoux and Schützenberger in [5]. A matrix $A \in RG_n$ is the *sup* of its rectrices, a rectrice of A being a $B_{r,s,a,n}$ matrix X with no $B_{r,s,a,n}$ matrix strictly between X and A. In sections 5.2 and 5.3, we present the notions of *Key* and *generalized Key* : the keys and triangles we have in [5] are Keys and generalized Keys with no zero entry. The Keys form a poset K_n, the generalized Keys form a lattice KG_n and we have : $L(K_n) = KG_n$. In section 5.4, we show that P_n and K_n are isomorphic posets : so RG_n and KG_n are isomorphic lattices. We describe this isomorphism $A \mapsto K(A)$: we find the rectrices of A and we obtain the rectrices of $K(A)$.

In section 6.1, we show that there is a bijection between RG_n and the set of alternating matrices At_n (which contains the classical alternating sign matrices). In section 6.2, we show that there is a bijection between At_n and KG_n : we obtain then a bijection between RG_n and KG_n. We show in section 6.3 that this bijection is an isomorphism of lattice.

This article is written from a PhD thesis [3] for which the director was Christophe Reutenauer.

2 Preliminaries on posets and MacNeille completion

Let $\phi : P \to Q$ be a function between two posets. We say that ϕ is a *morphism of poset* if $x \leq_P y \Leftrightarrow \phi(x) \leq_Q \phi(y)$. Note that ϕ is necessarily injective. We say also that ϕ is an *embedding* of P into Q.

All posets P considered here are *finite* with elements 0 and 1 such that: $\forall x \in P,\ 0 \leq x \leq 1$.

MacNeille [7] gave the construction for a poset P of a lattice $L(P)$ which contains P as a subposet. We find this construction in [2]. We define :

$$\forall X \subseteq P : X^- = \{y \in P \mid \forall x \in X, y \geq x\};\ X^+ = \{y \in P \mid \forall x \in X, y \leq x\}$$

$$L(P) = \{X \subseteq P \mid X^{-+} = X\},\ \textit{with}\ Y \leq Z\ \Leftrightarrow\ Y \subseteq Z$$

Theorem 2.1 ([2], theorem 2.16) *$L(P)$ is a lattice :*

$$\forall X \in L(P), X \wedge Y = (X \cap Y)^{-+} = X \cap Y;\ \ X \vee Y = (X \cup Y)^{-+}$$

We simply write x^- for $\{x\}^-$; and x^+ for $\{x\}^+$. We define :

$$\varphi : P \to L(P),\ x \mapsto x^+$$

Theorem 2.2 ([2], theorem 2.33)
(i) φ is an embedding *of P into $L(P)$;*
(ii) if $X \subseteq P$ and $\wedge X$ exists in P, then $\varphi(\wedge X) = \wedge(\varphi(X))$;
(iii) if $X \subseteq P$ and $\vee X$ exists in P, then $\varphi(\vee(\wedge X) = \vee(\varphi(X))$.

Theorem 2.3 ([2], theorem 2.36 (i)) $\forall X \in L(P)$ *:*

$$\exists\ Q, R \subseteq P \text{ such that } X = \vee(\varphi(Q)) = \wedge(\varphi(R)).$$

We give now some general properties of embeddings of posets into lattices, which allow to characterize the MacNeille completions and which will be used in the sequel.

Theorem 2.4
(i) Let P be a finite poset;
(ii) let be f an embedding of P into a lattice T;
(iii) let g be an embedding of P into a lattice S, such that :

$$\begin{aligned} \forall s \in S,\quad s &= \vee\{g(x) \mid x \in P \text{ and } g(x) \le s\} \\ &= \wedge\{g(x) \mid x \in P \text{ and } g(x) \ge s\}\}; \end{aligned}$$

then T contains S as a subposet : more precisely there is an embedding h of S into T such that $h \circ g = f$, where h is defined by :

$$h : S \to T, s \mapsto \vee_T\{f(x) \mid x \in P \text{ and } g(x) \le s\}.$$

Lemma 2.5 ([2], Lemma 2.35) *Let f be an embedding of a finite poset P into a lattice S, such that : $\forall s \in S,\ \exists\ Q, R \subseteq P$ such that $s = \vee(f(Q)) = \wedge(f(R))$; then*

$$\begin{aligned} \forall s \in S,\quad s &= \vee\{f(x) \mid x \in P \text{ and } f(x) \le s\} \\ &= \wedge\{f(x) \mid x \in P \text{ and } f(x) \ge s\}\}. \end{aligned}$$

Theorem 2.6 *Let P be a finite poset; then $L(P)$ is the smallest lattice that contains P as a subposet. More precisely, if f an embedding of P into a lattice T, then card($L(P)$) $\le$ card(T).*

Theorem 2.7 ([2], Theorem 2.33 (iii)) *Let P be a finite poset; let f be an embedding of P into a lattice S, such that :*

$$\forall s \in S,\ \exists Q, R \subseteq P \text{ such that } s = \vee(f(Q)) = \wedge(f(R));$$

then the lattices $L(P)$ and S are isomorphic.

In the Appendix, we give a proof of Theorems 2.4, 2.6 and 2.7, since the statements of Theorems 2.4 and 2.6 in [2] are slightly different, and for the reader's convenience.

An element $x \in P$ is *join-irreducible* if $\forall Y \subseteq P,\ x \notin Y \Rightarrow x \neq sup(Y)$. The set of join-irreducibles is denoted $B(P)$ and is called the *base* of P in [5]. We have : $x \in B(P)$ iff $\forall y_1, \ldots, y_n \in P,\ x = y_1 \vee \ldots \vee y_n \Rightarrow \exists i,\ x = y_i$.

An element $x \in P$ is *meet-irreducible* if $\forall Y \subseteq P,\ x \notin Y \Rightarrow x \neq inf(Y)$. The set of meet-irreducibles is denoted $C(P)$ and is called the *cobase* of P in [5]. We have : $x \in C(P)$ iff $\forall y_1, \ldots, y_n \in P,\ x = y_1 \wedge \ldots \wedge y_n \Rightarrow \exists i,\ x = y_i$.

An element $x \in P$ is an *upper-dissector* of P if $\exists$ an element of P, denoted $\beta(x)$, such that $P - x^- = \beta(x)^+$. The set of upper-dissectors is denoted $Cl(P)$. An element $\in Cl(P)$ is called *clivant* in [5].

Theorem 2.8 ([9], Proposition 12) *$Cl(P) \subseteq B(P)$.*

P is *dissective* if $Cl(P) = B(P)$.

Theorem 2.9 ([9], Proposition 28) *$B(P) = B(L(P))$; $Cl(P) = Cl(L(P))$.*

Theorem 2.10 ([9]) *If P is a lattice then $x \in B(P)$ iff x is the immediate successor of one and only one element of P.*

Theorem 2.11 ([9], Theorem 7) *L(P) is distributive iff P is dissective.*

3 Partial injective functions

3.1 Definition

A function $f : X \subseteq [n] = \{1, ..., n\} \to [n]$ is called a *partial injective function*. Let P_n be the set of partial injective functions. If $i \in [n] - dom(f)$, we write $f(i) = 0$. So we can represent f by a vector : $f = \begin{pmatrix} f(1) & f(2) & \ldots & f(n) \end{pmatrix}$.

We define an order on P_n. This order is a generalization of the *Bruhat order* of S_n, the poset of bijective functions $f : [n] \to [n]$. Let $f, g \in P_n$; we write $f \to g$ if :

$$\begin{array}{l}
1)\quad \exists\ i \in [n] \text{ such that} \\
\qquad a)\ f(j) = g(j)\ \forall\ j \neq i \\
\qquad b)\ f(i) < g(i) \\
or \\
2)\quad \exists\ i < j \in [n] \text{ such that} \\
\qquad a)\ f(k) = g(k)\ \forall\ k \neq i, j \\
\qquad b)\ g(j) = f(i) < f(j) = g(i)
\end{array}$$

This definition is due to Pennell, Putcha and Renner: see [10], sections 8.7 and 8.8.

Example 3.1 $\begin{pmatrix} 3 & 0 & \underline{2} & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & \underline{0} & 4 & 0 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & 4 & \underline{0} & \underline{5} \end{pmatrix} \to$
$\begin{pmatrix} 3 & \underline{1} & 4 & \underline{5} & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 5 & 4 & 1 & 0 \end{pmatrix}$.

A pair (i, j) is called an *inversion* of $f \in P_n$ if $i < j$ and $f(i) > f(j)$. We note $inv(f)$ the set of inversions of f.

Example 3.2 $inv \begin{pmatrix} 3 & 1 & 0 & 5 & 0 \end{pmatrix} = \{(1, 2), (1, 3), (1, 5), (2, 3), (2, 5), (4, 5)\}$.

To any $f \in P_n$, we define the *length* $L(f) = card(inv(f)) + \sum_{k=1}^{n} f(k)$. $L(f)$ is the number of inversions of f + the sum of the values of f.

We have : $f \to g \Rightarrow L(f) < L(g)$. So we can define a partial order on P_n : $f \leq g \Leftrightarrow \exists\ m \geq 0$ *and* $g_0, \ldots, g_m \in P_n$ *such that* $f = g_0 \to g_1 \to \ldots \to g_m = g$.

$\forall f \in P_n$, we have :

$$\mathbf{0}_{P_n} = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix} \leq f \leq \begin{pmatrix} n & n-1 & \ldots & 1 \end{pmatrix} = \mathbf{1}_{P_n}$$

$$0 = L(\mathbf{0}_{P_n}) \leq L(f) \leq L(\mathbf{1}_{P_n}) = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$$

The maximum element of P_n is not the identity map of $[n]$.

3.2 Diagram

To any $f \in P_n$, we associate its *graph*, which is the subset of all points $(i, f(i))$ in $\{1, \ldots, n\} \times \{0, \ldots, n\}$, where i is the number of the row and j the number of the column. We represent each point by a cross $\times$ and we obtain what we call the *planar representation* of f.

To any $f \in P_n$, we associate its *north-east diagram* $NE(f)$: the planar representation of f is a part of $NE(f)$; in addition, we put in each square $[i, i+1] \times [j, j+1] \subseteq [0, n+1] \times [0, n+1], 0 \leq i, j \leq n$, the number of $\times$ that lie above and to the right, i.e., in the north-east sector, of the square. We note this number $NE(f)([i, i+1] \times [j, j+1])$ and we have :

$$NE(f)([i, i+1] \times [j, j+1]) = card\{k \leq i \mid f(k) > j\}$$

Example 3.3 $f = \begin{pmatrix} 3 & 0 & 2 & 4 & 1 \end{pmatrix}$

$$NE(f) = \begin{array}{ccccccccccccc}
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & \cdot & & \cdot & & \cdot & & \times & & \cdot & & \cdot & \\
 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 \\
2 & \times & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \\
 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 \\
3 & \cdot & & \cdot & & \times & & \cdot & & \cdot & & \cdot & \\
 & & 2 & & 2 & & 1 & & 0 & & 0 & & 0 \\
4 & \cdot & & \cdot & & \cdot & & \cdot & & \times & & \cdot & \\
 & & 3 & & 3 & & 2 & & 1 & & 0 & & 0 \\
5 & \cdot & & \times & & \cdot & & \cdot & & \cdot & & \cdot & \\
 & & 4 & & 3 & & 2 & & 1 & & 0 & & 0
\end{array}$$

And finally, to any $f \in P_n$, we associate a square matrix of size n $M(f)$. The entries of $M(f)$ are numbers in the squares of $NE(f)$. Precisely, $M(f)[i, j] = NE(f)([i, i+1] \times [j-1, j])$, $i, j = 1, \ldots, n$.

Example 3.4 $f = \begin{pmatrix} 3 & 0 & 2 & 4 & 1 \end{pmatrix}$

$$M(f) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

3.3 The sets of matrices R_n and RG_n

We define two sets of matrices RG_n and R_n, and we will show that $R_n = \{M(f) \mid f \in P_n\}$.

RG_n is a set of square matrices of size n with entries $\in \{0, 1, ..., n\}$. We consider that $A \in RG_n$ has a row, numbered 0, and a column, numbered $n+1$, of zeros. $A \in RG_n$ if 1) the rows of A, from left to right, are decreasing, ending by 0 in column $n+1$; 2) the columns of A, from top to bottom, are increasing, starting by 0 in row 0; and 3) any two adjacent numbers on a row or on a column are equal or differ by 1.

Example 3.5

$$A = \begin{array}{c} \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \in RG_4$$

We say that $A \in RG_n$ has the *pattern* $\boxed{\begin{matrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mp} \end{matrix}}$ *in position* r, s if $A[r, s] = a_{11}, \dots, A[r, s+p-1] = a_{1p}, \dots, A[r+m-1, s] = a_{m1}, \dots, A[r+m-1, s+p-1] = a_{mp}$.

$\boxed{\begin{matrix} a & a \\ a+1 & a \end{matrix}}$ is called *plus pattern*; $\boxed{\begin{matrix} a+1 & a \\ a+1 & a+1 \end{matrix}}$ is called *minus pattern*; $\boxed{\begin{matrix} a & a \\ a & a \end{matrix}}$, $\boxed{\begin{matrix} a & a \\ a+1 & a+1 \end{matrix}}$, $\boxed{\begin{matrix} a+1 & a \\ a+1 & a \end{matrix}}$, $\boxed{\begin{matrix} a+1 & a \\ a+2 & a+1 \end{matrix}}$ are called *zero pattern.*

The next two lemmas will be proved later.

Lemma 3.6 *If $A \in RG_n$ has plus patterns (or minus patterns) in position r_1, s and r_2, s, with $r_1 < r_2$, then $\exists\ r', r_1 < r' < r_2$ such that A has a minus pattern (respectively plus pattern) in position r', s;*

if $A \in RG_n$ has plus patterns (or minus patterns) in position r, s_1 and r, s_2, with $s_1 < s_2$, then $\exists\ s', s_1 < s' < s_2$ such that A has a minus pattern (respectively plus pattern) in position r, s'.

We rephrase this lemma by saying that the patterns plus and minus, horizontally and vertically, alternate in a matrix $A \in RG_n$.

Lemma 3.7 *$\forall A \in RG_n,\ A[r,s]$ = the number of plus patterns - the number of minus patterns that lie above and to the right of the position r,s.*

We define R_n by saying that $A \in R_n \subseteq RG_n$ if A does not have any minus pattern.

Theorem 3.8 *$\forall f \in P_n,\ M(f) \in R_n$.*

Proof : $NE(f)([r,r+1]\times[s-1,s]) = NE(f)([r,r+1]\times[s,s+1]) + 1$ (= a+1 in the diagram below) iff there is a $\times$ above, i.e., $\exists r' \leq r$ such that $f(r') = s$:

$$
\begin{array}{lccccc}
 & & & & s & \\
 & \cdot & & & \cdot & \\
NE(f) = & r' & \ldots & & \times & \\
 & \cdot & & & \cdot & \\
 & r & \ldots & & \cdot & \\
 & & & a+1 & & a
\end{array}
$$

It follows that $M(f)$ does not have any minus pattern because $M(f)[r,s] = M(f)[r,s+1]+1 \Rightarrow M(f)[r+1,s] = M(f)[r+1,s+1]+1$. This means $M(f) \in R_n$. Q.E.D.

To any $A \in R_n$, we associate $f_A = \{(r,s) \in [n]\times[n] \mid A$ has a plus pattern in position $r-1, s\}$.

Theorem 3.9 *$\forall A \in R_n,\ f_A \in P_n$ and $M(f_A) = A$.*

Proof : $f_A \in P_n$ because, see lemma 3.6, the plus patterns and the minus patterns, horizontally and vertically, alternate and because A does not have any minus pattern.

We have, see lemma 3.7, that $A[r,s]$ is the number of plus patterns that lie above and to the right of the position r,s. $NE(f_A)([r,r+1]\times[s-1,s]) = M(f_A)[r,s]$ is the number of $\times$ that lie above and to the right of the square $[r,r+1]\times[s-1,s]$. Thus $M(f_A) = A$. Q.E.D.

Example 3.10

$$
\text{If } A = \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 & 1 \end{array}\right) \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array} \text{ then } f_A = (3,1,5,0,2)
$$

3.4 Isomorphism between P_n and R_n

We consider the natural partial order on RG_n :

$$\forall A, B \in RG_n,\ A \leq B \Leftrightarrow A[i,j] \leq B[i,j]\ \forall i,j$$

To any couple (f,g), $f,g \in P_n$, we associate its *north-east diagram* $NE(f,g)$: the planar representation of f, with a $\times$ for the point $(i, f(i))$, and the planar representation of g, with a $\odot$ for the point $(i, g(i))$, are parts of $NE(f,g)$); in addition, we put in each square $[i,i+1] \times [j,j+1] \subseteq [0,n+1] \times [0,n+1], 0 \le i,j \le n$, the number of $\odot$ - the number of $\times$ that lie above and to the right, i.e., in the north-east sector, of the square. We note this number $NE(f,g)[i,i+1] \times [j,j+1]$ and we have :

$$NE(f,g)[i,i+1] \times [j,j+1] = card\{k \le i \mid g(k) > j\} \ - \ card\{k \le i \mid f(k) > j\}$$

Example 3.11 $f = (3,0,2,4,1)$ *and* $g = (3,4,5,0,0)$ *:*

$$NE(f,g) = \begin{array}{c|cccccccccccc}
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & \\
1 & . & 0 & . & 0 & . & 0 & \otimes & 0 & . & 0 & . & 0 \\
2 & \times & 0 & . & 0 & . & 0 & . & 0 & \odot & 0 & . & 0 \\
3 & . & 1 & . & 1 & \times & 1 & . & 1 & . & 0 & \odot & 0 \\
4 & \odot & 1 & . & 1 & . & 2 & . & 2 & \times & 1 & . & 0 \\
5 & \odot & 0 & \times & 0 & . & 1 & . & 1 & . & 1 & . & 0 \\
 & & -1 & & 0 & & 1 & & 1 & & 1 & & 0
\end{array}$$

Observe that the squares sharing a common edge have the same value or differ by ± 1 following the rules, called *rules of passage*:

$$\begin{array}{c} \times \\ \vdots \\ i \vdots i+1 \\ \vdots \end{array} \qquad \begin{array}{c} \times \\ \vdots \\ i \vdots i+1 \\ \vdots \\ \odot \end{array} \qquad \begin{array}{c} \odot \\ \vdots \\ i+1 \vdots i \\ \vdots \end{array} \qquad \begin{array}{c} \odot \\ \vdots \\ i+1 \vdots i \\ \vdots \\ \times \end{array} \qquad \odot \cdots \overset{i+1}{\underset{i}{\cdots}} \cdots \times \qquad \times \cdots \overset{i}{\underset{i+1}{\cdots}} \cdots \odot$$

We show that P_n and R_n are isomorphic posets. The idea of the proof is essentially the idea of the proof of Proposition 7.1 of [4].

Theorem 3.12 $\forall f,g \in P_n,\ f \le_{P_n} g \ \Leftrightarrow \ M(f) \le_{R_n} M(g)$.

Proof : $(\Rightarrow)$ It is easy to see : $f \to g$ in $P_n \Rightarrow M(f) <_{R_n} M(g)$. Hence the implication follows.

$(\Leftarrow)$ Suppose $M(f) < M(g)$. We show : $\exists f' \in P_n$ such that $f < f'$ and $M(f') \le M(g)$. We conclude by induction that $f < g$.

1) Suppose : $\exists\ i$ such that $g(i) < f(i)$.
We will show : $\exists\ l < i$ such that
(I) $f(l) < f(i)$ and
(II) $NE(f,g)([r,r+1] \times [s,s+1]) > 0,\ \forall\ r,s$ such that $l \le r < i,\ f(l) \le s < f(i)$:

$$NE(f,g) = \begin{array}{c|cccccccc}
 & 0 & \cdots & g(i) & \cdots & f(l) & \cdots & f(i) & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
l & \cdot & \cdots & \cdot & \cdots & \times & \cdots & \cdot & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & > 0 & \vdots & \\
i & \cdot & \cdots & \odot & \cdots & \cdot & \cdots & \times & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

We will have then that $f'(x) = \begin{cases} f(x) & if \quad x \neq i,l \\ f(i) & if \quad x = l \\ f(l) & if \quad x = i \end{cases}$ is such that $f < f'$; and furthermore we will have $M(f') \leq M(g)$ because, if $l \leq r < i$, $f(l) \leq s < f(i)$, then :

$$NE(f',g)([r,r+1] \times [s,s+1]) = NE(f,g)([r,r+1] \times [s,s+1]) - 1$$

By the rules of passage, we have $NE(f,g)([i-1,i] \times [k',k'+1]) > 0$, $\forall k'$ such that $g(i) \leq k' < f(i)$. Let k, $0 < k \leq g(i)$, be the integer such that : 1) $NE(f,g)([i-1,i] \times [k',k'+1]) > 0$, $\forall k'$ such that $k \leq k' < g(i)$, and 2) $NE(f,g)([i-1,i] \times [k-1,k]) = 0$; if there is no such k, set $k = 0$:

$$NE(f,g) = \begin{array}{c|cccccccc}
 & 0 & \cdots & k & \cdots & g(i) & \cdots & f(i) & \cdots \\
\vdots & \vdots & & {}_0\vdots{}_1 & > 0 & \vdots & > 0 & \vdots & \\
i & \cdot & \cdots & \cdot & \cdots & \odot & \cdots & \times & \cdots
\end{array}$$

Let j be integer such that $NE(f,g)[j',j'+1] \times [k',k'+1] > 0$, $\forall\, j',k'$ such that $j \leq j' < i$, $k \leq k' < f(i)$. Then $\exists\, k''$, $k < k'' \leq f(i)$ such that $NE(f,g)[j,j+1] \times [k''-1,k''] = 1$ and $NE(f,g)[j-1,j] \times [k''-1,k''] = 0$:

$$NE(f,g) = \begin{array}{c|cccccccccc}
 & 0 & \cdots & k & \cdots & k'' & \cdots & g(i) & \cdots & f(i) & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
j & \cdot & \cdots & \cdot & \cdots & {}^{0}_{1}\cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
\vdots & \vdots & & {}_0\vdots{}_1 & & & > 0 & & & \vdots & \\
i & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \odot & \cdots & \times & \cdots
\end{array}$$

Applying the rules of passage, we have : $f(j) < k''$ and $\exists l' < i$ such that $f(l') = k$.

If $f(j) \geq k$, we have $l = j$. If $l' \geq j$, we have $l = l'$. If $k = 0$ then $k = 0 \leq f(j) < k''$ and we have $l = j$. In all those cases, we have the conclusion desired.

Suppose $f(j) < k$ and $l' < j$.

Then applying the rules of passage, we obtain with $a = NE(f,g)[j-1,j] \times [k-1,k] \geq 0$ and $b = NE(f,g)[i-1,i] \times [k''-1,k''] > 0$:

$$
NE(f,g) = \begin{array}{c|ccccccccccccc}
 & 0 & \cdots & & k & & \cdots & k'' & \cdots & g(i) & \cdots & f(i) & \cdots \\
\vdots & \vdots & & & \vdots & & & \vdots & & \vdots & & \vdots & \\
 & & & \mathrm{a} & & \mathrm{a+1} & 0 & & & & & & \\
j & \cdot & \cdots & & \cdot & & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots \\
 & & & \mathrm{a+1} & & \mathrm{a+2} & 1 & & & & & & \\
\vdots & \vdots & & & \vdots & & & & > 0 & & & \vdots & \\
 & & & 0 & & 1 & \mathrm{b} & & & & & & \\
i & \cdot & \cdots & & \cdot & & \cdots & \cdot & \cdots & \odot & \cdots & \times & \cdots
\end{array}
$$

The number of $\odot$ - the number of $\times$ inside the rectangle of corners $(i,k), (i,k''), (j,k)$, (j,k'') is $1-(a+2)-b+1 = -a-b \leq -b \leq -1$. This means : $\exists\ l',\ j < l' < i$ such that $k < f(l') < k''$. We have $l = l'$ and we have the conclusion desired.

2) Suppose : $\forall\ i,\ g(i) \geq f(i)$, i.e., on each row of $NE(f,g)$, we have $\cdots \times \cdots \odot \cdots$ or $\cdots \otimes \cdots$.

Let i be such that 1) $f(i) < g(i)$ and 2) $\nexists\ j, j \neq i$, such that $f(j) < g(j)$ and $g(i) < g(j)$. By the rules of passage, we have $NE(f,g)([r,r+1] \times [s,s+1]) > 0,\ \forall\ r,s$ such that $r \geq i,\ f(i) \leq s < g(i)$:

$$
NE(f,g) = \begin{array}{c|cccccc}
 & 0 & \cdots & f(i) & \cdots & g(i) & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & \\
i & \cdot & \cdots & \times & \cdots & \odot & \cdots \\
\vdots & \vdots & & \vdots & > 0 & \vdots &
\end{array}
$$

The fact that g is injective and the way we defined i imply that $f'(x) = \begin{cases} f(x) & if \quad x \neq i \\ g(i) & if \quad x = i \end{cases}$ is in P_n. We have $f' > f$ and furthermore $M(f') \leq M(g)$ because, if $r \geq i,\ f(i) \leq s < g(i)$, then

$$NE(f',g)([r,r+1] \times [s,s+1]) = NE(f,g)([r,r+1] \times [s,s+1]) - 1$$

Q.E.D.

4 MacNeille completion of P_n

4.1 The lattice RG_n

$(RG_n, \leq)$ is a lattice with $\forall A, A' \in RG_n$:

$$(A \vee A')[i,j] = max\{A[i,j], A'[i,j]\}$$

$$(A \wedge A')[i,j] = min\{A[i,j], A'[i,j]\}$$

$R_n \subseteq RG_n$ is not a lattice : we can see in Figure 1 that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \vee_{R_2} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ does not exist and that $L(R_2) = RG_2$.

We will show : $\forall n,\ L(R_n) = RG_n$.

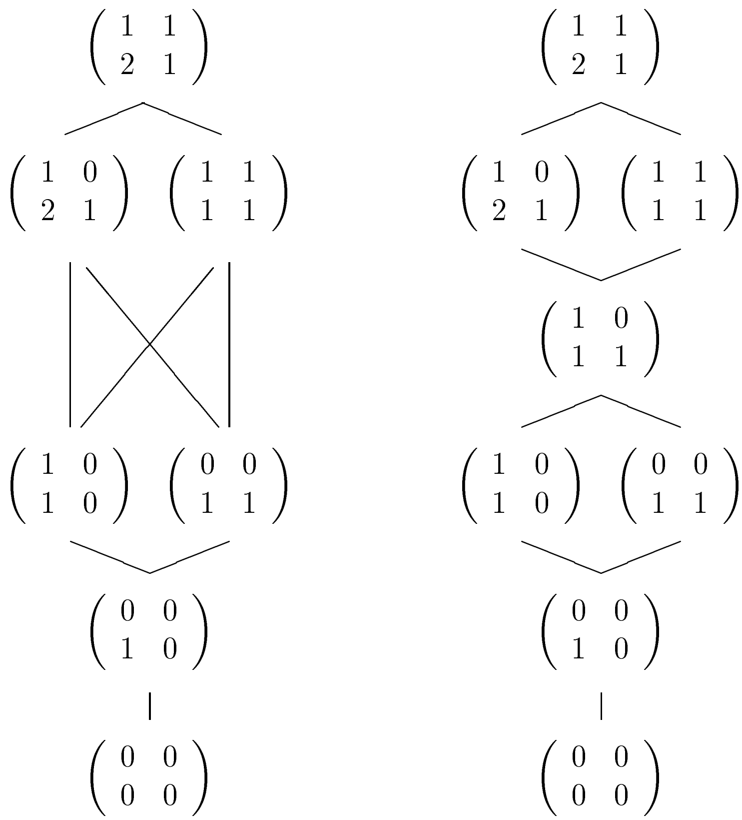


Figure 1: The poset R_2 and the lattice RG_2

4.2 The matrices $B_{r,s,a,n}$

$\forall r, s, a$ such that $1 \leq r, s \leq n,\ 0 < a \leq min\{r, n+1-s\}$, let $B_{r,s,a,n}$ be the matrix such that : 1) $B_{r,s,a,n}[r,s] = a$ and 2) $B_{r,s,a,n}[i,j]$, $(i,j) \neq (r,s)$, is the smallest value we can have in order that $B_{r,s,a,n} \in RG_n$.

Example 4.1

$$B_{4,3,3,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 \\ 3 & 3 & \underline{3} & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \end{pmatrix}$$

The following lemma is easy to prove. Details may be found in [3].

Lemma 4.2 *$\forall r, s, a$, such that $1 \leq r, s \leq n,\ 0 < a \leq min\{r, n+1-s\}$,*
1) $B_{r,s,a,n} = inf\{A \in RG_n \mid A[r,s] \geq a\}$: $A[r,s] \geq a \Rightarrow A \geq B_{r,s,a,n}$;
2) $A \ngeq B_{r,s,a,n} \Leftrightarrow A[r,s] < a$;
3) $B_{r,s,a,n} \in R_n$.

Theorem 4.3 *$\forall A \in RG_n$, $A = sup\{B_{r,s,a,n} \mid 1 \leq r, s \leq n,\ A[r,s] = a\}$.*

Proof : $\forall r,s$, such that $A[r,s] > 0$, $A \geq B_{r,s,A[r,s],n}$. Therefore $A \geq sup\{B_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\}$.

Suppose $A[i,j] \neq 0$; then $A[i,j] \geq (sup\{B_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\})[i,j] \geq B_{i,j,A[i,j],n}[i,j] = A[i,j]$. Therefore $A[i,j] = (sup\{B_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\})[i,j]$ and $A = sup\{B_{r,s,a,n} \mid 1 \leq r,s \leq n, A[r,s]=a\}$. Q.E.D.

Corollary 4.4 *$\forall A \in RG_n$, $\exists Q \subseteq R_n$ such that $A = sup(Q)$.*

Proof : Take $Q = \{B_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\}$. Q.E.D.

4.3 The matrices $C_{r,s,a,n}$

$\forall r,s,a$ such that $1 \leq r,s \leq n$, $0 \leq a < min\{r, n+1-s\}$, let $C_{r,s,a,n}$ be the matrix such that : 1) $C_{r,s,a,n}[r,s] = a$ and 2) $C_{r,s,a,n}[i,j]$, $(i,j) \neq (r,s)$, is the greatest value we can have in order that $C_{r,s,a,n} \in RG_n$.

Example 4.5 *$C_{6,4,1,8}$ and $C_{3,4,2,8}$ are respectively :*

$$\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 2&2&2&1&1&1&1&1\\ 3&3&2&1&1&1&1&1\\ 4&3&2&1&1&1&1&1\\ 4&3&2&1&1&1&1&1\\ 4&3&2&\underline{1}&1&1&1&1\\ 5&4&3&2&2&2&2&1\\ 6&5&4&3&3&3&2&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&1\\ 3&3&3&\underline{2}&2&2&2&1\\ 4&4&4&3&3&3&2&1\\ 5&5&5&4&4&3&2&1\\ 6&6&6&5&4&3&2&1\\ 7&7&6&5&4&3&2&1\\ 8&7&6&5&4&3&2&1 \end{pmatrix}$$

The following lemma is easy to prove. Details may be found in [3].

Lemma 4.6 *$\forall r,s,a$, such that $1 \leq r,s \leq n$, $0 \leq a < min\{r, n+1-s\}$,*
1) $C_{r,s,a,n} = sup\{A \in RG_n \mid A[r,s] \leq a\}$: $A[r,s] \leq a \Rightarrow A \leq C_{r,s,a,n}$;
2) $A \nleq C_{r,s,a,n} \Leftrightarrow A[r,s] > a$;
3) $C_{r,s,a,n} \in R_n$.

Theorem 4.7 *$\forall A \in RG_n$, $A = inf\{C_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\}$.*

Proof : $\forall r,s$, such that $A[r,s] < min\{r, n+1-s\}$, $A \leq C_{r,s,A[r,s],n}$. Therefore $A \leq inf\{C_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\}$.

Suppose $A[i,j] \neq min\{r, n+1-s\}$; then $A[i,j] \leq (inf\{C_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\})[i,j] \leq C_{i,j,A[i,j],n}[i,j] = A[i,j]$. Therefore $A[i,j] = (inf\{C_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\})[i,j]$ and $A = inf\{B_{r,s,a,n} \mid 1 \leq r,s \leq n, A[r,s]=a\}$. Q.E.D.

Corollary 4.8 *$\forall A \in RG_n$, $\exists R \subseteq R_n$ such that $A = inf(R)$.*

Proof : Take $R = \{C_{r,s,a,n} \mid 1 \leq r,s \leq n,\ A[r,s]=a\}$. Q.E.D.

Corollaries 4.4 and 4.8 and Theorem 2.7 give :

Theorem 4.9 *$L(R_n) \cong RG_n$, i.e., the MacNeille completion of R_n is isomorphic with RG_n.*

4.4 The base and cobase of R_n

Lemma 4.10 *$\forall r, s, a$ such that $1 \le r, s \le n,\ 0 < a \le min\{r, n+1-s\}$, $B_{r,s,a,n} \in B(R_n)$.*

Proof : $B(R_n) = B(RG_n)$ because (see Theorem 2.9) $L(R_n) \cong RG_n$; $B_{r,s,a,n} \in B(RG_n)$ if $B_{r,s,a,n}$ is the immediate successor of one and only one matrix $A \in RG_n$ (see Theorem 2.10).

Let A be the matrix such that $A[i,j] = B_{r,s,a,n}[i,j]$ $\forall (i,j) \ne (r,s)$ et $A[r,s] = a-1$.

$A \in RG_n$ because $\boxed{\begin{array}{ccc} & a-1 & \\ a & a & a-1 \\ & a & \end{array}}$ in $B_{r,s,a,n}$ becomes $\boxed{\begin{array}{ccc} & a-1 & \\ a & a-1 & a-1 \\ & a & \end{array}}$ in A.

We have $A \le Y \le B_{r,s,a,n} \Rightarrow Y[r,s] = a$ or $a-1$ $\Rightarrow Y = B_{r,s,a,n}$ or $Y = A$. Therefore $B_{r,s,a,n}$ is an immediate successor of A. Furthermore $Z < B_{r,s,a,n} \Rightarrow \forall (i,j) \ne (r,s)$, $Z[i,j] \le B_{r,s,a,n}[i,j] = A[i,j]$ and (see Lemma 4.2) $Z[r,s] \le a-1$. So $Z < B_{r,s,a,n} \Rightarrow Z \le A$, which shows that A is the only matrix for which $B_{r,s,a,n}$ is an immediate successor. Q.E.D.

Lemma 4.11 *$\forall r, s, a$ such that $1 \le r, s \le n,\ 0 \le a < min\{r, n+1-s\}$, $C_{r,s,a,n} \in C(R_n)$.*

Proof: Similar to the proof of the preceding lemma. Details in [3].

Theorem 4.12 *The matrices $B_{r,s,a,n}$ form exactly the base of R_n.*

Proof : By Lemma 4.10, we only need to show : $A \in B(R_n) \Rightarrow A$ is a matrix $B_{r,s,a,n}$. By Theorem 4.3, $A = sup\{B_{r,s,a,n} \mid 1 \le r, s \le n,\ A[r,s] = a\}$. Because $A \in B(R_n)$, A is one of these matrices. Q.E.D.

Theorem 4.13 *The matrices $C_{r,s,a,n}$ form exactly the cobase of R_n.*

Proof: Similar to the proof of the preceding theorem. Details in [3].

Theorem 4.14 *$\forall r, s, a$ such that $1 \le r, s \le n,\ 0 < a \le min\{r, n+1-s\}$, we have : $RG_n - B^-_{r,s,a,n} = C^+_{r,s,a-1,n}$, i.e., $B(RG_n) \subseteq Cl(RG_n)$.*

Proof : Let $A \in RG_n$; by Lemma 4.2, $A[r,s] \ge a \Leftrightarrow A \ge B_{r,s,a,n}$; by Lemma 4.6, $A[r,s] \le a-1 \Leftrightarrow A \le C_{r,s,a-1,n}$. Q.E.D.

Corollary 4.15 *$B(R_n) = Cl(R_n)$, i.e., R_n is dissective.*

Proof The conclusion follows from the preceding theorem and from Theorem 2.8. Q.E.D.

Theorem 4.16 *RG_n is a distributive lattice.*

Proof : The conclusion follows from the preceding corollary and from Theorem 2.11. Q.E.D.

4.5 The base and cobase of P_n

We have $R_n \cong P_n$. So $B(P_n) = \{f_A \mid A \in B(R_n)\}$ and $C(P_n) = \{f_A \mid A \in C(R_n)\}$.

Theorem 4.17 *$f \in B(P_n)$ iff f is an increasing function for which $dom(f)$ and $im(f)$ are intervals of integers.*

Proof : Let $A = B_{r,s,a,n}$, $1 \leq r, s \leq n$, $0 < a \leq min\{r, n+1-s\}$; then :

$$f_A = \begin{pmatrix} 1 & \cdots & r-a & r-a+1 & \cdots & r & r+1 & \cdots & n \\ 0 & \cdots & 0 & s & \cdots & s+a-1 & 0 & \cdots & 0 \end{pmatrix}$$

We see : $dom(f) = [r-a+1, r]$ and $im(f) = [s, s-a+1]$. Q.E.D.

Example 4.18 *If $A = B_{4,3,3,5}$ (see Example 4.1) then*

$$f_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 0 \end{pmatrix}$$

Let $A = C_{r,s,a,n}$, $1 \leq r, s \leq n$, $0 \leq a < min\{r, n+1-s\}$; if $a+s-1 < r$, i.e., if $C_{r,s,a,n}[n,1] < n$, then $f_A =$

$$\begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+s & \cdots & r & r+1 & \cdots & n \\ n & \cdots & n-a+1 & s-1 & \cdots & 0 & \cdots & 0 & n-a & \cdots & r-a+1 \end{pmatrix}$$

Example 4.19 *If $A = C_{6,4,1,8}$ (see Example 4.5) then*

$$f_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 2 & 1 & 0 & 0 & 7 & 6 \end{pmatrix}$$

if $a+s-1 \geq r$, i.e., if $C_{r,s,a,n}[n,1] = n$, then $f_A =$

$$\begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & r & r+1 \\ n & \cdots & n-a+1 & s-1 & \cdots & a+s-r & n-a \\ \\ \ldots & r+1+n-a-s & r+1+n-a-s+1 & \ldots & n & \\ \ldots & s & a+s-r-1 & \ldots & 1 & \end{pmatrix}$$

Example 4.20 *If $A = C_{3,4,2,8}$ (see Example 4.5) then*

$$f_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 3 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$$

4.6 Injection of P_n in S_{2n} with Bruhat order

We show that there exists a morphism of poset from P_n to S_{2n}. This result was suggested by Lascoux.

To any $f \in P_n$, we associate an element $f' \in P_{2n}$:

$$f'(i) = \begin{cases} f(i) + n & if\ 1 \le i \le n\ and\ i \in dom(f) \\ 0 & otherwise \end{cases}$$

Example 4.21 $f = \begin{pmatrix} 0 & 2 & 4 & 0 \end{pmatrix} \mapsto f' = \begin{pmatrix} 0 & 6 & 8 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$M(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} \mapsto M(f') = \begin{pmatrix} 0 & 0 & 0 & 0 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ 1 & 1 & 1 & 1 & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} \\ 2 & 2 & 2 & 2 & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{1} & \boldsymbol{1} \\ 2 & 2 & 2 & 2 & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{1} & \boldsymbol{1} \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}$$

As shown in the example, the submatrix of size n in the north-east corner of $M(f')$ is $M(f)$.

Lemma 4.22 $\forall f, g \in P_n,\ f \le_{P_n} g \ \Leftrightarrow\ f' \le_{P_{2n}} g'$.

Proof : We have the conclusion of the lemma because 1) $f \le_{P_n} g \Leftrightarrow M(f) \le_{R_n} M(g)$; 2) the submatrix of size n in the north-east corner of $M(f')$ is $M(f)$; 3) the submatrix of size n in the north-west corner of $M(f')$ is n copies of the first column of $M(f)$; 4) the submatrix of size n in the south-east corner of $M(f')$ is n copies of the last row of $M(f)$; 5) all the entries of the submatrix of size n in the south-west corner of $M(f')$ are $M(f)[n, 1]$. Q.E.D.

Lemma 4.23 $\forall f \in P_n,\ f \vee \boldsymbol{1}_{[n]} \in S_n$ *(where* $\boldsymbol{1}_{[n]}$ *is the identity function).*

Proof : We have:

$$M(\mathbf{1}_{[n]}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \vdots & & & & \vdots \\ n-2 & n-3 & n-4 & \dots & 0 \\ n-1 & n-2 & n-3 & \dots & 0 \\ n & n-1 & n-2 & \dots & 1 \end{pmatrix}$$

The minus pattern $\boxed{\begin{matrix} i+1 & i \\ i+1 & i+1 \end{matrix}}$ can be obtained in only one way as the supremum of two non minus patterns :

$$\boxed{\begin{matrix} i+1 & i \\ i+1 & i+1 \end{matrix}} = \boxed{\begin{matrix} i+1 & i \\ i+1 & i \end{matrix}} \vee \boxed{\begin{matrix} i & i \\ i+1 & i+1 \end{matrix}}$$

Observe that $M(\mathbf{1}_{[n]})$ does not have these two non minus patterns; so $M(f) \vee M(\mathbf{1}_{[n]}) \in R_n$ and $f \vee \mathbf{1}_{[n]} \in P_n$. Since $M(\mathbf{1}_{[n]})[n,1] = n$, $f \vee \mathbf{1}_{[n]} \in S_n$. Q.E.D.

Theorem 4.24 *$P_n \to S_{2n}$, $f \mapsto f' \vee \mathbf{1}_{[2n]}$, is a morphism of poset.*

Proof : By lemma 4.23, $f' \vee \mathbf{1}_{[2n]} \in S_{2n}$.

We have : $f \leq g \Leftrightarrow$ (by Lemma 4.22) $f' \leq g' \Rightarrow f' \vee \mathbf{1}_{[2n]} \leq g' \vee \mathbf{1}_{[2n]}$ because $g' \vee \mathbf{1}_{[2n]} \geq g' \geq f'$.

And $f' \vee \mathbf{1}_{[2n]} \leq g' \vee \mathbf{1}_{[2n]} \Leftrightarrow M(f' \vee \mathbf{1}_{[2n]}) \leq M(g' \vee \mathbf{1}_{[2n]}) \Rightarrow$ the submatrix of size n in the north-east corner of $M(f' \vee \mathbf{1}_{[2n]})$ is $\leq$ the submatrix of size n in the north-east corner of $M(g' \vee \mathbf{1}_{[2n]}) \Rightarrow$ the submatrix of size n in the north-east corner of $M(f')$ is $\leq$ the submatrix of size n in the north-east corner of $M(g')$ (because the submatrix of size n in the north-east corner of $\mathbf{1}_{[2n]}$ is the matrix 0) $\Rightarrow M(f) \leq M(g) \Rightarrow f \leq g$.

We have proved : $f \leq g \Leftrightarrow f' \vee \mathbf{1}_{[2n]} \leq g' \vee \mathbf{1}_{[2n]}$. Q.E.D.

Example 4.25

$$f = \begin{pmatrix} 0 & 2 & 4 & 0 \end{pmatrix} \mapsto f' \vee \mathbf{1}_{[2n]} = \begin{pmatrix} 1 & 6 & 8 & 2 & 3 & 4 & 5 & 7 \end{pmatrix}$$

5 Rectrices and corectrices

5.1 Rectrices and corectrices of RG_n

Let $A \in RG_n$; recall that $A^+ = \{X \in RG_n \mid X \leq A\}$ and that $A^- = \{X \in RG_n \mid X \geq A\}$. So by Theorem 4.3 and by Theorem 4.12, $A = sup(A^+ \cap B(R_n))$; and by Theorem 4.7 and by Theorem 4.13, $A = inf(A^- \cap C(R_n))$. Following [5], a *rectrice* of A is a maximal element of $(A^+ \cap B(R_n))$ and a *corectrice* of A is a minimal element of $(A^- \cap C(R_n))$.

Following [5], we say that $A \in RG_n$ has an *essential point* $\boxed{\begin{array}{ccc} & a-1 & \\ a & a & a-1 \\ & a & \end{array}}$ in position r, s, of value $a > 0$, if $A[r-1, s] = A[r, s+1] = a - 1$, $A[r, s-1] = A[r, s] = A[r+1, s] = a$. In other terms, A has an essential point in position r, s, of value $a > 0$, if we can replace $A[r, s] = a$ by $a - 1$ and still have a matrix $\in RG_n$. Hence A may have an essential point in position r, s, with r or $s \in \{1, n\}$. In brief, we will say that A has an essential point rsa.

Note that $B_{r,s,a,n}$ has one and only one essential point rsa.

Theorem 5.1 *$B_{r,s,a,n}$ is a rectrice of $A \Leftrightarrow A$ has an essential point rsa.*

Proof: ($\Leftarrow$) $A[r, s] = a \Rightarrow$ (by Lemma 4.2) $B_{r,s,a,n} \in (A^+ \cap B(R_n))$. Suppose $X \in (A^+ \cap B(R_n))$ with $A \geq X \geq B_{r,s,a,n}$. We find that X has an essential point rsa. Since X has only one essential point, $X = B_{r,s,a,n}$. Hence $B_{r,s,a,n}$ is a rectrice of A.

($\Rightarrow$) $B_{r,s,a,n}$ is a rectrice of A and $A = sup(A^+ \cap B(R_n)) \Rightarrow A[r, s] = a$.

Suppose and $A[r-1, s] = a$ (with $r > 1$). We have then: $Z = B_{r-1,s,a,n} \in (A^+ \cap B(R_n))$ with $Z \ngeq B_{r,s,a,n}$; by Theorem 4.2, $Z[r, s] < a$. Contradiction and $A[r-1, s] = a - 1$.

In the same way, we show that $A[r,s-1]=a$ (if $s>1$); $A[r+1,s]=a$ (if $r<n$); and $A[r,s+1]=a-1$ (if $s<n$). So A has an essential point rsa. Q.E.D.

Corollary 5.2 *$A = sup\{B_{r,s,a,n} \mid A$ has an essential point $rsa\}$.*

Proof: $A = sup(A^+ \cap B(R_n)) = sup\{B_{r,s,a,n} \mid B_{r,s,a,n}$ is a rectrice of $A\} = sup\{B_{r,s,a,n} \mid A$ has an essential point $rsa\}$. Q.E.D.

We say that $A \in RG_n$ has an *coessential point* $\boxed{\begin{array}{ccc} & a & \\ a+1 & a & a \\ & a+1 & \end{array}}$ in position r,s of value $0 \le a < min\{r, n+1-s\}$, if $A[r-1,s] = A[r,s] = A[r,s+1] = a$, $A[r,s-1] = A[r+1,s] = a+1$. In other terms, A has an coessential point rsa if we can replace $A[r,s]=a$ by $a+1$ and still have a matrix $\in RG_n$. Hence A may have an essential point in position r,s, with r or $s \in \{1,n\}$. In brief, we will say that A has an coessential point rsa.

Note that $C_{r,s,a,n}$ has one and only one coessential point rsa.

Theorem 5.3 *$C_{r,s,a,n}$ is a corectrice of $A \Leftrightarrow A$ has an coessential point rsa.*

Proof: Similar to the proof of Theorem 5.1. Details in [3].

Corollary 5.4 *$A = inf\{C_{r,s,a,n} \mid A$ has an coessential point $rsa\}$.*

Proof: Similar to the proof of Corollary 5.2. Details in [3].

Example 5.5

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 & 1 \end{pmatrix}$$

The essential points of A are : 131, 212, 241, 332, 351, 514. The coessential points of A are : 140, 221, 250, 312, 422, 541.

If we know the rectrices (or the essential points) of A, we can rebuild A : 1) $A[r,s]=a$ for all rectrices $B_{r,s,a,n}$ and 2) $A[i,j]$, $ij*$ not an essential point, is the smallest value we can have in order that $A \in RG_n$.

Example 5.6 *Suppose the rectrices of A are : $B_{2,3,1,4}$, $B_{4,2,3,4}$; then*

$$\begin{pmatrix} * & * & * & * \\ * & * & \underline{1} & * \\ * & * & * & * \\ * & \underline{3} & * & * \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & \underline{1} & 0 \\ 2 & 2 & 1 & 0 \\ 3 & \underline{3} & 2 & 1 \end{pmatrix}$$

If we know the corectrices (or the coessential points) of A, we can rebuild A : 1) $A[r,s] = a$ for all corectrices $C_{r,s,a,n}$ and 2) $A[i,j]$, $ij*$ not an coessential point, is the greatest value we can have in order that $A \in RG_n$.

Example 5.7 *Suppose the corectrices of A are : $C_{2,3,0,4}$, $C_{4,2,2,4}$; then*

$$\begin{pmatrix} * & * & * & * \\ * & * & \underline{0} & * \\ * & * & * & * \\ * & \underline{2} & * & * \end{pmatrix} \textit{ and } A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & \underline{0} & 0 \\ 3 & 2 & 1 & 1 \\ 3 & \underline{2} & 2 & 1 \end{pmatrix}$$

5.2 The sets of Keys K_n and generalized Keys KG_n

$k = (k_j)_{j=1,\dots,n} \in KG_n$ if k_j is an injective partial functions $k_j : \{1,\dots,j\} \to [n]$, $i \mapsto k_j(i) = k_{ij}$, such that 1) $dom(k_j) = \{1,\dots,j'\}, j' \le j$; 2) k_j is decreasing; 3) $k_{i+1,j+1} \le k_{ij} \le k_{i,j+1}$, $j = 1,\dots n-1$, $1 \le i \le j$, with the convention that $k_j(i) = k_{ij} = 0$ if $j' < i \le j$. An element $k \in KG_n$ will be called a *generalized Key.*

We represent k like this : $k = \begin{matrix} k_{11} & k_{12} & \cdots & k_{1n} \\ & k_{22} & \cdots & k_{2n} \\ & & \ddots & \vdots \\ & & & k_{nn} \end{matrix}$

Example 5.8

$$\begin{matrix} 2 & 5 & 5 & 5 & 5 & 5 & 7 \\ & 2 & 3 & 3 & 3 & 4 & 4 \\ & & 2 & 2 & 2 & 2 & 3 \\ & & & 0 & 1 & 1 & 2 \\ & & & & 0 & 0 & 1 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{matrix} \in KG_7$$

We define a partial order on KG_n : $k \le k' \Leftrightarrow k_{ij} \le k'_{ij}\ \forall i,j$. KG_n is a lattice : $sup(k,k')_{ij} = max(k_{ij}, k'_{ij})$ *and* $min(k,k')_{ij} = inf(k_{ij}, k'_{ij})$.

We define K_n by saying that $k \in K_n \subseteq KG_n$ if $k_{ij} = k_{i+1,j+1}$ or $k_{ij} = k_{i,j+1}$, $j = 1,\dots n-1$, $1 \le i \le j$. K_n is not a lattice.

An element $k \in K_n$ will be called a *Key.* In this section and in the next, we state results without proofs : details may be found in [3]. They generalize results that we can find in [5], where we deal with *keys* and with *triangles.* A key is a Key where the functions k_j are injective functions (not only partial injective functions) : a key has no zero entry. A triangle is a generalized Key with no zero entry.

To any $f \in P_n$, we can associate bijectively an element $K(f) \in K_n$. An example will show how.

Example 5.9

$$P_6 \ni f = \begin{pmatrix} 2 & 5 & 3 & 0 & 0 & 4 \end{pmatrix} \leftrightarrow k_f = \begin{array}{cccccc} 2 & 5 & 5 & 5 & 5 & 5 \\ & 2 & 3 & 3 & 3 & 4 \\ & & 2 & 2 & 2 & 3 \\ & & & 0 & 0 & 2 \\ & & & & 0 & 0 \\ & & & & & 0 \end{array} \in K_6$$

5.3 The Keys $b[r, s, a, n]$ and $c[r, s, a, n]$

$\forall r, s, a$ such that $1 \le s \le n,\ 1 \le r \le s,\ 0 < a \le n+1-r$, let $b[r, s, a, n]$ be the Key such that : 1) $b[r, s, a, n]_{rs} = a$ and 2) $b[r, s, a, n]_{ij},\ ij \neq rs$, is the smallest value we can have in order that $b[r, s, a, n] \in KG_n$.

Example 5.10

$$b[3,4,2,5] = \begin{array}{ccccc} 0 & 2 & 3 & 4 & 4 \\ & 0 & 2 & 3 & 3 \\ & & 0 & \underline{2} & 2 \\ & & & 0 & 0 \\ & & & & 0 \end{array}$$

Lemma 5.11 *$\forall r, s, a$, such that $1 \le s \le n$, $1 \le r \le s$, $0 < a \le n+1-r$,*
1) $b[r, s, a, n] = inf\{k \in KG_n \mid k_{rs} \ge a\}$: $k_{rs} \ge a \Rightarrow k \ge b[r, s, a, n]$;
2) $k \not\ge b[r, s, a, n] \Leftrightarrow k_{rs} < a$;
3) $b[r, s, a, n] \in K_n$.

Theorem 5.12 *$\forall k \in KG_n$, $k = sup\{b[r, s, a, n] \mid k_{rs} = a\}$.*

Corollary 5.13 *$\forall k \in KG_n$, $\exists Q \subseteq K_n$ such that $k = sup(Q)$.*

$\forall r, s, a$ such that $1 \le s \le n,\ 1 \le r \le s,\ 0 \le a < n+1-r$, let $c[r, s, a, n]$ be the Key such that : 1) $c[r, s, a, n]_{rs} = a$ and 2) $c[r, s, a, n]_{ij},\ ij \neq rs$, is the greatest value we can have in order that $c[r, s, a, n] \in KG_n$.

Example 5.14

$$c[3,4,2,5] = \begin{array}{ccccc} 5 & 5 & 5 & 5 & 5 \\ & 4 & 4 & 4 & 4 \\ & & 2 & \underline{2} & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{array}, \quad c[2,4,1,5] = \begin{array}{ccccc} 5 & 5 & 5 & 5 & 5 \\ & 1 & 1 & \underline{1} & 4 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{array}$$

Lemma 5.15 *$\forall r, s, a$ such that $1 \le s \le n$, $1 \le r \le s$, $0 \le a < n+1-r$,*
1) $c[r, s, a, n] = sup\{k \in KG_n \mid k_{rs} \le a\}$: $k_{rs} \le a \Rightarrow k \le b[r, s, a, n]$;
2) $k \not\le c[r, s, a, n] \Leftrightarrow k_{rs} > a$;
3) $c[r, s, a, n] \in K_n$.

Theorem 5.16 *$\forall k \in KG_n$, $k = inf\{c[r,s,a,n] \mid k_{rs} = a\}$.*

Corollary 5.17 *$\forall k \in KG_n$, $\exists R \subseteq K_n$ such that $k = inf(R)$.*

Theorem 5.18 *$L(K_n) \cong KG_n$, i.e., the MacNeille completion of K_n is isomorphic with KG_n.*

Theorem 5.19 *The Keys $b[r,s,a,n]$ form exactly the base of K_n; the Keys $c[r,s,a,n]$ form exactly the cobase of K_n.*

5.4 Rectrices and corectrices of KG_n

A *rectrice* of $k \in KG_n$ is a maximal element of $(k^+ \cap B(K_n))$ and a *corectrice* is a minimal element of $(k^- \cap C(K_n))$.

We say that $k \in KG_n$ has an *essential point* $\boxed{\begin{matrix} b & a & \\ & c & d \end{matrix}}$ in position r,s of value $0 < a \leq n+1-r$, if : $k_{rs} = a > b = k_{r,s-1}$, $a > d = k_{r+1,s+1}$ and $(a > c+1 = k_{r+1,s}+1$ or $c = 0)$. In other terms, k has an essential point in position r,s of value $0 < a \leq n+1-r$, if we can replace $k_{rs} = a$ by $a-1$ and still have an element $\in KG_n$. In brief, we will say that k has an essential point rsa.

Note that $b[r,s,a,n]$ has one and only one essential point rsa.

Theorem 5.20 *$b[r,s,a,n]$ is a rectrice of $k \Leftrightarrow k$ has an essential point rsa.*

Corollary 5.21 *$k = sup\{b[r,s,a,n] \mid k$ has an essential point $rsa\}$.*

We say that $k \in KG_n$ has an *coessential point* $\boxed{\begin{matrix} b & a & \\ & c & d \end{matrix}}$ in position r,s of value $0 < a \leq n+1-r$, if : $k_{rs} = a > b = k_{r,s-1}$, $a > d = k_{r+1,s+1}$ and $(a > c+1 = k_{r+1,s}+1$ or $c = 0)$. In other terms, k has an coessential point rsa if we can replace $k_{rs} = a$ by $a+1$ and still have an element $\in KG_n$. In brief, we will say that k has an essential point rsa.

Note that $c[r,s,a,n]$ has one and only one coessential point rsa.

Theorem 5.22 *$c[r,s,a,n]$ is a corectrice of $k \Leftrightarrow k$ has an coessential point rsa.*

Corollary 5.23 *$k = inf\{kc[r,s,a,n] \mid k$ has an coessential point $rsa\}$.*

If we know the rectrices (or the essential points) of k, or if we know the corectrices (or the coessential points) of k, we can rebuild k.

Example 5.24 *Suppose the rectrices of k are : $b[1,2,3,4]$, $b[3,3,1,4]$; then*

$$\begin{matrix} * & \underline{3} & * & * \\ & * & * & * \\ & & \underline{1} & * \\ & & & * \end{matrix} \quad \textit{and } k = \quad \begin{matrix} 1 & \underline{3} & 3 & 3 \\ & 1 & 2 & 2 \\ & & \underline{1} & 1 \\ & & & 0 \end{matrix}$$

Example 5.25 *Suppose the corectrices of* k *are :* $c[1,2,3,4]$, $c[3,3,0,4]$*; then*

$$\begin{array}{cccc} * & \underline{3} & * & * \\ & * & * & * \\ & & \underline{0} & * \\ & & & * \end{array} \quad \text{and } k = \quad \begin{array}{cccc} 3 & \underline{3} & 4 & 4 \\ & 2 & 3 & 3 \\ & & \underline{0} & 2 \\ & & & 0 \end{array}$$

We will show in the next section that the function between P_n and K_n, $f \leftrightarrow K(f)$, as illustrated in Example 5.9, is in fact an isomorphism of posets. So $A(\in R_n) \leftrightarrow f_A(\in P_n) \leftrightarrow K(f_A)(\in K_n)$ are isomorphisms of posets.

We have $B_{r,s,a,n}(\in B(R_n)) \leftrightarrow f_{B_{r,s,a,n}}(\in B(P_n)) \leftrightarrow K(f_{B_{r,s,a,n}}) = b[a,r,s,n](\in B(K_n))$. If $A \in R_n$ has an essential point rsa, then $K(f_A) = K(A)$ has an essential point ars.

Example 5.26

$$B_{4,2,3,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & \underline{3} & 2 & 1 & 0 \\ 3 & 3 & 3 & 2 & 1 \end{pmatrix} \leftrightarrow f_{B_{4,2,3,5}} = \begin{pmatrix} 0 & 2 & 3 & 4 & 0 \end{pmatrix}$$

$$\leftrightarrow K(f_{B_{4,2,3,5}}) = \quad \begin{array}{ccccc} 0 & 2 & 3 & 4 & 4 \\ & 0 & 2 & 3 & 3 \\ & & 0 & \underline{2} & 2 \\ & & & 0 & 0 \\ & & & & 0 \end{array} \quad = b[3,4,2,5]$$

Example 5.27 *The essential points of* $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & \underline{1} & 0 \\ 2 & 2 & 1 & 0 \\ 3 & \underline{3} & 2 & 1 \end{pmatrix}$ *are : 231, 423. The essential points of* $K(A)$ *are : 123, 342. So*

$$\begin{array}{cccc} . & \underline{3} & . & . \\ & . & . & . \\ & & . & \underline{2} \\ & & & . \end{array} \quad \text{and } K(A) = \quad \begin{array}{cccc} 0 & \underline{3} & 3 & 4 \\ & 0 & 2 & 3 \\ & & 0 & \underline{2} \\ & & & 0 \end{array}$$

We have also $C_{r,s,a,n}(\in C(R_n)) \leftrightarrow f_{C_{r,s,a,n}}(\in C(P_n)) \leftrightarrow K(f_{C_{r,s,a,n}}) = c[a+1,r,s-1,n](\in C(K_n))$. If $A \in R_n$ has a coessential point rsa, then $K(f_A) = K(A)$ has an coessential point $a+1,r,s-1$.

Example 5.28

$$C_{4,2,1,5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & \underline{1} & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 \end{pmatrix} \leftrightarrow f_{C_{4,2,1,5}} = \begin{pmatrix} 5 & 1 & 0 & 0 & 4 \end{pmatrix}$$

$$\leftrightarrow K(f_{C_{4,2,1,5}}) = \begin{array}{ccccc} 5 & 5 & 5 & 5 & 5 \\ & 1 & 1 & \underline{1} & 4 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{array} = c[2,4,1,5]$$

Example 5.29 *The coessential points of* $A = \begin{pmatrix} \underline{0} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & \underline{1} & 1 \\ 3 & 3 & 2 & 1 \end{pmatrix}$ *are : 110, 331. The coessential points of* $K(A)$ *are : 110, 232. So*

$$\begin{array}{cccc} \underline{0} & . & . & . \\ & . & \underline{2} & . \\ & & . & . \\ & & & . \end{array} \quad \text{and } K(A) = \quad \begin{array}{cccc} \underline{0} & 4 & 4 & 4 \\ & 0 & \underline{2} & 3 \\ & & 0 & 2 \\ & & & 0 \end{array}$$

5.5 Isomorphism between Keys and partial injective functions

We show that K_n and P_n are isomorphic posets. Theorem 5.30 is a generalization of Proposition 2.1.11 in [8] and of Proposition 1.19 of [6]. Moreover there is a little gap in the proofs of these propositions. We will show where while giving the proof of Theorem 5.30.

Theorem 5.30 $\forall f, g \in P_n,\ f \leq_{P_n} g \ \Leftrightarrow\ K(f) \leq_{K_n} K(g)$.

Proof : ($\Rightarrow$) It is easy to see : $f \to g$ in $P_n \Rightarrow K(f) <_{K_n} K(g)$. Hence the implication follows.

($\Leftarrow$) Suppose $K(f) < K(g)$. We show : $\exists f' \in P_n$ such that $f < f'$ and $K(f) < K(f') \leq K(g)$ or $\exists g' \in P_n$ such that $g' < g$ and $K(f) \leq K(g') < K(g)$. We conclude by induction that $f < g$.

Let $s \geq 0$ be the smallest integer such that the columns $1, \ldots, s-1$ of $K(f)$ and $K(g)$ are identical. Let a and b be the integers such that : $0 \leq a = f(s) < g(s) = b$.

(a) suppose : $\exists\ s' > s$ such that $a < f(s') = c \leq b$. We take the smallest s' and we then have : $\forall s''$ such that $s < s'' < s'$, $f(s'') \leq a$ or $f(s'') > b$.

In [8] and in [6], s' exists because f is bijective : s' is such that $f(s') = b$; the function $f'(x) = \begin{cases} f(x) & if \quad x \neq s, s' \\ b & if \quad x = s \\ a & if \quad x = s' \end{cases}$ is such that $f < f'$, but we cannot conclude that $K(f') \leq K(g)$:

Example 5.31 *Let* $f = \begin{pmatrix} 1 & 3 & 4 & 2 \end{pmatrix}$ *and* $g = \begin{pmatrix} 4 & 2 & 3 & 1 \end{pmatrix}$

$$K(f) = \begin{array}{cccc} 1 & 3 & 4 & 4 \\ & 1 & 3 & 3 \\ & & 1 & 2 \\ & & & 1 \end{array} \leq K(g) = \begin{array}{cccc} 4 & 4 & 4 & 4 \\ & 2 & 3 & 3 \\ & & 2 & 2 \\ & & & 1 \end{array}$$

$$f < f' = \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix} \text{ but } K(f') = \begin{array}{cccc} 4 & 4 & 4 & 4 \\ & 3 & 3 & 3 \\ & & 1 & 2 \\ & & & 1 \end{array} \not\leq K(g)$$

Let $a_0 = a, a_1, \ldots, a_m, a_{m+1}$ be the numbers in successive rows in column s of $K(f)$ $\boxed{\begin{array}{c} a_{m+1} \\ a_m \\ \vdots \\ a_1 \\ a \end{array}}$ such that $a_m < c < a_{m+1}$.

The function $f'(x) = \begin{cases} f(x) & if \quad x \neq s, s' \\ c & if \quad x = s \\ a & if \quad x = s' \end{cases}$ is such that :

1) $f < f'$ because $a < c$;

2) $K(f) < K(f')$ because $\boxed{\begin{array}{c} a_{m+1} \\ a_m \\ \vdots \\ a_1 \\ a \end{array}}$ in columns $s, s+1, \ldots, s'-1$ of $K(f)$ has been replaced by $\boxed{\begin{array}{c} a_{m+1} \\ c \\ a_m \\ \vdots \\ a_1 \end{array}}$ in $K(f')$;

3) $K(f') \leq K(g):$ we have in columns s of respectively $K(f')$ and $K(g)$ $\boxed{\begin{array}{cc} & b \\ \vdots & \\ a_{m+1} & \\ c & a_{m+1} \\ a_m & a_m \\ \vdots & \\ a_1 & a_1 \end{array}}$,

so the column s of $K(f')$ is $\leq$ the column s of $K(g)$; furthermore $K(f) < K(g)$ and the way we defined s' imply that the number of integers $> b$ in columns s'' of $K(f')$, $s \leq s'' < s'$, is $\leq$ the number of integers $> b$ in columns s'' of $K(g)$, $s \leq s'' < s'$; this means that $c, a_m, \ldots, a_1$, in columns s'' of $K(f')$, $s \leq s'' < s'$, are on rows which are the same or are above the rows where are $a_{m+1}, a_m, \ldots, a_1$, in columns s'' of $K(g)$, $s \leq s'' < s'$: thus the columns s'' of $K(f')$, $s \leq s'' < s'$ are $\leq$ the columns s'' of $K(g)$, $s \leq s'' < s'$.

(b) suppose : $\exists\ s' > s$ such that $a \leq g(s') = d < b$. We take the smallest s' and we have then: $\forall s''$ such that $s < s'' < s$, $g(s'') < a$ or $g(s'') > b$.

The function $g'(x) = \begin{cases} g(x) & if \quad x \neq s, s' \\ d & if \quad x = s \\ b & if \quad x = s' \end{cases}$ is such that :

1) $g' < g$;
2) $K(g') < K(g)$;
3) $K(f) \leq K(g')$.

(c) suppose : $\nexists\ s' > s$ such that $a < f(s') = c \leq b$ or such that $a \leq g(s') = d < b$. This implies : $b \notin \mathrm{im}(f)$ and $a \notin \mathrm{im}(g)$.

The function $f'(x) = \begin{cases} f(x) & if \quad x \neq s \\ b & if \quad x = s \end{cases}$ is such that :

1) $f < f'$;
2) $K(f) < K(f')$;
3) $K(f') \leq K(g)$.

The proof is complete. Q.E.D.

6 Alternating matrices : At_n

6.1 Bijection between RG_n and At_n

The set of *alternating matrices* is denoted At_n. At_n is a set of square matrices of size n with entries $\in \{-1, 0, 1\}$. $A \in At_n$ if 1) the sum on each row and on each column is 0 or 1; 2) the 1 and -1 alternate on each row and on each column; 3) the first non-zero entry (if any) on each column is 1; 4) the last non-zero entry (if any) on each row is 1.

Note that an alternating sign matrix, see [1], is an alternating matrix for which the sum on each row and on each column is 1.

The pattern $\boxed{\begin{matrix} a \\ a+1 \end{matrix}}$ in a matrix $\in RG_n$ is followed by : $\boxed{\begin{matrix} a \\ a+1 \end{matrix}}$, $\boxed{\begin{matrix} a-1 \\ a \end{matrix}}$ or $\boxed{\begin{matrix} a \\ a \end{matrix}}$. The pattern $\boxed{\begin{matrix} a \\ a \end{matrix}}$ in a matrix $\in RG_n$ is followed by : $\boxed{\begin{matrix} a \\ a \end{matrix}}$, $\boxed{\begin{matrix} a-1 \\ a-1 \end{matrix}}$ or $\boxed{\begin{matrix} a-1 \\ a \end{matrix}}$.

So the pattern $\boxed{\begin{matrix} a \\ a+1 \end{matrix}}$ is the beginning of a pattern zero or a pattern plus, and the pattern plus $\boxed{\begin{matrix} a & a \\ a+1 & a \end{matrix}}$ is followed by a pattern zero or by a pattern minus :

$$\boxed{\begin{matrix} a & a \\ a+1 & a+1 \end{matrix}}, \boxed{\begin{matrix} a & a-1 \\ a+1 & a \end{matrix}}, \boxed{\begin{matrix} a & a \\ a+1 & a \end{matrix}};$$

$$\boxed{\begin{matrix} a & a & a \\ a+1 & a & a \end{matrix}}, \boxed{\begin{matrix} a & a & a-1 \\ a+1 & a & a-1 \end{matrix}}, \boxed{\begin{matrix} a & a & a-1 \\ a+1 & a & a \end{matrix}};$$

and the pattern $\boxed{\begin{matrix} a \\ a \end{matrix}}$ is the end of a pattern zero or a pattern plus, and the pattern minus $\boxed{\begin{matrix} a+1 & a \\ a+1 & a+1 \end{matrix}}$ is followed by a pattern zero or by a pattern plus :

$$\boxed{\begin{matrix} a & a \\ a & a \end{matrix}}, \boxed{\begin{matrix} a+1 & a \\ a+1 & a \end{matrix}}, \boxed{\begin{matrix} a & a \\ a+1 & a \end{matrix}};$$

$$\boxed{\begin{matrix} a+1 & a & a \\ a+1 & a+1 & a+1 \end{matrix}}, \boxed{\begin{matrix} a+1 & a & a-1 \\ a+1 & a+1 & a \end{matrix}}, \boxed{\begin{matrix} a+1 & a & a \\ a+1 & a+1 & a \end{matrix}}.$$

The work we did horizontally, we can make it vertically. So we have proved Lemma 3.6 : the patterns plus and minus, horizontally and vertically, alternate in a matrix $A \in RG_n$.

Furthermore, because the row 0 of $A \in RG_n$ is a row of zeros and the column $n+1$ a column of zeros, the first non-zero (if any) pattern on a column is 1 and the last non-zero (if any) pattern on a row is 1.

So the matrix A', $A'[r,s] = \begin{cases} +1 & \text{if } A \text{ has a pattern plus in position } r-1, s \\ -1 & \text{if } A \text{ has a pattern minus in position } r-1, s \\ 0 & \text{if } A \text{ has a pattern zero in position } r-1, s \end{cases}$, is an alternating matrix.

Example 6.1 : $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix}$, $A' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}$

Theorem 6.2 *Card(RG_n) = card(At_n).*

Proof : The function $RG_n \to At_n,\ A \mapsto A'$ in a bijection : $A[r,s]$ is the number of 1 - the number of -1 in position r', s' of A', $r' < r$ and $s' \geq s$. This a consequence of lemma 3.7 : $\forall A \in RG_n,\ A[r,s] =$ the number of plus patterns - the number of minus patterns that lie above and to the right of the position r, s. Thus card(RG_n) = card(At_n). Q.E.D.

Proof of Lemma 3.7 : We define :

$$|r,s| = card\{(r',s') \mid r' < r, s' \geq s, A \text{ has a pattern plus in position } r', s'\}$$

$$- card\{(r',s') \mid r' < r, s' \geq s, A \text{ has a pattern minus in position } r', s'\};$$

We prove that $A[r,s] = |r,s|$.

If A has the pattern $\boxed{\begin{matrix} a \\ a+1 \end{matrix}}$ in position $r-1, s$, it is the beginning of a pattern zero or a pattern plus; if it is a pattern zero, it is followed by pattern(s) zero and by a pattern plus; the number of patterns plus to the right of $\boxed{\begin{matrix} a \\ a+1 \end{matrix}}$ is one more than the number of patterns minus because the patterns plus and minus alternate, ending by a pattern plus. So $A[r,s] = A[r-1,s] + 1 \ \Rightarrow\ |r,s| = |r-1,s| + 1$.

If A has the pattern $\boxed{\begin{matrix} a \\ a \end{matrix}}$ in position $r-1, s$, it is the beginning of a pattern zero or a pattern minus; if it is a pattern zero, it is followed by pattern(s) zero and, possibly, by a pattern minus; the number of patterns plus to the right of $\boxed{\begin{matrix} a \\ a \end{matrix}}$ is the same than the number of patterns minus because the patterns plus and minus alternate, ending by a pattern plus. So $A[r,s] = A[r-1,s] \ \Rightarrow\ |r,s| = |r-1,s|$.

We have also : $A[r,s+1] = A[r,s] - 1 \ \Rightarrow\ |r,s| = |r,s+1| - 1$ and $A[r,s+1] = A[r,s] \ \Rightarrow\ |r,s| = |r,s+1|$.

Since $A[1,1] = 1$ if A has a pattern plus in position $0, s$, s being unique, and $A[1,1] = 0$, otherwise we have $A[1,1] = |1,1|$. We then have the conclusion of the lemma by double induction on r and s. Q.E.D.

6.2 Bijection between At_n and KG_n

Here is a bijection between KG_n and At_n that generalizes the bijection we find in [1], page 57, between alternating sign matrices and triangles.

To any $A' \in At_n$, we associate a square matrix X_A of size n in which $X_A[i,j] = \sum_{k=1}^{j} A'[i,k]$. $X_A[i,j]$ is the sum of the entries from rows 1 to i of the jth column of A'. We recover A' from X_A : $A'[i,j] = X_A[i,j] - X_A[i-1,j]$.

Suppose row j of X_A has a 1 in columns $j_1 < j_2 < \ldots < j_r$. Let $k(A)_j : \{1,\ldots,j\} \to [n]$ a partial injective function defined like this : $k(A)_j(1) = k(A)_{1j} = j_r, k(A)_j(2) = k(A)_{2j} = j_{r-1}, \ldots, k(A)_j(r) = k(A)_{rj} = j_1$ and $k(A)_j(r+1) = k(A)_{r+1,j} = \ldots = k(A)_j(j) = k(A)_{jj} = 0$. We have then (see [3]) :

Theorem 6.3 $\forall A' \in At_n,\ k(A) = (k(A)_j)_{j=1,\ldots,n} \in KG_n$.

Theorem 6.4 $At_n \to KG_n,\ A' \mapsto k(A)$ *is a bijection.*

Example 6.5

$$A' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix},\ X_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix},\ k(A) = \begin{matrix} 3 & 3 & 4 & 4 \\ & 1 & 2 & 3 \\ & & 0 & 1 \\ & & & 0 \end{matrix}$$

6.3 Isomorphism between RG_n and KG_n

Since R_n, P_n and K_n are isomorphic posets, by Theorem 2.7, $L(R_n)$, $L(P_n)$ and $L(K_n)$ are isomorphic lattices. Since $L(R_n)$ and RG_n are isomorphic lattices and since $L(K_n)$ and KG_n are isomorphic lattices, RG_n and KG_n are isomorphic lattices. We give here another way to see this isomorphism.

Let $A \in RG_n$. Since $A = inf\{C_{r,s,a,n} \mid A$ has an coessential point $rsa\}$ (see Corollary 5.4), A is the greatest matrix $\in RG_n$ that has the coessential points the matrix A has. If $A < B$ in RG_n, then A has a coessential point, say rsa, that B does not have because B cannot have the coessential points of A and be $> A$.

The matrix $C[i,j] = \begin{cases} A[i,j]+1 & if\ (i,j)=(r,s) \\ A[i,j] & otherwise \end{cases}$ is an immediate successor of A and it is easy to prove that $C \leq B$. Thus we have :

Theorem 6.6 $A < B \Rightarrow \sum_{i,j} A[i,j] < \sum_{i,j} B[i,j]$.

Corollary 6.7 *B is an immediate successor of A iff $A < B$ and* $1 + \sum_{i,j} A[i,j] = \sum_{i,j} B[i,j]$.

Corollary 6.8 *The number of immediate successors of $A \in RG_n$ is the number of coessential points of A.*

Corollary 6.9 *The number of immediate predecessors of $A \in RG_n$ is the number of essential points of A.*

Corollary 6.10 *RG_n is a graded lattice of rank* $\frac{n(n+1)(2n+1)}{6}$.

proof : We have the conclusion of the corollary because $inf(RG_n) = 0$ and $sup(RG_n) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ n-2 & n-2 & n-2 & \ldots & 1 \\ n-1 & n-1 & n-2 & \ldots & 1 \\ n & n-1 & n-2 & \ldots & 1 \end{pmatrix}$. Q.E.D.

Theorem 6.11 *Suppose A has a coessential point rsa; suppose B is an immediate successor of A such that $B[r,s] = a+1$; then X_A and X_B have the same entries except $X_A[r,s] = X_B[r,s+1] = 1$ and $X_A[r,s+1] = X_B[r,s] = 0$: X_A has the pattern $\boxed{1 \quad 0}$ in position r,s and X_B has the pattern $\boxed{0 \quad 1}$ in position r,s.*

Proof : Since A has a coessential point rsa, $A[r-1,s-1] = a+1$ or a; $A[r-1,s+1] = a$ or $a-1$; $A[r+1,s-1] = a+1$ or $a+2$; $A[r+1,s+1] = a+1$ or a. There are 16 possibilities.

Let us look at one of these possibilities. Suppose A has the pattern $\boxed{\begin{array}{ccc} a & a & a \\ a+1 & a & a \\ a+1 & a+1 & a \end{array}}$ in position $r-1, s-1$; then B has the pattern $\boxed{\begin{array}{ccc} a & a & a \\ a+1 & a+1 & a \\ a+1 & a+1 & a \end{array}}$ in position $r-1, s-1$.

The matrices A' and B' have the same entries except that A' has the pattern $\boxed{\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}}$ in position r,s and B' the pattern $\boxed{\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}}$ in position r,s. We obtain then that the matrices X_A and X_B have the same entries except that X_A has the pattern $\boxed{1 \quad 0}$ in position r,s and X_B the pattern $\boxed{0 \quad 1}$ in position r,s.

The other 15 possibilities give the same result. Q.E.D.

Suppose A has a coessential point rsa; suppose B is an immediate successor of A such that $B[r,s] = a+1$; suppose $k(A)_{tr} = s$, i.e., suppose $card\{l \mid l \geq s \text{ and } X_A[r,l] = 1\} = t$; then the real meaning of theorem 6.11 is that $k(B)_{tr} = s+1$, i.e., $k(B)$ is an immediate successor of $k(A)$.

And this proves : $RG_n \to KG_n,\ A \mapsto k(A)$ is an isomorphism of lattices. Q.E.D.

Example 6.12

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & \underline{1} & 1 \\ 2 & 2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & \underline{2} & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},\ B' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ X_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$k(A) = \begin{array}{ccc} 3 & 3 & 3 \\ & \underline{1} & 2 \\ & & 0 \end{array},\quad k(B) = \begin{array}{ccc} 3 & 3 & 3 \\ & \underline{2} & 2 \\ & & 0 \end{array}$$

7 Appendix

Proof of the theorem 2.4 : Since f and g are embeddings, we have $\forall x \in P$, $\{y \in P \mid y \leq x\} = \{y \in P \mid g(y) \leq g(x)\} = \{y \in P \mid f(y) \leq f(x)\}$; thus $(h \circ g)(x) = \vee\{f(y) \mid y \in P$ and $g(y) \leq g(x)\} = \vee\{f(y) \mid y \in P$ and $f(y) \leq f(x)\} = f(x)$, and $h \circ g = f$.

We prove : $\forall s, t \in S,\ s \leq t \ \Rightarrow\ h(s) \leq h(t)$. We have : $s \leq t \Rightarrow \{x \in P \mid g(x) \leq s\} \subseteq \{x \in P \mid g(x) \leq t\} \Rightarrow h(s) = \vee\{f(x) \mid x \in P$ and $g(x) \leq s\} \leq \vee\{f(x) \mid x \in P$ and $g(x) \leq t\} = h(t)$.

We have : $t \not\leq s \ \Rightarrow\ (\exists x \in P$ such that $g(x) \leq t$ and $g(x) \not\leq s)$, because $(\forall y \in P,\ g(y) \leq t \Rightarrow\ g(y) \leq s)\ \Rightarrow\ t = \vee\{g(y) \mid y \in P \textit{ and } g(y) \leq t\} \leq \vee\{g(y) \mid y \in P \textit{ and } g(y) \leq s\} = s$.

Suppose $t \not\leq s$ and let x be such that $g(x) \leq t$ and $g(x) \not\leq s$. We prove : $h(s) < (h(s) \vee f(x))$. Suppose $h(s) = (h(s) \vee f(x))$, i.e., suppose $f(x) \leq h(s)$. Let $z \in P$ be such that $g(z) \geq s$. Then $f(z) \geq \vee\{f(y) \mid y \in P \textit{ and } g(y) \leq s\} = h(s) \geq f(x)$; thus $z \geq x$ which imply that $g(x) \leq \wedge\{g(y) \mid y \in P \textit{ and } g(y) \geq s\} = s$. Contradiction.

We prove now : $s < t \ \Rightarrow\ h(s) < h(t)$. Since $t \not\leq s$, $\exists x \in P$ such that $g(x) \leq t$ and $g(x) \not\leq s$, and such that $h(s) < (h(s) \vee f(x))$. We have : $s < t \ \Rightarrow\ h(s) \leq h(t)$; and we have : $g(x) \leq t \ \Rightarrow\ f(x) = h(g(x)) \leq h(t)$. Thus $h(s) < (h(s) \vee f(x)) \leq h(t)$.

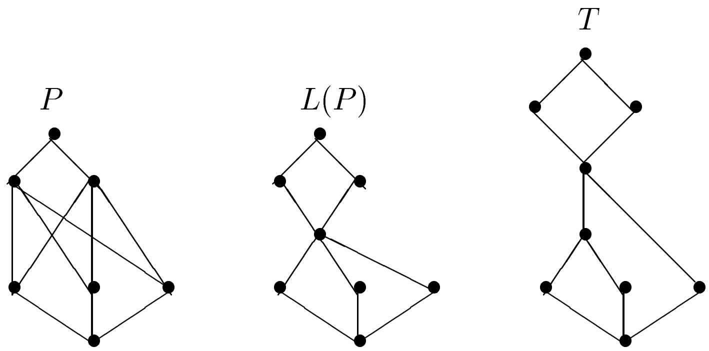
We prove now : $h(s) = h(t) \ \Rightarrow\ s = t$. Suppose $t \not\leq s$; then $\exists x \in P$ be such that $g(x) \leq t$ and $g(x) \not\leq s$, and such that $h(s) < (h(s) \vee f(x))$. We have : $g(x) \leq t \Rightarrow f(x) = h(g(x)) \leq h(t)$. Thus $h(s) < (h(s) \vee f(x)) = (h(t) \vee f(x)) \leq h(t)$. Contradiction. Thus $t \leq s$; similarly we have $s \leq t$. Thus $s = t$.

We prove finally : $h(s) < h(t) \ \Rightarrow\ s < t$. Suppose $s \not\leq t$; then $\exists x \in P$ such that $g(x) \leq s$ and $g(x) \not\leq t$. We have : $g(x) \not\leq t \Rightarrow \exists y \in P$ such that $t \leq g(y)$ and $g(x) \not< g(y)$, because $t = \wedge\{g(z) \mid z \in P$ and $g(z) \geq t\}$. Then $f(x) = h(g(x)) \leq h(s) < h(t) \leq h(g(y)) = f(y)$, which imply $x < y$ and $g(x) < g(y)$. Contradiction. And since $h(s) = h(t) \ \Rightarrow\ s = t$, we have $h(s) < h(t) \ \Rightarrow\ s < t$. Q.E.D.

Proof of the theorem 2.6 : The function $h : L(P) \to T,\ X \mapsto \vee_T\{f(x) \mid x \in P \textit{ and } \varphi(x) \leq X\}$, where $\varphi : P \to L(P), x \mapsto x^+$, is injective. Thus $card(L(P)) \leq card(T)$. Q.E.D.

Proof of the theorem 2.7 : The function $h : S \to L(P),\ s \mapsto \vee_{L(P)}\{\varphi(x) \mid x \in P \textit{ and } f(x) \leq s\}$, where $\varphi : P \to L(P), x \mapsto x^+$, is injective. Thus $card(S) \leq card(L(P))$. Thus $card(L(P)) = card(S)$ and h is an isomorphism. Q.E.D.

To prove that for a poset P, a lattice $T \supseteq P$ is isomorphic with $L(P)$, we must have $\forall t \in T, t = \vee\{x \in P \mid x \leq t\}$, and $\forall t \in T, t = \wedge\{x \in P \mid x \geq t\}$. In the example that follows, T contains P as a subposet, $\forall t \in T, t = \vee\{x \in P \mid x \leq t\}$, but $T \not\cong L(P)$.



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