Derangements and Euler's difference table for $C_\ell \wr S_n$

Hilarion L. M. Faliharimalala[1] and Jiang Zeng[2]

[1]Département de Mathématiques et Informatique
Université d'Antananarivo, 101 Antananarivo, Madagascar
hilarion@ist-tana.mg

[2]Université de Lyon, Université Lyon 1,
Institut Camille Jordan, UMR 5208 du CNRS,
F-69622, Villeurbanne Cedex, France
zeng@math.univ-lyon1.fr

Submitted: Dec 26, 2007; Accepted: Apr 22, 2008; Published: Apr 28, 2008
Mathematics Subject Classifications: 05A18; 05A15, 05A30

Abstract

Euler's difference table associated to the sequence $\{n!\}$ leads naturally to the counting formula for the derangements. In this paper we study Euler's difference table associated to the sequence $\{\ell^n n!\}$ and the generalized derangement problem. For the coefficients appearing in the later table we will give the combinatorial interpretations in terms of two kinds of k-successions of the group $C_\ell \wr S_n$. In particular for $\ell = 1$ we recover the known results for the symmetric groups while for $\ell = 2$ we obtain the corresponding results for the hyperoctahedral groups.

1 Introduction

The *problème de rencontres* in classical combinatorics consists in counting permutations without fixed points (see [6, p. 9–12]). On the other hand one finds in the works of Euler (see [11]) the following table of differences:

$$g_n^n = n! \quad \text{and} \quad g_n^m = g_n^{m+1} - g_{n-1}^m \quad (0 \le m \le n-1).$$

Clearly this table leads naturally to an explicit formula for g_n^0, which corresponds to the number of derangements of $[n] = \{1, \ldots, n\}$. As $n!$ is the cardinality of the symmetric group of $[n]$, Euler's difference table can be considered to be an array associated to the symmetric group.

In the last two decades much effort has been made to extend various enumerative results on symmetric groups to other Coxeter groups, the wreath product of a cyclic group with a symmetric group, and more generally to complex reflection groups. The reader is referred to [1, 2, 10, 9, 12, 14, 4, 5, 3] and the references cited there for the recent works in this direction.

In this paper we shall consider the *problème de rencontres* in the group $C_\ell \wr S_n$ via Euler's difference table. For a fixed integer $\ell \geq 1$, we define Euler's difference table for $C_\ell \wr S_n$to be the array $(g_{\ell,n}^m)_{n,m\geq 0}$ defined by

$$\begin{cases} g_{\ell,n}^n = \ell^n\, n! & (m=n); \\ g_{\ell,n}^m = g_{\ell,n}^{m+1} - g_{\ell,n-1}^m & (0 \leq m \leq n-1). \end{cases} \tag{1.1}$$

The first values of these numbers for $\ell = 1$ and $\ell = 2$ are given in Table 1.

$n\backslash m$	0	1	2	3	4	5
0	1					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

$(g_{1,n}^m)$

$n\backslash m$	0	1	2	3	4	5
0	1					
1	1	$2^1\, 1!$				
2	5	6	$2^2\, 2!$			
3	29	34	40	$2^3\, 3!$		
4	233	262	296	336	$2^4\, 4!$	
5	2329	2562	2824	3120	3456	$2^5\, 5!$

$(g_{2,n}^m)$

Table 1: Values of $g_{\ell,n}^m$ for $0 \leq m \leq n \leq 5$ and $\ell = 1$ or 2.

The $\ell = 1$ case of (1.1) corresponds to Euler's difference table, where $g_{1,n}^n$ is the cardinality of S_n and $g_{1,n}^0$ is the number of *derangements*, i.e., the fixed point free permutations in S_n. The combinatorial interpretation for the general coefficients $g_{1,n}^m$ was first studied by Dumont and Randrianarivony [11] and then by Clarke et al [8]. More recently Rakotondrajao [15, 16] has given further combinatorial interpretations of these coefficients in terms of *k-successions* in symmetric groups.

As $g_{2,n}^n = 2^n n!$ is the cardinality of the hyperoctahedral group B_n, Chow [9] has given a similar interpretation for $g_{2,n}^0$ in terms of derangements in the hyperoctahedral groups.

For positive integers ℓ and n the group of *colored permutations* of n digits with ℓ colors is the wreath product $G_{\ell,n} = C_\ell \wr S_n = C_\ell^n \rtimes S_n$, where C_ℓ is the ℓ-cyclic group generated by $\zeta = e^{2i\pi/\ell}$ and S_n is the symmetric group of the set $[n]$. By definition, the multiplication in $G_{\ell,n}$, consisting of pairs $(\epsilon, \sigma) \in C_\ell^n \times S_n$, is given by the following rule: for all $\pi = (\epsilon, \sigma)$ and $\pi' = (\epsilon', \sigma')$ in $G_{\ell,n}$,

$$(\epsilon, \sigma) \cdot (\epsilon', \sigma') = ((\epsilon_1 \epsilon'_{\sigma^{-1}(1)}, \epsilon_2 \epsilon'_{\sigma^{-1}(2)}, \ldots, \epsilon_n \epsilon'_{\sigma^{-1}(n)}),\ \sigma \circ \sigma').$$

One can identify $G_{\ell,n}$ with a permutation group of the colored set:

$$\Sigma_{\ell,n} := C_\ell \times [n] = \{\zeta^j i \mid i \in [n], 0 \leq j \leq \ell - 1\}$$

via the morphism $(\epsilon, \sigma) \longmapsto \pi$ such that for any $i \in [n]$ and $0 \leq j \leq \ell - 1$,

$$\pi(i) = \epsilon_{\sigma(i)}\sigma(i) \qquad \text{and} \qquad \pi(\zeta^j\, i) = \zeta^j \pi(i).$$

Clearly the cardinality of $G_{\ell,n}$ equals $\ell^n n!$.

We can write a signed permutation $\pi \in G_{\ell,n}$ in two-line notation. For example, if $\pi = (\epsilon, \sigma) \in G_{4,11}$, where $\epsilon = (\zeta^2, 1, 1, \zeta, \zeta^2, \zeta, \zeta, \zeta, 1, \zeta, \zeta^3)$ and

$$\sigma = 3 \quad 5 \quad 1 \quad 9 \quad 6 \quad 2 \quad 7 \quad 4 \quad 11 \quad 8 \quad 10,$$

we write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & \zeta^2 5 & \zeta^2 1 & 9 & \zeta 6 & 2 & \zeta 7 & \zeta 4 & \zeta^3 11 & \zeta 8 & \zeta 10 \end{pmatrix}.$$

For small j, it is convenient to write j bars over i instead of $\zeta^j i$. Thus, the above permutation can be written in one-line form as $\pi = 3 \;\; \bar{\bar{5}} \;\; \bar{\bar{1}} \;\; 9 \;\; \bar{6} \;\; 2 \;\; \bar{7} \;\; \bar{4} \;\; \bar{\bar{\bar{11}}} \;\; \bar{8} \;\; \overline{10}$, or in cyclic notation as

$$\pi = (\bar{\bar{1}},\ 3)\ (2,\ \bar{\bar{5}},\ \bar{6})\ (\bar{4},\ 9,\ \bar{\bar{\bar{11}}},\ \overline{10},\ \bar{8})\ (\bar{7}).$$

Note that when using cyclic notation to determine the image of a number, one ignores the sign on that number and then considers only the sign on the next number in the cycle. Thus, in this example, we ignore the sign ζ^2 on the 5 and note that then 5 maps to $\zeta 6$ since the sign on 6 is ζ. Furthermore, throughout this paper we shall use the following conventions:

i) If $\pi = (\epsilon, \sigma) \in G_{\ell,n}$, let $|\pi| = \sigma$ and $\mathrm{sgn}_\pi(i) = \epsilon_i$ for $i \in [n]$. For example, if $\pi = \bar{\bar{4}}\,\bar{3}\,1\,\bar{2}$ then $\epsilon = (1, \zeta, \zeta, \zeta^2)$ and $\mathrm{sgn}_\pi(4) = \zeta^2$.

ii) For $i \in [n]$ and $j \in \{0, 1, \ldots, \ell - 1\}$ define $\zeta^j i + k = \zeta^j(i + k)$ for $0 \leq k \leq n - i$, and $\zeta^j i - k = \zeta^j(i - k)$ for $0 \leq k \leq i$. For example, we have $\bar{\bar{4}} + 1 = \bar{\bar{5}}$ in $G_{4,11}$.

iii) We use the following total order on $\Sigma_{\ell,n}$: for $i, j \in [\ell]$ and $a, b \in [n]$,

$$\zeta^i a < \zeta^j b \Longleftrightarrow [i > j] \quad \text{or} \quad [i = j \text{ and } a < b].$$

It is not hard to see that the coefficient $g^m_{\ell,n}$ is divisible by $\ell^m m!$. This prompted us to introduce $d^m_{\ell,n} = g^m_{\ell,n}/\ell^m m!$. We derive then from (1.1) the following allied array $(d^m_{\ell,n})_{n,m\geq 0}$:

$$\begin{cases} d^n_{\ell,n} = 1 & (m = n); \\ d^m_{\ell,n} = \ell(m+1)\, d^{m+1}_{\ell,n} - d^m_{\ell,n-1} & (0 \leq m \leq n-1). \end{cases} \tag{1.2}$$

$n\backslash m$	0	1	2	3	4	5
0	1					
1	0	1				
2	1	1	1			
3	2	3	2	1		
4	9	11	7	3	1	
5	44	53	32	13	4	1

$(d_{1,n}^m)$

$n\backslash m$	0	1	2	3	4	5
0	1					
1	1	1				
2	5	3	1			
3	29	17	5	1		
4	233	131	37	7	1	
5	2329	1281	353	65	9	1

$(d_{2,n}^m)$

Table 2: Values of $d_{\ell,n}^m$ for $0 \le m \le n \le 5$ and $\ell = 1$ or 2.

The first terms of these coefficients for $\ell = 1, 2$ are given in Table 2.

One can find the $\ell = 1$ case of (1.2) and the table $(d_{1,n}^m)$ in Riordan's book [17, p. 188]. Recently Rakotondrajao [16] has given a combinatorial interpretation for the coefficients $d_{1,n}^m$ in the symmetric group S_n.

The aim of this paper is to study the coefficients $g_{\ell,n}^m$ and $d_{\ell,n}^m$ in the colored group $G_{\ell,n}$, i.e., the wreath product of a cyclic group and a symmetric group. This paper merges from the two papers [8] and [16]. In the same vein as in [8] we will give a q-version of (1.1) in a forthcoming paper.

2 Main results

We first generalize the notion of k-succession introduced by Rakotondrajao [16] in the symmetric group to $G_{\ell,n}$.

Definition 1 (k-circular succession). *Given a permutation $\pi \in G_{\ell,n}$ and a nonnegative integer k, the value $\pi(i)$ is a k-circular succession at position $i \in [n]$ if $\pi(i) = i + k$. In particular a 0-circular succession is also called fixed point.*

Remark 2. *Some words are in order about the requirement $\pi(i) = i+k$ in this definition. The "wraparound" is not allowed, i.e., $i+k$ is not to be interpreted $\bmod n$, also $i+k$ needs to be uncolored, i.e., $i+k \in [n]$, in order to count as a k-circular succession.*

Denote by $\mathcal{C}^k(\pi)$ the set of k-circular successions of π and let $c^k(\pi) = \#\,\mathcal{C}^k(\pi)$. In particular $FIX(\pi)$ denotes the set of fixed points of π. For example, for the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{1} & 5 & \bar{\bar{9}} & \bar{6} & 8 & \bar{7} & \bar{\bar{\bar{3}}} & \bar{\bar{4}} & \bar{2} \end{pmatrix} \in G_{4,9},$$

the values 5 and 8 are the two 3-circular successions at positions 2 and 5. Thus $C^3(\pi) = \{5, 8\}$.

The following is our main result on the combinatorial interpretation of the coefficients $g_{\ell,n}^m$ in terms of k-circular successions.

Theorem 3. *For any integer k such that $0 \le k \le m$, the entry $g_{\ell,n}^m$ equals the number of permutations in $G_{\ell,n}$ whose k-circular successions are included in $[m]$. In particular, by taking $k = 0$ and $k = m$, respectively, either of the following holds.*

(i) The entry $g_{\ell,n}^m$ is the number of permutations in $G_{\ell,n}$ whose fixed points are included in $[m]$.

(ii) The entry $g_{\ell,n}^m$ is the number of permutations in $G_{\ell,n}$ without m-circular succession.

For example, the permutations in $G_{2,2}$ whose fixed points are included in $[1]$ are:

$$21, \quad 1\bar{2}, \quad \bar{2}1, \quad 2\bar{1}, \quad \bar{1}\bar{2}, \quad \bar{2}\bar{1};$$

while those without 1-circular succession are:

$$12, \quad \bar{1}2, \quad 1\bar{2}, \quad \bar{1}\bar{2}, \quad \bar{2}1, \quad \bar{2}\bar{1}.$$

Note that Dumont and Randrianarivony [11] proved the $\ell = 1$ case of (i), while Rakotondrajao [16] proved the $\ell = 1$ case of (ii).

Let $c_{\ell,n,m}^k$ be the number of colored permutations in $G_{\ell,n}$ with m k-circular successions.

Theorem 4. *Let n, k and m be integers such that $n \ge 1$, $k \ge 0$ and $m \ge 0$. Then*

$$c_{\ell,n+1,m}^{k+1} = c_{\ell,n+1,m}^k + c_{\ell,n,m}^k - c_{\ell,n,m-1}^k, \tag{2.1}$$

where $c_{\ell,n,-1}^k = 0$.

Definition 5 (k-linear succession). *For $\pi \in G_{\ell,n}$, the value $|\pi(i)|$ $(2 \le i \le n)$ is a k-linear succession $(k \ge 1)$ of π at position i if $\pi(i) = \pi(i-1) + k$.*

Denote by $\mathcal{L}^k(\pi)$ the set of k-linear successions of π and let $l^k(\pi) = \#\mathcal{L}^k(\pi)$. Let $l_{\ell,n,m}^k$ be the number of colored permutations in $G_{\ell,n}$ with m k-linear successions. For example, 9 and 3 are the two 2-linear successions of the permutation $\pi = \bar{5}\,\bar{2}\,4\,7\,9\,\bar{1}\,\bar{3}\,\bar{\bar{8}}\,6 \in G_{4,9}$.

Definition 6 (Skew k-linear succession). *For $\pi = (\varepsilon, \sigma) \in G_{\ell,n}$, the value $\sigma(i)$ $(1 \le i \le n)$ is a skew k-linear succession $(k \ge 1)$ of π at position i if*

$$\pi(i) = \pi(i-1) + k,$$

where, by convention, $\sigma(0) = 0$ and $\varepsilon(0) = 1$.

Denote by $\mathcal{L}^{*k}(\pi)$ the set of skew k-linear successions of π and $l^{*k}(\pi) = \#\mathcal{L}^{*k}(\pi)$. The number of permutations in $G_{\ell,n}$ with m skew k-linear successions is $l^{*k}_{\ell,n,m}$. Obviously we have the following relation:

$$\mathcal{L}^{*k}(\pi) = \begin{cases} \mathcal{L}^k(\pi), & \text{if } \pi(1) \neq k; \\ \mathcal{L}^k(\pi) \cup \{k\}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Let δ be the bijection from $G_{\ell,n}$ onto itself defined by:

$$\pi = \pi_1\ \pi_2\ \cdots\ \pi_n \longmapsto \delta(\pi) = \pi_n\ \pi_1\ \pi_2\ \cdots\ \pi_{n-1}. \tag{2.3}$$

Theorem 7. *For any integer $k \geq 0$ there is a bijection Φ from $G_{\ell,n}$ onto itself such that for $\pi \in G_{\ell,n}$,*

$$\mathcal{C}^{k+1}(\pi) = \mathcal{L}^{k+1}(\Phi(\pi)), \tag{2.4}$$

and

$$\mathcal{C}^{k}(\delta(\pi)) = \mathcal{L}^{*(k+1)}(\Phi(\pi)). \tag{2.5}$$

Thanks to the transformation Φ the two statistics c^k and l^k are equidistributed on the group $G_{\ell,n}$ for $k \geq 1$. So we can replace the left-hand sides of (2.1) by $l^{k+1}_{\ell,n+1,m}$ and derive the following interesting result.

Corollary 8. *Let n, k and m be integers such that $n \geq 1$, $k \geq 0$ and $m \geq 0$. Then*

$$l^{k+1}_{\ell,n+1,m} = c^k_{\ell,n+1,m} + c^k_{\ell,n,m} - c^k_{\ell,n,m-1}, \tag{2.6}$$

where $c^k_{\ell,n,-1} = 0$.

Our proof of the last two theorems is a generalization of that given by Clarke et al [8], where the $(k,\ell) = (0,1)$ case of Corollary 8 is proved. Note that the $(k,\ell,m) = (0,2,0)$ case of (2.6) is the main result of a recent paper by Chen and Zhang [7].

In order to interpret the entry $d^m_{\ell,n}$ we need the following definition.

Definition 9. *For $0 \leq m \leq n$, a permutation π in $G_{\ell,n}$ is called m-increasing-fixed if it satisfies the following conditions:*

i) $\forall i \in [m]$, $sgn_\pi(|\pi|(i)) = 1$;

ii) $\mathrm{FIX}(\pi) \subseteq [\mathrm{m}]$;

iii) $\pi(1) < \pi(2) < \cdots < \pi(m)$.

Let $I^m_{\ell,n}$ be the set of m-increasing-fixed permutations in $G_{\ell,n}$. For example,

$$I^2_{2,3} = \{1\,2\,\bar{3},\quad 1\,3\,2,\quad 1\,3\,\bar{2},\quad 2\,3\,1,\quad 2\,3\,\bar{1}\}.$$

Theorem 10. *For $0 \leq m \leq n$, the entry $d_{\ell,n}^m$ equals the cardinality of $I_{\ell,n}^m$.*

Proof. Let $F_{\ell,n}^m$ be the set of permutations with fixed points included in $[m]$ in $G_{\ell,n}$. By Theorem 2 the cardinality of $F_{\ell,n}^m$ equals $g_{\ell,n}^m$. We define a mapping $f : (\tau, \pi) \mapsto \tau \odot \pi$ from $G_{\ell,m} \times F_{\ell,n}^m$ to $F_{\ell,n}^m$ as follows:

$$\tau \odot \pi = \pi(\tau^{-1}(1))\pi(\tau^{-1}(2)) \ldots \pi(\tau^{-1}(m))\pi(m+1) \ldots \pi(n).$$

Clearly f defines a group action of $G_{\ell,m}$ on the set $F_{\ell,n}^m$. We can choose an element π in each orbit such that

$$\forall i \in [m], \quad \mathrm{sgn}_\pi(|\pi|(i)) = 1 \quad \text{and} \quad \pi(1) < \pi(2) < \cdots < \pi(m).$$

As the cardinality of the group $G_{\ell,m}$ is $\ell^m m!$, we derive that the number of the orbits equals $g_{\ell,n}^m / \ell^m m!$. □

Rakotondrajao [16] gave a different interpretation for $d_{\ell,n}^m$ when $\ell = 1$. We can generalize her result as in the following theorem.

Definition 11. *For $0 \leq m \leq n$, a permutation π in $G_{\ell,n}$ is called m-isolated-fixed if it satisfies the following conditions:*

i) $\forall i \in [m]$, $sgn_\pi(i) = 1$;

ii) $\mathrm{FIX}(\pi) \subseteq [\mathrm{m}]$;

iii) each cycle of π has at most one point in common with $[m]$.

Let $D_{\ell,n}^m$ be the set of m-isolated-fixed permutations in $G_{\ell,n}$. For example,

$$D_{2,3}^2 = \{(1)(2)(\bar{3}),\ (1,3)(2),\ (1,\bar{3})(2),\ (1)(2,3),\ (1)(2,\bar{3})\}.$$

Note that $\pi = \bar{3}\ 1\ 2 \notin D_{2,3}^2$ because 1 and 2 are in the same cycle.

Theorem 12. *For $0 \leq m \leq n$, the entry $d_{\ell,n}^m$ equals the cardinality of $D_{\ell,n}^m$.*

As we will show in Section 7 there are more recurrence relations for $g_{\ell,n}^m$ and $d_{\ell,n}^m$. In particular, we shall prove an explicit formula for the ℓ-derangement numbers:

$$d_{\ell,n}^0 = g_{\ell,n}^0 = n! \sum_{i=0}^{n} \frac{(-1)^i \ell^{n-i}}{i!}, \tag{2.7}$$

which implies immediately the following recurrence relation:

$$d_{\ell,n}^0 = \ell n d_{\ell,n-1}^0 + (-1)^n \qquad (n \geq 1). \tag{2.8}$$

Note that (2.8) is the ℓ-version of a famous recurrence for derangements. Using the combinatorial interpretation for $g^m_{\ell,n}$ and $d^m_{\ell,n}$ it is possible to derive bijective proofs of these recurrence relations. However we will just give combinatorial proofs for (2.8) and two other recurrences by generalizing the combinatorial proofs of Rakotondrajao [16] for $\ell = 1$ case, and leave the others for the interested readers.

The rest of this paper is organized as follows: The proofs of Theorems 2, 3, 6 and 11 will be given in Sections 3, 4, 5 and 6, respectively. In Section 7 we give the generating function of the coefficients $g^m_{\ell,n}$'s and derive more recurrence relations for the coefficients $g^m_{\ell,n}$'s and $d^m_{\ell,n}$'s. Finally, in Section 8 we give combinatorial proofs of three remarkable recurrence relations of $d^m_{\ell,n}$'s.

3 Proof of Theorem 3

Let m and k be integers such that $n \geq m \geq k \geq 0$. Denote by $G^m_{\ell,n}(k)$ the set of permutations in $G_{\ell,n}$ whose k-circular successions are bounded by m and $s^m_{\ell,n} = \# G^m_{\ell,n}(k)$. We show that the sequence $(s^m_{\ell,n})$ satisfies (1.1).

By definition, we have immediately $G^n_{\ell,n}(k) = G_{\ell,n}$ and then $s^n_{\ell,n} = \ell^n\, n!$. Now, suppose $m < n$, then $G^{m+1}_{\ell,n}(k) \setminus G^m_{\ell,n}(k)$ is the set of permutations in $G^{m+1}_{\ell,n}(k)$ whose maximal k-circular succession is $m+1$. It remains to show that the cardinality of the latter set equals $s^m_{\ell,n-1}$. To this end, we define a simple bijection $\rho : \pi \mapsto \pi'$ from $G^{m+1}_{\ell,n}(k) \setminus G^m_{\ell,n}(k)$ to $G^m_{\ell,n-1}(k)$ as follows.

Starting from any $\pi = \pi_1 \pi_2 \dots \pi_n$ in $G^{m+1}_{\ell,n}(k) \setminus G^m_{\ell,n}(k)$, we construct π' by deleting $\pi_{m+1-k} = m+1$ and replacing each letter π_i by $\pi_i - 1$ if $|\pi_i| > m+1$. Conversely, starting from $\pi' = \pi'_1 \pi'_2 \dots \pi'_{n-1}$ in $G^m_{\ell,n-1}(k)$, one can recover π by inserting $m+1$ between π'_{m-k} and π'_{m-k+1} and then replacing each letter π'_i by $\pi'_i + 1$ if $|\pi'_i| > m$. For example, if $\pi = 3\,\bar{9}\,5\,\bar{\bar{8}}\,\bar{7}\,\bar{\bar{6}}\,2\,\bar{1}\,4 \in G^5_{3,9}(2)$, then $\pi' = 3\,\bar{8}\,\bar{\bar{7}}\,\bar{6}\,\bar{\bar{5}}\,2\,\bar{1}\,4 \in G^4_{3,8}(2)$. Note that π' has a k-circular succession $j \geq m+1$ if and only if $j+1 \geq m+2$ is a k-circular succession of π. Therefore, the maximal k-circular succession of π is $m+1$ if and only if the k-circular successions of π' are bounded by m. This completes the proof.

Remark 13. *The above argument does not explain why $g^m_{\ell,n}$ is independent from k $(0 \leq k \leq m)$. We can provide such an argument as follows. Consider the following simple bijection d which consists in transforming $\pi = \pi_1\pi_2\pi_3 \cdots \pi_n$ into $d(\pi) = \pi' = \pi_2\pi_3 \cdots \pi_n\pi_1$. Clearly the k-successions of π are bounded by m if and only if the $(k+1)$-successions of π' are bounded by m. Hence, denoting by d^j the composition of j-times of d, the application of $d^{k_2-k_1}$ permits to pass from k_1-successions to k_2-successions if $k_1 < k_2$. In particular if we apply m times the mapping d to a permutation whose fixed points are bounded by m then we obtain a permutation without m-succession and vice versa.*

4 Proof of Theorem 4

Let $S_n^k(x)$ be the counting polynomial of the statistic c^k on the group $G_{\ell,n}$, i.e.,

$$S_n^k(x) = \sum_{\pi \in G_{\ell,n}} x^{c^k(\pi)} = \sum_{m=0}^{n} c_{\ell,n,m}^k x^m. \tag{4.1}$$

Then (2.1) is equivalent to the following equation:

$$S_{n+1}^{k+1}(x) = S_{n+1}^k(x) + (1-x)S_n^k(x). \tag{4.2}$$

By (2.3) it is readily seen that

$$C^{k+1}(\pi) = \begin{cases} C^k(\delta(\pi)), & \text{if } \pi_n \neq k+1; \\ C^k(\delta(\pi)) \setminus \{k+1\}, & \text{otherwise.} \end{cases} \tag{4.3}$$

It follows that

$$\begin{aligned} S_{n+1}^{k+1}(x) &= \sum_{\substack{\pi \in G_{\ell,n+1} \\ \pi(1)=k+1}} x^{c^k(\pi)-1} + \sum_{\substack{\pi \in G_{\ell,n+1} \\ \pi(1)\neq k+1}} x^{c^k(\pi)} \\ &= \sum_{\substack{\pi \in G_{\ell,n+1} \\ \pi(1)=k+1}} x^{c^k(\pi)-1} + \sum_{\pi \in G_{\ell,n+1}} x^{c^k(\pi)} - \sum_{\substack{\pi \in G_{\ell,n+1} \\ \pi(1)=k+1}} x^{c^k(\pi)}. \end{aligned} \tag{4.4}$$

For any $\pi \in G_{\ell,n+1}$ such that $\pi(1) = k+1$ we can associate bijectively a permutation $\pi' \in G_{\ell,n}$ such that $c^k(\pi) = c^k(\pi') + 1$ as follows: $\forall i \in [n]$,

$$\pi'(i) = \begin{cases} \pi(i+1), & \text{if } \pi(i+1) \leq k; \\ \pi(i+1) - 1, & \text{if } \pi(i+1) > k. \end{cases}$$

Therefore we can rewrite (4.4) as (4.2).

We can also derive Theorem 3 from Theorem 2. First we prove a lemma.

Lemma 14. *For $0 \leq k \leq n-m$ there holds*

$$c_{\ell,n,m}^k = \binom{n-k}{m} g_{\ell,n-m}^k. \tag{4.5}$$

Proof. To construct a permutation π in $G_{\ell,n}$ with m k-circular successions we can first choose m positions $i_1, \ldots, i_m$ of k-circular successions among the first $n-k$ ones and then construct a permutation π_0 of order $n-m$ without k-circular successions on the remaining $n-m$ positions, where and in what follows we shall assume that $i_1 < i_2 < \cdots < i_m$. More precisely, there is a bijection $\theta : \pi \mapsto (I, \pi_0)$, where $I = \{i_1, \ldots, i_m\}$, from the set of the colored permutations of order n with m k-successions to the product of the set of all

m-subsets of $[n-k]$ and the set of colored permutations of order $n-m$ without k-circular successions.

Denote by $G_{\ell,n,k,i}$ the set of all permutations in $G_{\ell,n}$ whose maximal position of k-circular successions equals i. Define the mapping $R_i : \pi \mapsto \pi'$ from $G_{\ell,n,k,i}$ to $G_{\ell,n-1}$ such that the linear form of π' is obtained from $\pi = \pi_1 \dots \pi_n$ by removing the letter $(i+k)$ and replacing each colored letter π_j by $\pi_j - 1$ if $|\pi_j| > i+k$. It is readily seen that the map R_i is a bijection and $c^k(\pi') = c^k(\pi) - 1$. Indeed it is easy to see that $j+k$ is a k-circular succession of π different of $i+k$ if and only if $j+k$ is a k-circular succession of π'. Hence, $\pi_0 = R_{i_1} \circ R_{i_2} \circ \cdots \circ R_{i_m}(\pi)$ is a colored permutation without k-circular succession in $G_{\ell,n-m}$.

Conversely given a subset $I = \{i_1, i_2, \cdots, i_m\}$ of $[n-k]$ and a colored permutation π_0 without k-circular succession in $G_{\ell,n-m}$ we can construct

$$\pi = \theta^{-1}(I, \pi_0) = R_{i_m}^{-1} \circ R_{i_{m-1}}^{-1} \circ \cdots \circ R_{i_1}^{-1}(\pi_0),$$

where $R_i^{-1}(\pi')$ is obtained from $\pi' = \pi'_1 \dots \pi'_{n-1}$ by inserting the integer $(i+k)$ between π'_{i-1} and π'_i and replacing each colored letter π'_j by $\pi'_j + 1$ if $|\pi'_j| \geq i+k$. Therefore

$$c^k_{\ell,n,m} = \binom{n-k}{m} c^k_{\ell,n-m,0}. \tag{4.6}$$

By Theorem 2 (ii) we have $c^k_{\ell,n,0} = g^k_{\ell,n}$. Substituting this in (4.6) yields then (4.5).

□

By (4.5) we see that (2.1) is equivalent to

$$\binom{n-k}{m} g^{k+1}_{\ell,n+1-m} = \binom{n+1-k}{m} g^k_{\ell,n+1-m} + \binom{n-k}{m} g^k_{\ell,n-m} - \binom{n-k}{m-1} g^k_{\ell,n+1-m}.$$

Since $g^{k+1}_{\ell,n+1-m} - g^k_{\ell,n-m} = g^k_{\ell,n-m+1}$ by (1.1), we can rewrite the last equation as

$$\binom{n-k}{m} g^k_{\ell,n+1-m} = \binom{n+1-k}{m} g^k_{\ell,n+1-m} - \binom{n-k}{m-1} g^k_{\ell,n+1-m},$$

which is obvious in view of the identity $\binom{n-k}{m} = \binom{n+1-k}{m} - \binom{n-k}{m-1}$. This completes the proof of Theorem 3.

5 Proof of Theorem 7

There is a well-known bijection on the symmetric groups transforming the cyclic structure into linear structure (see [13] and [18, p. 17]). We need a variant of this transformation, say $\varphi : S_n \to S_n$, as follows.

Given a permutation $\sigma \in S_n$ written as a product of cycles, arrange the cycles in the decreasing order of their maximum elements from left to right with the maximum element at the end of each cycle. We then obtain $\varphi(\sigma)$ by erasing the parentheses. Conversely, starting from a permutation written in one-line form $\sigma' = a_1 a_2 \ldots a_n$, find out the *right-to-left maxima* of σ' from right to left and decompose the word σ' into blocks by putting a bar at the right of each right-to-left maximum and construct a cycle of σ with each block.

For example, if $\sigma = (3, 1, 4, 6, 9)(5, 7, 8)(2) \in S_9$ then $\varphi(\sigma) = 3\ 1\ 4 6\ 9\ 5\ 7\ 8\ 2$. Conversely, starting from $\sigma' = 3\,1\,4\,6\,\mathbf{9}\,5\,7\,\mathbf{8}\,\mathbf{2}$, so the right-to-left maxima are 9,8 and 2, then the decomposition into blocks is $3\ 1\ 4\ 6\ 9|5\ 7\ 8|2|$ and we recover σ by putting parentheses around each block.

Lemma 15. *For $k \geq 1$, the mapping φ transforms the k-circular successions to k-linear successions and vice versa.*

Proof. Indeed, an integer p is a k-circular succession of σ if and only if there is an integer $i \in [n]$ such that $\sigma(i) = i + k$, so i and $i + k$ are two consecutive letters in the one-line form of $\varphi(\sigma)$. Conversely if i and $i + k$ are two consecutive letters in the one-line form of a permutation τ then i cannot be a right-to-left maximum, so i and $i + k$ are in the same cycle of $\varphi^{-1}(\tau)$, say σ, and then $\sigma(i) = i + k$.

□

We now construct a bijection $\Phi : \pi \mapsto \pi'$ from $G_{\ell,n}$ onto itself such that

$$C^k(\pi) = L^k(\pi') \quad (k \geq 1). \tag{5.1}$$

Let $\sigma = |\pi|$ and $\sigma' = |\pi'|$.

Bijection Φ: First define $\sigma' = \varphi(\sigma)$: Factorize σ as product of r disjoint cycles $C_1, \ldots, C_r$. Suppose that ℓ_i and g_i are, respectively, the length and greatest element of the cycle C_i $(1 \leq i \leq r)$ such that $g_1 > g_2 > \cdots > g_r$. Then

$$\sigma' = \sigma(g_1)\ \cdots\ \sigma^{\ell_1 - 1}(g_1)\ g_1\ \sigma(g_2)\ \cdots\ \sigma^{\ell_2 - 1}(g_2)\ g_2\ \cdots\ \sigma(g_r)\ \cdots\ \sigma^{\ell_r - 1}(g_r)\ g_r. \tag{5.2}$$

Let $T_\sigma = \{\sigma(g_i),\ i \in [r]\}$. It remains to define $\mathrm{sgn}_{\pi'}(\sigma^j(g_i))$ for all $i \in [r]$ and $1 \leq j \leq \ell_i$. We proceed by induction on j as follows: For each $i \in [r]$ let

$$\mathrm{sgn}_{\pi'}(\sigma(g_i)) = \mathrm{sgn}_{\pi}(\sigma(g_i)),$$

and for $j = 2, \ldots, \ell_j$ define

$$\mathrm{sgn}_{\pi'}(\sigma^j(g_i)) = \mathrm{sgn}_{\pi'}(\sigma^{j-1}(g_i)) \cdot \mathrm{sgn}_{\pi}(\sigma^j(g_i)). \tag{5.3}$$

It is easy to establish the inverse of Φ.

Bijection Φ^{-1}: Starting from π' we can recover $\sigma = |\pi|$ by applying φ^{-1} to σ'. Suppose σ' is given as in (5.2) and $g_1, \dots, g_r$ are the left-to-right-maxima. We then determine $\mathrm{sgn}_\pi(\sigma^j(g_i))$ for all $i \in [r]$ and $1 \le j \le \ell_i$ as follows: For each $i \in [r]$ let

$$\mathrm{sgn}_\pi(\sigma(g_i)) = \mathrm{sgn}_{\pi'}(\sigma(g_i)),$$

and for $j = 2, \dots, \ell_j$ define

$$\mathrm{sgn}_\pi(\sigma^j(g_i)) = \mathrm{sgn}_{\pi'}(\sigma^j(g_i))(\mathrm{sgn}_{\pi'}(\sigma^{j-1}(g_i)))^{-1}, \tag{5.4}$$

where $\mathrm{sgn}_\pi^{-1}(i)$ is the inverse of $\mathrm{sgn}_\pi(i)$ in the cyclic group C_ℓ.

Lemma 16. *For $k \ge 1$, the mapping Φ transforms a k-circular succession of π to a k-linear succession of π' and vice versa.*

Proof. By Lemma 15 we have the equivalence: $\sigma(i)$ is a k-circular succession of σ if and only if $\sigma(i)$ is a k-linear succession of σ'. It remains to verify that if $\sigma(i)$ is a k-circular succession of σ then

$$\mathrm{sgn}_\pi(\sigma(i)) = 1 \Leftrightarrow \mathrm{sgn}_{\pi'}(\sigma(i)) = \mathrm{sgn}_{\pi'}(i) \tag{5.5}$$

Note that if $\sigma(i)$ is a k-circular succession of σ then i and $\sigma(i)$ must be in the same cycle and that $\sigma(i)$ cannot be in T_σ for, otherwise, i would be the greatest element of the cycle but this is impossible because $\sigma(i) = i + k$ ($k \ge 1$).

Now, assume that $\sigma(i)$ is a k-circular succession of σ.

(i) Suppose $\mathrm{sgn}_\pi(\sigma(i)) = 1$. As $\sigma(i) \notin T_\sigma$ and $\sigma^{-1}(\sigma(i)) = i$, we have

$$\mathrm{sgn}_{\pi'}(\sigma(i)) = \mathrm{sgn}_{\pi'}(i) \cdot \mathrm{sgn}_\pi(\sigma(i)) = \mathrm{sgn}_{\pi'}(i).$$

(ii) Suppose that $\mathrm{sgn}_{\pi'}(\sigma(i)) = \mathrm{sgn}_{\pi'}(i)$. As $\sigma(i) \notin T_\sigma$ and $\sigma^{-1}(\sigma(i)) = i$, we have

$$\mathrm{sgn}_\pi(\sigma(i)) = \mathrm{sgn}_{\pi'}(\sigma(i))/\mathrm{sgn}_{\pi'}(i) = 1.$$

Hence (5.5) is established. $\square$

Obviously the above lemma is equivalent to (2.4). We obtain (2.5) by combining (2.4), (2.2) and (4.3).

We conclude this section with an example. Consider

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{3} & 4 & \bar{9} & \bar{8} & 7 & \bar{5} & 6 & \bar{\bar{2}} & \bar{\bar{1}} \end{pmatrix} \in G_{4,9}.$$

Factorizing $\sigma = |\pi|$ into cycles we get $\sigma = (1,3,9)(2,4,8)(6,5,7)$, then

$$\sigma' = 1\ 3\ 9\ 2\ 4\ 8\ 6\ 5\ 7 \quad \text{and} \quad T_\sigma = \{1, 2, 6\}.$$

The signs of $\sigma'(i)$ for $i \in [9]$ are computed as follows:

$$\begin{aligned}
&\mathrm{sgn}_{\pi'}(1) = \mathrm{sgn}_{\pi}(1) = \zeta^2 \quad \text{for} \quad 1 \in T_\sigma;\\
&\mathrm{sgn}_{\pi'}(3) = \mathrm{sgn}_{\pi'}(1) \cdot \mathrm{sgn}_{\pi}(3) = \zeta^2.\zeta = \zeta^3 \quad \text{for} \quad 3 \notin T_\sigma;\\
&\mathrm{sgn}_{\pi'}(9) = \mathrm{sgn}_{\pi'}(3) \cdot \mathrm{sgn}_{\pi}(9) = \zeta^3.\zeta = 1 \quad \text{for} \quad 9 \notin T_\sigma;\\
&\mathrm{sgn}_{\pi'}(2) = \mathrm{sgn}_{\pi'}(2) = \zeta^2 \quad \text{for} \quad 2 \in T_\sigma;\\
&\mathrm{sgn}_{\pi'}(4) = \mathrm{sgn}_{\pi'}(2) \cdot \mathrm{sgn}_{\pi}(4) = \zeta^2 \cdot 1 = \zeta^2 \quad \text{for} \quad 4 \notin T_\sigma;\\
&\mathrm{sgn}_{\pi'}(8) = \mathrm{sgn}_{\pi'}(4) \cdot \mathrm{sgn}_{\pi}(8) = \zeta^2 \cdot \zeta = \zeta^3 \quad \text{for} \quad 8 \notin T_\sigma;\\
&\mathrm{sgn}_{\pi'}(6) = \mathrm{sgn}_{\pi}(6) = 1 \quad \text{for} \quad 6 \in T_\sigma;\\
&\mathrm{sgn}_{\pi'}(5) = \mathrm{sgn}_{\pi'}(6) \cdot \mathrm{sgn}_{\pi}(5) = 1 \cdot \zeta = \zeta \quad \text{for} \quad 8 \notin T_\sigma;\\
&\mathrm{sgn}_{\pi'}(7) = \mathrm{sgn}_{\pi'}(5) \cdot \mathrm{sgn}_{\pi}(7) = \zeta \cdot 1 = \zeta \quad \text{for} \quad 7 \notin T_\sigma.
\end{aligned}$$

Thus we have $\pi \mapsto \pi' = \bar{\bar{1}}\,\bar{\bar{\bar{3}}}\,9\,\bar{\bar{2}}\,\bar{\bar{4}}\,\bar{\bar{\bar{8}}}\,6\,\bar{5}\,\bar{7}$. We have $C^2(\pi) = \{4,7\} = L^{*2}(\pi')$.

Conversely, starting from π', we can recover σ by φ^{-1} and the signs of $\sigma(i)$ $(i \in [n])$ by (5.4). As $\sigma = (139)(248)(657)$ and $T_\sigma = \{1,2,6\}$, we have, for example,

$$\mathrm{sgn}_{\pi}(9) = \mathrm{sgn}_{\pi'}(9) \cdot \mathrm{sgn}_{\pi'}^{-1}(3) = 1 \cdot \zeta = \zeta$$

for $9 \notin T_\sigma$.

6 Proof of Theorem 11

We shall give two proofs by using Theorems 9 and 2, respectively.

6.1 First Proof

We shall define a mapping $\varphi : \pi \mapsto \pi'$ from $D^m_{\ell,n}$ to $I^m_{\ell,n}$ in two steps. First we establish the correspondence $|\pi| \mapsto |\pi'|$ and then determine the sign transformation. Define the permutation $|\pi'| = |\pi'|(1)\dots|\pi'|(n)$ such that $|\pi'|(1)\dots|\pi'|(m)$ is the increasing rearrangement of $|\pi|(1),\dots,|\pi|(i_m)$ and $|\pi'|(m+1)\dots|\pi'|(n) = |\pi|(m+1)\dots|\pi|(n)$. Conversely, starting from $\pi' \in I^m_{\ell,n}$, for each $i \in [m]$ we construct the cycle of $|\pi|$ containing i by

$$(|\pi'|^{-s}(i),\dots,|\pi'|^{-2}(i),|\pi'|^{-1}(i),i),$$

where s is the smallest non negative integer such that $|\pi'|^{-s}(i) \in T$, where

$$T := \{|\pi'|(i), i \in [m]\},$$

and by convention $|\pi'|^0(i) = i$. In particular if $i \in [m] \cap T$, then $s = 0$ and i is a fixed point of $|\pi|$. The other cycles remain unaltered.

For example, for $\pi = (1)(2, \bar{7}, \bar{\bar{6}})(3, \bar{\bar{5}}, 9)(4)(\bar{\bar{8}}) \in D^4_{3,9}$ (i.e., $n = 9$, $\ell = 3$, $m = 4$), we have $|\pi| = (1)(2,7,6)(3,5,9)(4)(8)$ and $|\pi'| = 145792683 = (1)(2476)(359)(8)$, so $T = \{1, 4, 5, 7\}$.

Now, we describe the sign transformation. For each $i \in [m]$, since the letter i of π (m-isolated-fixed) as well as the letter $\pi(i)$ of π' (m-increasing-fixed) are uncolored, the transformation of the signs is obtained by exchanging the sign of $|\pi|(i)$ and that of i, namely

$$\mathrm{sgn}_{\pi'}(i) = \mathrm{sgn}_{\pi}(|\pi|(i)) \quad \text{and} \quad \mathrm{sgn}_{\pi}(i) = \mathrm{sgn}_{\pi'}(|\pi|(i)) = 1 \quad \forall i \in [m];$$

the signs of other letters reamain unaltered, i.e.,

$$\mathrm{sgn}_{\pi'}(i) = \mathrm{sgn}_{\pi}(i) \qquad \forall i \in [n] \setminus (T \cup [m]).$$

Continuing the above example, we have $\mathrm{sgn}_{\pi'}(2) = \mathrm{sgn}_{\pi}(\pi(2)) = \mathrm{sgn}_{\pi}(7) = \zeta^1$; $\mathrm{sgn}_{\pi'}(3) = \mathrm{sgn}_{\pi}(\pi(3)) = \mathrm{sgn}_{\pi}(5) = \zeta^2$, hence $\pi' = 1\,4\,5\,7\,9\,\bar{2}\,\bar{\bar{6}}\,\bar{\bar{8}}\,\bar{\bar{3}}$.

6.2 Second proof

Let $G^m_{\ell,n} := G^m_{\ell,n}(0)$ be the set of permutations in $G_{\ell,n}$ whose fixed points are included in $[m]$.

Let $\pi = (\varepsilon, \sigma)$ be a permutation in $G^m_{\ell,n}$, written as a product of disjoint cycles. For each $i \in [m]$, let s be the smallest integer ≥ 1 such that $\sigma^s(i) \in [m]$ and $w_\pi(i) = [\varepsilon_{\sigma(i)}\sigma(i)] \ldots [\varepsilon_{\sigma^{s-1}(i)}\sigma^{s-1}(i)]$. Clearly $w_\pi(i) = \emptyset$ if $s = 1$. Let Ω_π be the product of cycles of π which have no common point with $\{i\zeta^j | i \in [m], 0 \leq j \leq \ell - 1\}$ and π_m be the permutation in $G_{\ell,m}$ obtained from π by deleting the cycles in Ω_π and letters in $w_\pi(i)$ for $i \in [m]$. For example, if $\pi = (\bar{\bar{1}}\,\bar{4}\,7\,\bar{\bar{3}}\,2\,\bar{6}\,5)(\bar{8})(\bar{\bar{9}}) \in G^3_{3,9}$ then $\pi_3 = (\bar{\bar{1}}\,\bar{\bar{3}}\,2)$,

$$w_\pi(1) = \bar{4}\,7, \quad w_\pi(2) = \bar{6}\,5, \quad w_\pi(3) = \varnothing \quad \text{and} \quad \Omega_\pi = (\bar{8})\,(\bar{\bar{9}}).$$

Let $T(\pi) = (w_\pi(1), w_\pi(2), \cdots, w_\pi(m), \Omega_\pi)$ and define the relation $\sim$ on $G^m_{\ell,n}$ by

$$\pi_1 \sim \pi_2 \Leftrightarrow T(\pi_1) = T(\pi_2).$$

Clearly this is an equivalence relation. To determine the equivalence class $\mathcal{C}_\pi$ of each permutation $\pi \in G^m_{\ell,n}$ we consider the mapping $\theta : (\tau, \pi) \mapsto \theta(\tau, \pi)$ from $G_{\ell,m} \times G^m_{\ell,n}$ to $G^m_{\ell,n}$, where the cyclic factorization of $\theta(\tau, \pi)$ is obtained by inserting the word $w_\pi(i)$ after each letter $i\zeta^j$ appearing in a cycle of τ for each $i \in [m]$ and some $j : 0 \leq j \leq \ell - 1$, and then add the cycles in Ω_π. For example, if $\pi = (\bar{\bar{1}}\,\bar{4}\,7\bar{\bar{3}}\,2\bar{6}\,5)(\bar{8})(\bar{\bar{9}}) \in G^3_{3,9}$ and $\tau = (\bar{1}\,\bar{\bar{2}})(3) \in G_{3,3}$ then

$$\theta(\tau, \pi) = (\bar{1}\,\bar{4}\,7\,\bar{\bar{2}}\,\bar{6}\,5)(3)(\bar{8})(\bar{\bar{9}}).$$

Clearly $\mathcal{C}_\pi = \{\theta(\tau,\pi) | \tau \in G_{\ell,m}\}$. Indeed, by definition $\theta(\tau,\pi) \sim \pi$ for each $\tau \in G_{\ell,m}$ and $\pi \in G_{\ell,n}^m$ and conversely, if $\pi' \sim \pi$ then $\pi' = \theta(\pi'_m, \pi)$ for $T(\pi') = T(\pi)$. Moreover, suppose $\theta(\tau,\pi) = \theta(\tau',\pi) = \pi'$ for $\tau, \tau' \in G_{\ell,m}$, then $\tau = \tau' = \pi'_m$. Hence the cardinality of each equivalence class is $\ell^m m!$ and, by Theorem 2 the number of equivalence classes equals $d_{\ell,n}^m = g_{\ell,n}^m / \ell^m m!$. Choosing $\theta(\iota,\pi)$ as the representative of the class $\mathcal{C}_\pi$, where ι is the identity of $G_{\ell,m}$, yields the desired result.

7 Generating functions and further recurrence relations

For any function $f : \mathbb{Z} \to \mathbb{C}$ introduce the difference operator $\Delta f(n) = f(n) - f(n-1)$. Then it is easy to see by induction on $N \geq 0$ that

$$\Delta^N f(n) = \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(n-i) = \sum_{i=0}^{N} (-1)^{N-i} \binom{N}{i} f(n-N+i). \tag{7.1}$$

Proposition 17. *For $m \geq 0$ the following identities hold true:*

$$g_{\ell,n+m}^m = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \ell^{m+i} (m+i)!, \tag{7.2}$$

$$\sum_{n\geq 0} g_{\ell,n+m}^m \frac{u^n}{n!} = \frac{\ell^m m! \exp(-u)}{(1-\ell u)^{m+1}}, \tag{7.3}$$

$$\sum_{m,n\geq 0} g_{\ell,n+m}^m \frac{x^m}{m!}\frac{u^n}{n!} = \frac{\exp(-u)}{1-\ell x-\ell u}. \tag{7.4}$$

Proof. Setting $f(n) = g_{\ell,n}^n$ then $g_{\ell,n+m}^{n+m-i} = \Delta^i f(n+m)$ for $i \geq 0$. It follows from (7.1) that

$$g_{\ell,n+m}^m = \Delta^n f(n+m) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \ell^{m+i} (m+i)!. \tag{7.5}$$

Multiplying the above identity by $u^n/n!$ and summing over $n \geq 0$ we obtain

$$\sum_{n\geq 0} g_{\ell,n+m}^m \frac{u^n}{n!} = \ell^m m! \sum_{n,i\geq 0} (-1)^{n-i} \binom{m+i}{i} \frac{\ell^i u^n}{(n-i)!}.$$

Shifting n to $n+i$ yields

$$\sum_{n\geq 0} g_{\ell,n+m}^m \frac{u^n}{n!} = \ell^m m! \left(\sum_{n\geq 0} (-1)^n \frac{u^n}{n!}\right) \cdot \left(\sum_{i\geq 0} \binom{m+i}{i} (\ell u)^i\right),$$

which is clearly equal to the right-hand side of (7.3). Finally multiplying (7.3) by $x^m/m!$ and summing over $m \geq 0$ yields (7.4). $\square$

Setting $m = 0$ in (7.2) yields immediately formula (2.7).

Proposition 18. *For $\ell \geq 0$ and $0 \leq m \leq n$ there hold*

$$g_{\ell,n}^{m} = (\ell n - 1)g_{\ell,n-1}^{m} + \ell(n - m - 1)g_{\ell,n-2}^{m} \qquad (n \geq 2); \tag{7.6}$$
$$g_{\ell,n}^{m} = \ell(n - m)g_{\ell,n-1}^{m} + \ell m g_{\ell,n-1}^{m-1} \qquad (m \geq 1,\ n \geq 1); \tag{7.7}$$
$$g_{\ell,n}^{m} = \ell n g_{\ell,n-1}^{m} - \ell m g_{\ell,n-2}^{m-1} \qquad (m \geq 1,\ n \geq 2); \tag{7.8}$$

where $g_{\ell,0}^{0} = 1$, $g_{\ell,1}^{0} = \ell - 1$ and $g_{\ell,1}^{1} = \ell$.

Proof. Let $F(u)$ be the left-hand side of (7.3). Differentiating $F(u)$ and using the right-hand side of (7.3) we get

$$(1 - \ell u)F'(u) = [\ell(m + 1) - 1 + \ell u]F(u). \tag{7.9}$$

Equating the coefficients of $u^n/n!$ in (7.9) yields

$$g_{\ell,n+m+1}^{m} = [\ell(m + n + 1) - 1]g_{\ell,n+m}^{m} + \ell n g_{\ell,n+m-1}^{m},$$

which gives (7.6) by shifting $n + m + 1$ to n.

Next, multiplying the two sides of (7.3) by $1 - \ell u$ gives

$$(1 - \ell u)\sum_{n\geq 0} g_{\ell,n+m}^{m}\frac{u^n}{n!} = \frac{\ell^m m! \exp(-u)}{(1 - \ell u)^m} = \ell m \sum_{n\geq 0} g_{\ell,n+m-1}^{m-1}\frac{u^n}{n!}. \tag{7.10}$$

Equating the coefficients of $u^n/n!$ yields

$$g_{\ell,n+m}^{m} - \ell n g_{\ell,n+m-1}^{m} = \ell m g_{\ell,n+m-1}^{m-1}, \tag{7.11}$$

which is (7.7) by shifting $n + m$ to n.

Finally, we derive (7.8) from (7.7) and (1.1):

$$\begin{aligned} g_{\ell,n}^{m} &= \ell n g_{\ell,n-1}^{m} - \ell m(g_{\ell,n-1}^{m} - g_{\ell,n-1}^{m-1}) \\ &= \ell n g_{\ell,n-1}^{m} - \ell m g_{\ell,n-2}^{m-1}. \end{aligned}$$

The proof is thus completed. □

It is easy to convert the above relations for $g_{\ell,n}^{m}$ to those for $d_{\ell,n}^{m}$.

Proposition 19. *For $\ell \geq 0$ and $0 \leq m \leq n$ we have*

$$d_{\ell,n}^{m} = (\ell n - 1)d_{\ell,n-1}^{m} + \ell(n - m - 1)d_{\ell,n-2}^{m} \qquad (n \geq 2); \tag{7.12}$$
$$d_{\ell,n}^{m} = \ell(n - m)d_{\ell,n-1}^{m} + d_{\ell,n-1}^{m-1} \qquad (m \geq 1,\ n \geq 1); \tag{7.13}$$
$$d_{\ell,n}^{m} + d_{\ell,n-2}^{m-1} = \ell n d_{\ell,n-1}^{m} \qquad (m \geq 1,\ n \geq 2), \tag{7.14}$$

where $d_{\ell,0}^{0} = 1$, $d_{\ell,1}^{0} = \ell - 1$ and $d_{\ell,1}^{1} = 1$.

Proof. The equations (7.12), (7.13) and (7.14) follow directly from Proposition 18. □

8 Combinatorial proofs of three recurrence relations

Using the combinatorial interpretation for $d_{\ell,n}^n$ in Theorem 11 we now give combinatorial interpretations of (1.2), (2.8) and (7.14) by generalizing the proofs of Rakotondrajao [16], which correspond to the $\ell = 1$ case.

8.1 Combinatorial proof of (1.2)

We shall prove that the cardinality of $D_{\ell,n}^m$ satisfies the following recurrence:

$$d_{\ell,n}^n = 1 \quad \text{and} \quad d_{\ell,n}^{m-1} + d_{\ell,n-1}^{m-1} = \ell m\, d_{\ell,n}^m \qquad (1 \le m \le n). \tag{8.1}$$

First, the identity permutation is the only n-isolated-fixed permutation in $G_{\ell,n}$, so $d_{\ell,n}^n = 1$. To prove (8.1) we construct a bijection $\vartheta : \pi \longmapsto (\epsilon, \alpha, \pi')$ from $D_{\ell,n-1}^{m-1} \cup D_{\ell,n}^{m-1}$ to $\mathcal{C}_\ell \times [m] \times D_{\ell,n}^m$ as follows:

Let $\sigma = |\pi|$ and factorize π into disjoint cycles.

1. If $\pi \in D_{\ell,n-1}^{m-1}$, then $\epsilon = 1$, $\alpha = m$, the cycles of π' are obtained from those of π by substituting $\zeta^j i$ by $\zeta^j i + 1$ if $i \ge m$ and then adding the cycle (m).

2. If $\pi \in D_{\ell,n}^{m-1}$, let C_m be the cycle of σ containing m. Then $\epsilon = \mathrm{sgn}_\pi(m)$ and α is the smallest integer in C_m; let q be the smallest integer such that $\sigma^q(m) = \alpha$; then σ' is obtained from σ by deleting the letters $m, \sigma(m), \ldots, \sigma^{q-1}(m)$ from C_m and creating a new cycle $(m\, \sigma(m) \cdots \sigma^{q-1}(m))$. Finally, define the sign of $i \in [n]$ in π' by
$$\mathrm{sgn}_{\pi'}(i) = \begin{cases} \mathrm{sgn}_\pi(i), & \text{if } i \ne m; \\ 1, & \text{if } i = m. \end{cases}$$

For example, let $\ell = 3$, $n = 9$ and $m = 6$. If $\pi = (1\bar{\bar{6}})(2)(3)(4)(5)(\bar{7}\bar{\bar{8}}) \in D_{3,8}^5$, then

$$\epsilon = 1; \alpha = 6 \quad \textit{and} \quad \pi' = (1\bar{\bar{7}})(2)(3)(4)(5)(6)(\bar{8}\bar{\bar{9}});$$

if $\pi = (1)(2\bar{9}\bar{6}\bar{\bar{8}})(3)(4)(5)(\bar{7}) \in D_{3,9}^5$ then

$$\alpha = 2, \quad \epsilon = \zeta \quad \textit{and} \quad \pi' = (1)(2\bar{9})(3)(4)(5)(\bar{7})(6\bar{\bar{8}}).$$

It remains to show that ϑ is a bijection. Given (ϵ, α, π') let $\sigma' = |\pi'|$. We define the inverse $\vartheta^{-1} : (\epsilon, m, \pi') \mapsto \pi$ as follows:

1. If $\alpha = m$; $\epsilon = 1$ and $\pi'(m) = m$ then the cycles of π are obtained by deleting the cycle (m) and replacing $\zeta^j i$ by $\zeta^j i - 1$ if $i \ge m$ in π'.

2. If $\alpha = m$; $\epsilon = 1$ and $\pi'(m) \ne m$ then $\pi = \pi'$.

3. If $\alpha = m$; $\epsilon \neq 1$ then we get π from π' by replacing m by $\epsilon\, m$.

4. If $\alpha < m$; σ is obtained from σ' by removing the cycle which contains m and then inserting the word $m\sigma'(m)\sigma'^2(m)...\sigma'^{q-1}(m)$, where $\sigma'^q(m) = m$, in the cycle which contains the integer α just before the integer α. In other words, we define $m = \sigma(\sigma'^{-1}(\alpha))$, $\sigma^j(m) = \sigma'^j(m)$ for $1 \leq j \leq q-1$ and $\sigma^q(m) = \alpha$. Finally define $\mathrm{sgn}_\pi(i) = \mathrm{sgn}_{\pi'}(i)$ for $i \neq m$ and $\mathrm{sgn}_\pi(m) = \epsilon$.

To see that this is indeed the inverse of ϑ we just note the following simple facts:

- If $\pi \in D_{\ell,n-1}^{m-1}$ then $\epsilon = 1$, $\alpha = m$ and $m \in FIX(\pi')$. We have $\vartheta(\pi) \in E_1 = \{(1, m, \pi'), m \in FIX(\pi'), \pi' \in D_{\ell,n}^m\}$.

- If $\pi \in D_{\ell,n}^{m-1} \bigcap D_{\ell,n}^m$ then $\epsilon = 1$, $\alpha = m$ and $m \notin FIX(\pi')$. In this case $\pi' = \pi$ we have $\vartheta(\pi) \in E_2 = \{(1, m, \pi'), m \notin FIX(\pi'), \pi' \in D_{\ell,n}^m\}$.

- If $m \notin$ cycle of σ containing $i < m$ and $\mathrm{sgn}_\pi(m) \neq 1$ then $\alpha = m$, $\epsilon \neq 1$; and π' is obtained from π by just replacing ϵm by m. We have $\vartheta(\pi) \in E_3 = \{(\epsilon, m, \pi'), \epsilon \neq 1, \pi \in D_{\ell,n}^m\}$.

- If $m \in$ cycle of σ containing $i < m$ then $\alpha < m$. In this case the image π' is defined by the second case of the construction of ϑ. We have $\vartheta(\pi) \in E_4 = \mathcal{C}_\ell \times [m-1] \times D_{\ell,n}^m$.

Clearly $\{E_1, E_2, E_3, E_4\}$ is a partition of $D_{\ell,n}^m$.

8.2 Combinatorial proof of (2.8)

We shall prove the following version of (2.8):

$$\begin{aligned} \ell n d_{\ell,n-1}^0 - 1 &= d_{\ell,n}^0 && \text{if } n \text{ is odd,} \\ \ell n d_{\ell,n-1}^0 &= d_{\ell,n}^0 - 1 && \text{if } n \text{ is even.} \end{aligned}$$

Denote by $\mathcal{D}_{\ell,n}$ the set of derangements in $G_{\ell,n}$. Let $E_n = \emptyset$ if n is odd and $E_n = \{(1\ 2)(3\ 4)\cdots(n-1\ n)\}$ if n is even. Introduce also $F_n = \emptyset$ if n is even and $F_n = \{1\} \times \{n\} \times E_{n-1}$ if n is odd. We are going to define a mapping $\tau_\ell : (\varepsilon, k, \pi) \longmapsto \pi'$ from $(\mathcal{C}_\ell \times [n] \times \mathcal{D}_{\ell,n-1}) \setminus F_n$ to $\mathcal{D}_{\ell,n} \setminus E_n$, which implies the above identities.

Factorize π into disjoint cycles. We construct the cyclic factorization of π' by distinguishing several cases and by giving an example in $G_{4,9}$ for each case. Let $c(k)$ be the length of the cycle of π containing k and write $\widehat{k} = \mathrm{sgn}_\pi(k) \cdot k$.

1. If $k < n$, we obtain π' by inserting $\varepsilon\, n$ just after $\widehat{k}$ in a cycle of π.
   Example: $\varepsilon = \zeta^3$, $k = 3$, $\pi = (\bar{\bar{1}}\,\bar{4}\,2)(\bar{\bar{3}})(\bar{5}\,\bar{\bar{6}}\,8\,\bar{\bar{\bar{7}}})$ then $\pi' = (\bar{\bar{1}}\,\bar{4}\,2)(\bar{\bar{\bar{9}}}\bar{\bar{3}})(\bar{5}\,\bar{\bar{6}}\,8\,\bar{\bar{\bar{7}}})$.

2. If $k = n$ and $\varepsilon \neq 1$, we obtain π' by creating the cycle (εn).
   Example: $\varepsilon = \zeta^3$, $k = 9$, $\pi = (\bar{\bar{1}}\,\bar{4}\,2)(\bar{\bar{3}})(\bar{5}\,\bar{\bar{6}}\,8\,\bar{\bar{\bar{7}}})$ then $\pi' = (\bar{\bar{1}}\,\bar{4}\,2)(\bar{\bar{3}})(\bar{5}\,\bar{\bar{6}}\,8\,\bar{\bar{\bar{7}}})(\bar{\bar{\bar{9}}})$.

3. Suppose $k = n$ and $\varepsilon = 1$. Let $p \geq 0$ be the smallest integer such that the transposition $(2p+1,\, 2p+2)$ is not a cycle of π. In all the examples of this part we take $p = 2$.

   3.1 If $\mathrm{sgn}_\pi(2p+1) = 1$ then

   3.1.1 If $2p+2$ is a 1-circular succession of $|\pi|$ then π' is obtained by deleting $2p+1$ and creating the cycle $(n, 2p+1)$.
   Example: $\pi = (1\,2)(3\,4)(5\,\bar{\bar{6}})(\bar{7}\,\bar{\bar{\bar{8}}})$ then $\pi' = (1\,2)(3\,4)(\bar{\bar{6}})(\bar{7}\,\bar{\bar{\bar{8}}})(9\,5)$.

   3.1.2 If $2p+2$ is not a 1-circular succession of $|\pi|$ then:

   a) If $c(2p+1) = 2$ then π' is obtained by deleting the cycle $(2p+1, \pi(2p+1))$ and inserting $2p+1$ just before the letter $\widehat{2p+2}$ and creating the cycle $(\lambda n, |\pi|(2p+1))$ where $\lambda = \mathrm{sgn}_\pi(|\pi|(2p+1))$.
   Example: $\pi = (12)(34)(5\bar{\bar{\bar{8}}})(\bar{\bar{6}}\bar{7})$ then $\pi' = (12)(34)(5\,\bar{\bar{6}}\bar{7})(\bar{\bar{\bar{9}}}\,8)$.

   b) If $c(2p+1) > 2$. Let $a = |\pi|^{-1}(2p+1)$ and $\xi = \mathrm{sgn}_\pi(a)$ then π' is obtained by deleting the letter $\xi \cdot a$ and creating the cycle $(\xi n, a)$.
   Example: $\pi = (12)(34)(5\bar{\bar{\bar{8}}}\bar{\bar{6}}\bar{7})$ then $\pi' = (12)(34)(5\bar{\bar{\bar{8}}}\bar{\bar{6}})(\bar{9}\,7)$.

   3.2 If $\mathrm{sgn}_\pi(2p+1) = \gamma \neq 1$ then

   3.2.1 If $c(2p+1) = 1$ then π' is obtained by deleting the letter $\gamma \cdot (2p+1)$ and creating the cycle $(\gamma n, 2p+1)$.
   Example: $\pi = (12)(34)(\bar{\bar{5}})(\bar{6}\,\bar{\bar{8}}\,7)$ then $\pi' = (12)(34)(\bar{\bar{9}}\,5)(\bar{6}\,\bar{\bar{8}}\,7)$.

   3.2.2 If $c(2p+1) \neq 1$. Let $a = |\pi|^{-1}(2p+1)$ and $\gamma = \mathrm{sgn}_\pi(a)$ then π' is obtained by deleting the letter $\gamma \cdot a$ and creating the cycle $(\gamma n, a)$.
   Example: $\pi = (12)(34)(\bar{\bar{5}}\,\bar{\bar{8}}\,7\,\bar{6})$ then $\pi' = (12)(34)(\bar{\bar{5}}\,\bar{\bar{8}}\,7)(\bar{9}\,6)$.

Here is the inverse algorithm of $\tau_\ell : \pi' \mapsto (\varepsilon, k, \pi)$. Denote by $c(n)$ the length of the cycle of π' containing n. In what follows we write $\rho = \mathrm{sgn}_{\pi'}(n)$ and $\widetilde{k} = \mathrm{sgn}_{\pi'}(k) \cdot k$.

- If $c(n) \geq 3$ or $c(n) = 1$ or $c(n) = 2$ and $\mathrm{sgn}_{\pi'}(|\pi'|(n)) \neq 1$ then $\varepsilon = \mathrm{sgn}_{\pi'}(n)$ and $k = |\pi'|^{-1}(n)$ and we obtain π by deleting the letter $\widetilde{n}$.

- If $c(n) = 2$ and $\mathrm{sgn}_{\pi'}(|\pi'|(n)) = 1$ then $\varepsilon = 1$ and $k = n$. Let p be the smallest integer such that the transposition $(2p+1,\, 2p+2)$ is not a cycle of π'.

   a. If $\pi'(n) = 2p+1$ and $\rho = 1$ then we delete the cycle containing n and insert the letter $2p+1$ just before the letter $\widetilde{2p+2}$.

   b. If $\pi'(n) \neq 2p+1$ and $\mathrm{sgn}_{\pi'}(2p+1) = 1$ and $|\pi'|(2p+1) = 2p+2$, we first delete $2p+1$ and the cycle containing n, then create the cycle $(\rho \cdot \pi'(n), 2p+1)$.

c. If $|\pi'|(2p+1) \neq 2p+2$ and $\pi'(n) \neq 2p+1$ then we delete the cycle containing n and then insert the letter $\rho \cdot \pi'(n)$ before the letter $\widetilde{2p+1}$.

d. If $\pi'(n) = 2p+1$ and $\rho \neq 1$ then we delete the cycle containing n and create the cycle containing the single letter $2p+1$ with the sign ρ.

For example, the mapping $\tau_2 : (\mathcal{C}_2 \times [3] \times \mathcal{D}_{2,2}) \setminus F_3 \longrightarrow \mathcal{D}_{2,3} \setminus E_3$, where $E_3 = \emptyset$ and $F_3 = (1, 3, (1\,2))$, is given in the following table.

$\pi \setminus (\varepsilon, k)$	$(1,1)$	$(1,2)$	$(1,3)$	$(\zeta,1)$	$(\zeta,2)$	$(\zeta,3)$
(12)	(132)	(123)		$(1\bar{3}2)$	$(12\bar{3})$	$(12)(\bar{3})$
$(\bar{1}2)$	$(\bar{1}32)$	$(\bar{1}23)$	$(\bar{1})(32)$	$(\bar{1}\bar{3}2)$	$(\bar{1}2\bar{3})$	$(\bar{1}2)(\bar{3})$
$(1\bar{2})$	$(13\bar{2})$	$(1\bar{2}3)$	$(13)(\bar{2})$	$(1\bar{3}\bar{2})$	$(1\bar{2}\bar{3})$	$(1\bar{2})(\bar{3})$
$(\bar{1}\bar{2})$	$(\bar{1}3\bar{2})$	$(\bar{1}\bar{2}3)$	$(\bar{3}2)(\bar{1})$	$(\bar{1}\bar{3}\bar{2})$	$(\bar{1}\bar{2}\bar{3})$	$(\bar{1}\bar{2})(\bar{3})$
$(\bar{1})(\bar{2})$	$(\bar{1}3)(\bar{2})$	$(\bar{1})(\bar{2}3)$	$(\bar{3}1)(\bar{2})$	$(\bar{1}\bar{3})(\bar{2})$	$(\bar{1})(\bar{2}\bar{3})$	$(\bar{1})(\bar{2})(\bar{3})$

8.3 Combinatorial proof of (7.14)

By Theorem 12 the coefficient $d^m_{\ell,n}$ equals the cardinality of $D^m_{\ell,n}$. We are going to establish a bijection $\Phi : (\rho, \alpha, \pi) \longmapsto \pi'$ from $\mathcal{C}_\ell \times [n] \times D^m_{\ell,n-1}$ to $D^m_{\ell,n} \cup D^{m-1}_{\ell,n-2}$.

Let $\sigma = |\pi|$ and $\sigma' = |\pi'|$. The cyclic factorization of π' is obtained from that of π' as follows:

1. If $\alpha = n$, $\rho = 1$ and $1 \in FIX(\pi)$, we get $\pi' \in D^{m-1}_{\ell,n-2}$ by deleting the cycle (1) and decreasing all other letters by 1.

2. If $\alpha = n$ and $\rho \neq 1$ then we create the cycle (ρn). In this case $\pi' \in D^m_{\ell,n}$ and the cycle containing n is of length 1 but n is not a fixed point of π'.

3. If $\alpha = n$, $\rho = 1$ and $1 \notin FIX(\pi)$, then we delete $\pi(1)$ from its cycle and create a new cycle $(\gamma n, \sigma(1))$ where $\gamma = \mathrm{sgn}_\pi(\sigma(1))$. In this case $\pi'(n) > m$.

4. If $\alpha < n$ then we insert the letter ρn just before α.

To show that the mapping Φ is a bijection we construct its inverse as follows.

1. If $\pi' \in D^{m-1}_{\ell,n-2}$ then $\alpha = n$, $\rho = 1$ and π is obtained from π' by adding 1 to all letters and creating the cycle (1).

2. If $\pi' \in D^m_{\ell,n}$ and the cycle containing n is of length 1, then let $\alpha = n$, $\rho = \mathrm{sgn}_{\pi'}(n)$ and π is obtained from π' by deleting the letter ρn.

3. If $\pi' \in D^m_{\ell,n}$ and the cycle containing n is of length 2 with $\pi'(n) > m$ then let $\alpha = n$, $\rho = 1$ and π is obtained from π' by deleting the letter n and inserting the letter $\gamma\sigma'(n)$ just after 1 where $\gamma = \mathrm{sgn}_{\pi'}(n)$.

4. In all other cases, let $\alpha = \sigma'(n)$, $\rho = \mathrm{sgn}_{\pi'}(n)$ and π is obtained from π' by just deleting the letter ρn.

For $n = 9; m = 4; \ell = 3$ we give some examples to illustrate the above bijection.

- $\pi' = (1\bar{\bar{5}})(2)(3\bar{6})(4\bar{7}) \in D^{m-1}_{\ell,n-2}$ then $\alpha = 9$, $\rho = 1$ and $\pi = (1)(2\bar{\bar{6}})(3)(4\bar{7})(5\bar{8})$.
- $\pi' = (1\bar{\bar{5}})(2\bar{8})(3\bar{6})(4\bar{7})(\bar{\bar{9}}) \in D^m_{\ell,n}$ then $\alpha = 9$, $\rho = \zeta^2$ and $\pi = (1\bar{\bar{5}})(2\bar{8})(3\bar{6})(4\bar{7})$.
- $\pi' = (15)(2\bar{8})(3)(4\bar{7})(\bar{\bar{9}}6) \in D^m_{\ell,n}$ then $\alpha = 9$, $\rho = 1$ and $\pi = (1\bar{\bar{6}}5)(2\bar{8})(3)(4\bar{7})$.
- $\pi' = (1\bar{\bar{5}})(2\bar{8})(\bar{6})(4)(\bar{\bar{9}}\bar{7}3) \in D^m_{\ell,n}$ then $\alpha = 7$, $\rho = \zeta^2$ and $\pi = (1\bar{\bar{5}})(2\bar{8})(\bar{6})(4)(\bar{7}3)$.

Acknowledgments: This work was done during the visit of the first author to Institut Camille Jordan (Université Lyon 1) in the fall of 2007. This visit was supported by a scholarship of AUF (Agence universitaire de la francophonie). Both authors would like to thank the referee for his/her careful reading of a previous version of this paper.

References

[1] Adin (R. M.) and Roichman (Y.), The flag major index and group actions on polynomial rings, *European J. Combin.* **22** (2001), 431–446.

[2] Adin (R. M.), Brenti (F.) and Roichman (Y.), Equi-distribution over descent classes of the hyperoctahedral group, J. Combin. Theory Ser. A 113 (2006), no. 6, 917–933.

[3] Bagno (E.) and Biagioli (R.), Colored-descent representations of complex reflection groups $G(r, p, n)$, Israel J. Math. 160 (2007), 317–347.

[4] Bagno (E.) and Garber (D.), On the excedance number of colored permutation groups, Sém. Lothar. Combin. 53 (2004/06), Art. B53f, 17 pp. (electronic).

[5] Briggs (K.) and Remmel (J.), m-rook numbers and a generalization of a formula of Frobenius to $C_m \wr S_n$, J. Combin. Theory Ser. A 113 (2006), no. 6, 1138–1171.

[6] Comtet (L.), *Analyse combinatoire*, Tome second, Collection SUP, Presses universitaires de France, 1970.

[7] Chen (W.) and Zhang (J.), The skew and relative derangments of type B, Electronic J. Combin., 14 (2007), #N24.

[8] Clarke (R), Han (G.N.) and Zeng (J.), A combinatorial interpretation of the Seidel generation of q-derangement numbers, Ann. Comb. 1 (1997), no. 4, 313–327.

[9] Chow (C.), On derangement polynomials of type B, Sém. Lothar. Combin. 55 (2005/06), Art. B55b, 6 pp. (electronic).

[10] Chow (C.) and Gessel (I.), On the descent numbers and major indices for the hyperoctahedral group, Adv. in Appl. Math. 38 (2007), no. 3, 275–301.

[11] Dumont (D.) et Randrianarivony (A.), Dérangements et nombres de Genocchi, Discrete Math. 132 (1994), no. 1-3, 37–49.

[12] Foata (D.) and Han (G.N.), Signed words and permutations, IV; fixed and pixed points, preprint, to appear in Israel J. Math., 2006.

[13] Foata (D.) and Schützenberger (M. P.): *Théorie géométrique des polynômes eulériens*, Lecture Notes in Mathematics, Springer-Verlag, vol. 138, 1970.

[14] Haglund (J.), Loehr (N.) and Remmel (J. B.), Statistics on wreath products, perfect matchings, and signed words, European J. Combin. 26 (2005), no. 6, 835–868.

[15] Rakotondrajao (F.), k-fixed-points-permutations, *Integers, electronic journal of combinatorial number theory*, 7 (2007), #A36.

[16] Rakotondrajao (F.), On Euler's difference table, Proc. of FPSAC'07, Tianjin, 2007.

[17] Riordan (J.), *Introduction to combinatorial analysis*, originally published by Wiley, New York, 1958, and reprinted by Dover Publications, Inc., Mineola, NY, 2002

[18] Stanley (R.), *Enumerative Combinatorics*, Vol. 1, Cambridge Studies in Advanced Math. 49, 1997.