$\overline{\mathcal{R}}(3,4)=17$

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"Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

Paul Erdős [5]

Abstract

In this paper, we consider the on-line Ramsey numbers $\overline{\mathcal{R}}(k,l)$ for cliques. Using a high performance computing networks, we 'calculated' that $\overline{\mathcal{R}}(3,4)=17$. We also present an upper bound of $\overline{\mathcal{R}}(k,l)$, study its asymptotic behaviour, and state some open problems.

1 Introduction and definitions

In this paper, we consider the on-line Ramsey numbers introduced by Kurek and Ruciński [7] and corresponding to them the on-line Ramsey game. (The game was considered earlier by Beck [1] but not in terms of the numbers; Friedgut et al. [3] also studied a variant of this game but in the context of the random graph theory.) Let G, H be a fixed graphs. The game between two players, called the Builder and the Painter, is played on an unbounded set of vertices. In each of her moves the Builder draws a new edge which is immediately coloured red or blue by the Painter. The goal of the Builder is to force the Painter to create a red copy of G or a blue copy of H; the goal of the Painter is the opposite, he is trying to avoid it for as long as possible. The payoff to the Painter is the number of moves until this happens. The Painter seeks the highest possible payoff. Since this is a two-person, full information game with no ties, one of the players must have a winning strategy. *The on-line Ramsey number* $\overline{\mathcal{R}}(G,H)$ is the smallest payoff over all possible strategies of the Builder, assuming the Painter uses an optimal strategy. For simplicity, we use $\overline{\mathcal{R}}(k,l)$ for $\overline{\mathcal{R}}(K_k,K_l)$ and $\overline{\mathcal{R}}(G)$ for $\overline{\mathcal{R}}(G,G)$.

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Similar to the classical Ramsey numbers (see a dynamic survey of Radziszowski [11] which includes all known nontrivial values and bounds for Ramsey numbers), it is hard to compute the exact value of $\overline{\mathcal{R}}(G,H)$ unless G,H are trivial. In this relatively new area of small on-line Ramsey numbers, very little is known.

Recently, Grytczuk et al. [6], dealing with many labourious subcases, determined the on-line Ramsey numbers for a few short paths ($\overline{\mathcal{R}}(P_2) = 1$, $\overline{\mathcal{R}}(P_3) = 3$, $\overline{\mathcal{R}}(P_4) = 5$, $\overline{\mathcal{R}}(P_5) = 7$, $\overline{\mathcal{R}}(P_6) = 10$). It is clear that $\overline{\mathcal{R}}(P_n) \geq 2n-3$ for $n \geq 2$ since the Painter may color safely the first $n-2$ edges red, and the next $n-2$ edges blue. Also it is not hard to prove that $\overline{\mathcal{R}}(P_n) \leq 4n-7$ for $n \geq 2$ (see [6] for more details) but it seems that determining the exact values for longer paths requires computer support. The author of this paper was able to determine some new values, namely $\overline{\mathcal{R}}(P_7) = 12$, $\overline{\mathcal{R}}(P_8) = 15$, and $\overline{\mathcal{R}}(P_9) = 17$ (see [9, 8] for more details).

Kurek and Ruciński considered in [7] the most interesting case where G and H are cliques, but besides the trivial $\overline{\mathcal{R}}(2,k) = \binom{k}{2}$, they were able to determine only one more value, namely $\overline{\mathcal{R}}(3,3) = 8$ (the upper bound can be shown by mimicking the proof of the upper bound for classical Ramsey number $R(K_3,K_3)$; the proof of the lower bound is elegant and definitely nontrivial). In [7], it has been shown that

$$\overline{\mathcal{R}}(k,k) \leq 2k\binom{2k-2}{k-1} \sim \frac{1}{2\sqrt{\pi}}\sqrt{k}4^k.$$

In this paper, we show that $\overline{\mathcal{R}}(3,4) = 17$ (Section 3), provide a general upper bound for $\overline{\mathcal{R}}(k,l)$ which gives a slightly better asymptotic upper bound of $\frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}}$ for diagonal numbers (Section 2), and state some open problems (Section 4).

We also consider a new version of the on-line Ramsey numbers, related to the similar game we described before but the number of vertices is no longer unbounded. The Builder starts with an empty graph with k vertices. *The generalized on-line Ramsey number* $\overline{\mathcal{R}}_k(G,H)$ is defined as the minimum number of rounds in such a game if the Builder wins, otherwise $\overline{\mathcal{R}}_k(G,H) = \infty$ (that is, after $\binom{k}{2}$ moves the game is still not finished but the Builder has no more edges to present). Note that $\overline{\mathcal{R}}(G,H)$ moves are enough to win a game on unbounded set of vertices but it does mean the Builder does not use more than $2\overline{\mathcal{R}}(G,H)$ vertices in this game (in fact, this number is much smaller; see also Conjecture 4.3). Thus, $\overline{\mathcal{R}}_{2\overline{\mathcal{R}}(G,H)}(G,H) = \overline{\mathcal{R}}(G,H)$.

2 An upper bound for $\overline{\mathcal{R}}(k,l)$

In this section, we present a general upper bound for $\overline{\mathcal{R}}(k,l)$. The main result is the following Theorem 2.1 and the whole section is devoted to prove this result. (Of course, the first inequality is obvious.)

Theorem 2.1. *For all k,l, $2 \leq k \leq l$*

$$\overline{\mathcal{R}}(k,l) \leq \overline{\mathcal{R}}_{\binom{k+l-2}{l-1}}(k,l) \leq \frac{3}{2}\sum_{i=0}^{k-1}\binom{2i}{i} + \binom{k+l-1}{l-1} - \binom{2k-1}{k-1} - l - k + \frac{1}{2}.$$

In a table below we present the values of an upper bound of $\overline{\mathcal{R}}(k,l)$ for $3 \le k \le l \le 10$.

	3	4	5	6	7	8	9	10
3	8	17	31	51	78	113	157	211
4		36	70	125	208	327	491	710
5			139	264	473	802	1296	2010
6				515	976	1767	3053	5054
7					1899	3614	6616	11620
8						7045	13479	24918
9							26348	50657
10								99276

Table 1: Upper bounds of $\overline{\mathcal{R}}(k,l)$

Let us start from the following simple observation.

Lemma 2.2. *Assume that* $\overline{\mathcal{R}}_m(k-1,l) < \infty$ *and* $\overline{\mathcal{R}}_n(k,l-1) < \infty$. *Then*

$$\overline{\mathcal{R}}_{m+n}(k,l) \le m+n-1+\max\{\overline{\mathcal{R}}_m(k-1,l), \overline{\mathcal{R}}_n(k,l-1)\}.$$

Proof. We present a natural Builder's strategy forcing the Painter to create a red copy of K_k or a blue K_l after $m+n-1+\max\{\overline{\mathcal{R}}_m(k-1,l), \overline{\mathcal{R}}_n(k,l-1)\}$ moves. The Builder presents $m+n-1$ edges of a star $K_{1,m+n-1}$. By pigeonhole principle, the Painter must use either red at least m times or blue at least n times. If the red $K_{1,m}$ is created, then the Painter can use a strategy forcing a red K_{k-1} or a blue K_l on m leaves of a red star in $\overline{\mathcal{R}}_m(k-1,l)$ moves. Otherwise a strategy forcing a red K_k or a blue K_{l-1} can be used. □

Lemma 2.2 guarantees the existence of numbers $\overline{\mathcal{R}}(k,l)$ for any value of k and l. (Of course, it follows from the existence of classical Ramsey numbers $R(k,l)$ as well.) From this lemma, it is also possible to determine a recursive relation for the number of vertices $n(k,l)$ used in the described strategy of the Builder. We note that for $k \ge 2$, $n(2,k) = n(k,2) = k$, and for all $k,l \ge 3$

$$n(k,l) = n(k,l-1) + n(k-1,l)\,. \tag{1}$$

In fact, this recurrence is used in the proof that classical Ramsey numbers are well defined, given by Graham et al. [4].

From this relation, we can derive an explicit value of $n(k,l)$ by elementary methods. This is a known result but we present a proof for completeness.

Lemma 2.3. *For all* $k,l \ge 2$

$$n(k,l) = \binom{k+l-2}{l-1}.$$

Proof. Let $\rho(k,l) = \binom{k+l-2}{l-1}$. Since $\binom{u}{v} = \binom{u-1}{v-1} + \binom{u-1}{v}$, we have

$$\begin{aligned}\rho(k,l) &= \binom{k+l-2}{l-1} = \binom{k+l-3}{l-2} + \binom{k+l-3}{l-1} \\ &= \rho(k,l-1) + \rho(k-1,l)\,.\end{aligned}$$

This recursive relation is analogous to (1). Thus, together with the fact that for any $k \geq 2$

$$\rho(k,2) = \rho(2,k) = k = n(k,2) = n(2,k)\,,$$

this finishes the proof. $\square$

Immediately from Lemma 2.2 and Lemma 2.3 we get the following corollary.

Corollary 2.4. *For all $k, l \geq 3$*

$$\overline{\mathcal{R}}_{\binom{k+l-2}{l-1}}(k,l) \leq \binom{k+l-2}{l-1} - 1 + \max\left\{\overline{\mathcal{R}}_{\binom{k+l-3}{l-1}}(k-1,l), \overline{\mathcal{R}}_{\binom{k+l-3}{l-2}}(k,l-1)\right\}.$$

In order to study an upper bound of $\overline{\mathcal{R}}(k,l)$ we study the behaviour of $\tau(k,l)$ where $\tau(k,l)$ is defined by the recursive relation analogous to one in the corollary, namely,

$$\begin{aligned}\tau(2,k) &= \tau(k,2) = \binom{k}{2} \\ \tau(k,l) &= \tau(l,k) = \binom{k+l-2}{l-1} - 1 + \max\{\tau(k-1,l), \tau(k,l-1)\} \qquad (2)\end{aligned}$$

for all $k, l \geq 3$. It is clear that $\overline{\mathcal{R}}(k,l) \leq \overline{\mathcal{R}}_{\binom{k+l-2}{l-1}}(k,l) \leq \tau(k,l)$.

It is convenient to put $\tau(1,1) = \tau(2,1) = 0$. Now the following holds.

Theorem 2.5. *For all k, l such that $2 \leq k \leq l$*

$$\tau(k,l) = \binom{k+l-2}{l-1} - 1 + \tau(k,l-1)\,. \qquad (3)$$

Proof. Since $\tau(k-1,k) = \tau(k,k-1)$, (3) holds for $2 \leq k = l$. Thus it is enough to verify (3) for $2 \leq k < l$ and we use induction on k for that. For a basis step ($k = 2$), note that for any $l > 2$,

$$\binom{l}{l-1} - 1 + \tau(2,l-1) = l - 1 + \binom{l-1}{2} = \binom{l}{2} = \tau(2,l)\,.$$

For an induction step, fix $k_0 \geq 3$, suppose that (3) holds for all l and $2 \leq k_0 - 1 < l$, and take any $l > k_0$. By the induction hypothesis and simple property of the binomial coefficient,

$$\begin{aligned}\tau(k_0-1,l) &= \binom{k_0+l-3}{l-1} - 1 + \tau(k_0-1,l-1) \\ &\leq \binom{k_0+l-3}{l-2} - 1 + \tau(k_0-1,l-1)\,. \qquad (4)\end{aligned}$$

Using (2) and (4) we have that

$$\begin{aligned}
\tau(k_0, l-1) &= \binom{k_0+l-3}{l-2} - 1 + \max\{\tau(k_0-1, l-1), \tau(k_0, l-2)\} \\
&\geq \binom{k_0+l-3}{l-2} - 1 + \tau(k_0-1, l-1) \\
&\geq \tau(k_0-1, l),
\end{aligned}$$

and now (3) follows directly from (2). □

Now, we are ready to prove the main result of this section, namely, Theorem 2.1.

Proof of Theorem 2.1. From Theorem 2.5 it follows that for any $k \geq 2$

$$\begin{aligned}
\tau(k,k) &= \binom{2k-2}{k-1} + \binom{2k-3}{k-1} - 2 + \tau(k-1, k-1) \\
&= 3\binom{2k-3}{k-1} - 2 + \tau(k-1, k-1) \\
&= \sum_{i=3}^{k}\left(3\binom{2i-3}{i-1} - 2\right) + 1 \\
&= \frac{3}{2}\sum_{i=2}^{k-1}\binom{2i}{i} - 2k + 5 \\
&= \frac{3}{2}\sum_{i=0}^{k-1}\binom{2i}{i} - 2k + \frac{1}{2}\,. \qquad (5)
\end{aligned}$$

Thus, for any $l \geq k \geq 2$

$$\begin{aligned}
\tau(k,l) &= \sum_{m=k+1}^{l}\left(\binom{k+m-2}{m-1} - 1\right) + \tau(k,k) \\
&= \sum_{m=k}^{l-1}\binom{k+m-1}{m} - (l-k) + \tau(k,k) \\
&= \sum_{m=0}^{l-1}\binom{(k-1)+m}{m} - \sum_{m=0}^{k-1}\binom{(k-1)+m}{m} + \frac{3}{2}\sum_{i=0}^{k-1}\binom{2i}{i} - l - k + \frac{1}{2}
\end{aligned}$$

and the assertion follows from the fact that $\sum_{j=0}^{r}\binom{n+j}{j} = \binom{n+r+1}{r}$. □

From Theorem 2.1 we can easily derive an asymptotic upper bound for diagonal online Ramsey numbers.

Corollary 2.6.

$$\overline{\mathcal{R}}(k,k) \le \tau(k,k) \sim \frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}}\,.$$

Proof. From (5) and the Stirling formula we get that

$$\begin{aligned}\tau(k,k) &\sim \frac{3}{2}\sum_{i=0}^{k-1}\binom{2i}{i} = \frac{3}{2}\sum_{i=1}^{k-1}\frac{\sqrt{4\pi i}(2i/e)^{2i}}{2\pi i(i/e)^{2i}}(1+O(1/i))\\ &= \frac{3}{2\sqrt{\pi}}\sum_{i=1}^{k-1}\frac{4^i}{\sqrt{i}}(1+O(1/i))\,.\end{aligned}$$

Using summation by parts, sometimes called the Abel transformation, it follows that

$$\begin{aligned}\sum_{i=1}^{k-1}\frac{4^i}{\sqrt{i}} &= \frac{4^{k-1}}{\sqrt{k-1}} - \sum_{i=1}^{k-2}\frac{4^{i+1}-1}{3}\left(\frac{1}{\sqrt{i+1}}-\frac{1}{\sqrt{i}}\right)\\ &= \frac{4^{k-1}}{\sqrt{k-1}} - \sum_{i=1}^{k-2}O\left(\frac{4^i}{i^{3/2}}\right)\,.\end{aligned}$$

Thus,

$$\begin{aligned}\tau(k,k) &\sim \frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}} + \sum_{i=1}^{k-1}O\left(\frac{4^i}{i^{3/2}}\right) = \frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}} + \frac{4^k}{k^{3/2}}\cdot O\left(\sum_{i=1}^{k-1}\frac{1}{2^i}\right)\\ &\sim \frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}}\end{aligned}$$

since

$$\frac{4^{n-1}}{(n-1)^{3/2}} = \frac{4^n}{n^{3/2}}\cdot\frac{1}{4}\cdot\left(\frac{n}{n-1}\right)^{3/2} < \frac{1}{2}\cdot\frac{4^n}{n^{3/2}}$$

for $n \ge 3$. □

3 Games for red K_3 and blue K_4.

Note that, for any two graphs G, H and $k, l \in \mathbb{N}$, $k < l$

$$\overline{\mathcal{R}}_k(G,H) \ge \overline{\mathcal{R}}_l(G,H) \ge \overline{\mathcal{R}}(G,H)$$

since in the generalized version of the game the Builder has more restrictions to follow. Using a computer support we were able to find that $\overline{\mathcal{R}}_{12}(3,4) = 17$ (Theorem 3.1) and show that this implies that $\overline{\mathcal{R}}(3,4) = 17$ (Theorem 3.2). We also checked that $\overline{\mathcal{R}}_9(3,4) \ge 19$ (Theorem 3.3).

We implemented and ran programs written in C/C++ using backtracking algorithms. (The programs can be downloaded from [10].) Backtracking is a refinement of the brute force approach, which systematically searches for a solution to a problem among all available options. Since it is not possible to examine all possibilities, we used many advanced validity criteria to determine which portion of the solution space needed to be searched. For example, one can look at the coloured graph in every round and try to estimate the number of red (and blue) edges needed to create desired structure. This knowledge can be used to avoid considering the whole branch in the searching tree. If the Painter can use red colour and 'survive' additional k rounds, then there is no point to check whether using blue colour forces him to finish the game earlier.

Using a set of clusters (see Section 5 for more details), we were able to run (independently) the program from different initial graphs with given colouring of edges. In the table below we present the numbers of nonisomorphic coloured graphs with k edges that have been found by computer. Since the game we play is nonsymmetric we have to consider more initial graphs than in the symmetric version (see [9] where the symmetric game for paths was considered). If the number of edges is odd, we have exactly two times more graphs to consider. For the even case, this number is a little bit smaller than double.

k	# of symmetric graphs	# of nonsymmetric graphs
1	1	2
2	4	6
3	12	24
4	51	93
5	203	406
6	1,004	1,959
7	5,117	10,234
8	29,153	58,013
9	176,778	353,556
10	1,150,164	2,298,303

Table 2: Number of nonisomorphic coloured graphs with k edges

Having results from simulations starting from different initial graphs (even partial ones!) we are able to determine the exact value of the on-line Ramsey numbers. The relations between the partial results in different levels are complicated but can be found using a computer. The relations between levels 1 – 2, and 2 – 3 are described below. For simplicity, we present the symmetric case; the nonsymmetric one is studied in the same way.

There is only one possible coloured graph G_1^1 with one edge (up to isomorphism). Graphs with two and three edges are presented in Figure 1 and Figure 2, respectively. Let $x_i^m = x_i^m(G_i^m, k, l)$ denote the number of moves in a winning strategy of the Builder in the on-line Ramsey game, provided that after m moves a coloured graph is isomorphic

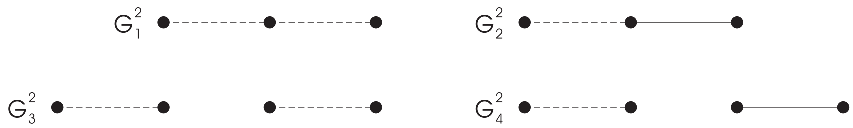

Figure 1: Coloured graphs with two edges

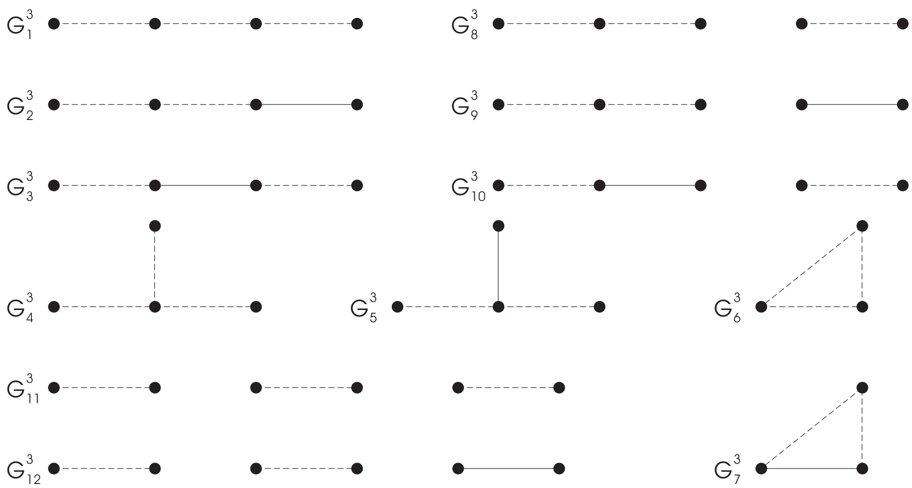

Figure 2: Coloured graphs with three edges

to G_i^m. Using the notation

$$\begin{aligned} x_1 \vee x_2 &= \max\{x_1, x_2\} \\ x_1 \wedge x_2 \wedge \cdots \wedge x_k &= \min\{x_1, x_2, \ldots, x_k\}, \end{aligned}$$

it is not hard to see that

$$x_1^1 = (x_1^2 \vee x_2^2) \wedge (x_3^2 \vee x_4^2),$$

and

$$\begin{aligned} x_1^2 &= (x_1^3 \vee x_2^3) \wedge (x_8^3 \vee x_9^3) \wedge (x_4^3 \vee x_5^3) \wedge (x_6^3 \vee x_7^3) \\ x_2^2 &= (x_3^3 \vee x_2^3) \wedge x_{10}^3 \wedge x_5^3 \wedge x_7^3 \\ x_3^2 &= (x_1^3 \vee x_3^3) \wedge (x_8^3 \vee x_{10}^3) \wedge (x_{11}^3 \vee x_{12}^3) \\ x_4^2 &= x_2^3 \wedge (x_9^3 \vee x_{10}^3) \wedge x_{12}^3 . \end{aligned}$$

Each "$\vee$" sign corresponds to the Painter's move, "$\wedge$" corresponds to the Builder's one. He tries to play as long as possible, choosing the maximum value, but she would like to win as soon as possible.

Since it is more difficult to study on-line Ramsey game on unbounded set of vertices, we consider a game on 12 vertices first and then we show that the Builder cannot win faster by playing on larger set.

Theorem 3.1. $\overline{\mathcal{R}}_{12}(3,4) = 17$

Proof. It follows from Theorem 2.1 that $\overline{\mathcal{R}}_{10}(3,4) \le 17$ ($\tau(3,4) = 17$, $n(3,4) = 10$). Thus, $\overline{\mathcal{R}}_{12}(3,4) \le \overline{\mathcal{R}}_{10}(3,4) \le 17$.

In order to show that $\overline{\mathcal{R}}_{12}(3,4) \ge 17$ we examined $2,298,303$ initial configurations with 10 edges. Exactly $280,993$ graphs with at most 12 vertices contain a red K_3 or a blue K_4 so we put $x_i^{10} \le 10$ for these graphs. $100,946$ graphs contain more than 12 vertices; we put $x_i^{10} = \infty$ for these graphs. For the rest, we run the simulation to check whether $x_i^{10} \le 16$. (Note that we can restrict our consideration to this interval since we know that $\overline{\mathcal{R}}_{12}(3,4) \le 17$.) The results are presented below. Next we verified that the

	# of initial configurations
$x_i^{10} \le 10$	$280,993$
$x_i^{10} = 11$	868
$x_i^{10} = 12$	$1,578$
$x_i^{10} = 13$	$8,043$
$x_i^{10} = 14$	$14,065$
$x_i^{10} = 15$	$43,695$
$x_i^{10} = 16$	$96,701$
$17 \le x_i^{10} < \infty$	$1,751,414$
$x_i^{10} = \infty$	$100,946$
total	$2,298,303$

Table 3: Results for a game on 12 vertices

Painter has a strategy to reach one of the 'good' configurations that allow him to survive the next six moves. ☐

Increasing the number of vertices for this game will not change her winning strategy provided she already has enough vertices. It is also clear that in order to force the Painter to create a red copy of K_3 or a blue copy of K_4, the Builder has to build relatively dense structure. Moreover, it seems that there is no point for her to have disconnected components at the end of the game (it might be a good idea to start with disconnected graphs at the beginning) so the final graph *should* be connected (see Conjecture 4.4). If this is proven, then there is a simple proof of Theorem 3.2. But since the conjecture is still open, we have to be content with the following proof which is definitely not from *the Book*.

Theorem 3.2. $\overline{\mathcal{R}}(3,4) = 17$

Proof. Since $\overline{\mathcal{R}}(3,4) \le \overline{\mathcal{R}}_{12}(3,4) = 17$ (see Theorem 3.1) it is enough to show that $\overline{\mathcal{R}}(3,4) \ge 17$. For a contradiction, let us suppose that $\overline{\mathcal{R}}(3,4) = m \le 16$. Consider a winning strategy of the Builder as a (binary) game tree of depth m; for every Builder's move from the winning strategy, the Painter can reply by using red (left child) or blue (right child) colour. (Note that, since he is playing perfectly, sometimes his move is determined, that is, the game tree is not complete.)

All graphs at the very last level of the tree (level m) must contain a red copy of K_3 or a blue copy of K_4. It is also clear that all graphs at the level $m-1$ do not have those structures but they contain a subgraph A presented in Figure 3. Continuing this way of thinking, we can try to investigate the shape of graphs with $m-2$ edges at the higher level. Those graphs cannot contain A (otherwise the Builder would be able to finish the game earlier) so we know that one edge e (blue or red) from A was added at the round $m-1$. Without loss of generality, we can assume that $e \in S = \{e_1, e_2, e_3\}$ (see Figure 3) since the other edges are isomorphic to one of those from S. We focus on the case corresponding to adding the edge e_1 only; the rest can be studied the same way.

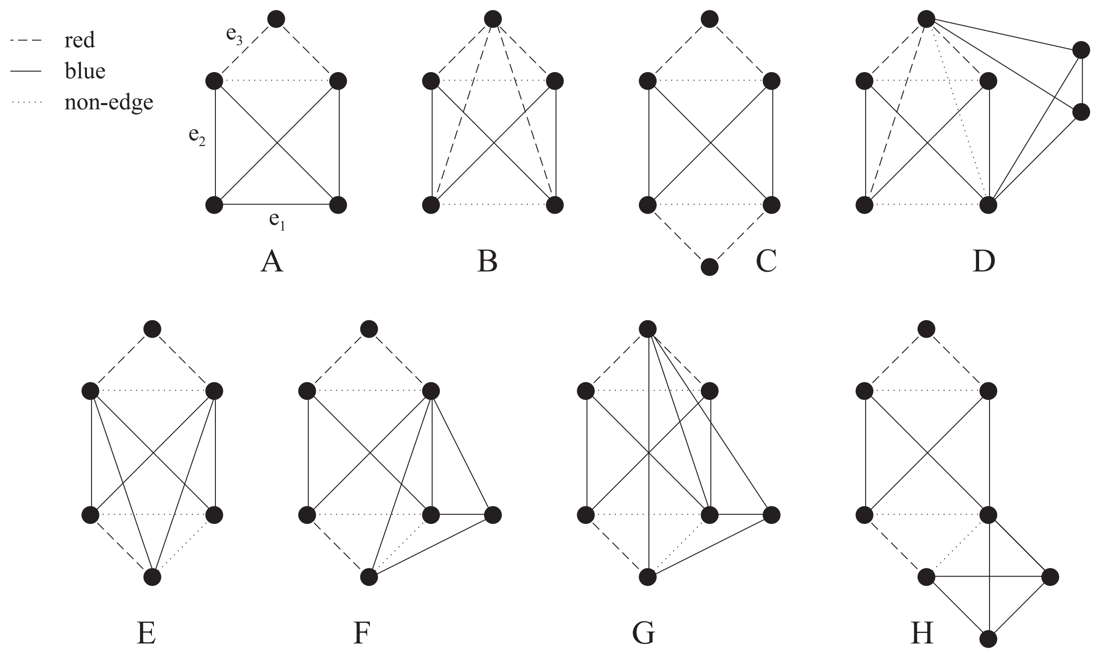

Figure 3: Desired subgraphs

Suppose that the Painter was forced to use blue at the round $m-1$, that is, using red would create a red triangle and the game would be finished. This means that B or C is a subgraph of a graph at the round $m-2$. Alternatively, if the Painter had a free choice of colours, one of the graphs $D-H$ must appear on this level; no matter which colour is used, the Painter cannot avoid A.

Note that subgraphs D, G, H have 12 edges and at most 8 vertices. Since all graphs on that level have exactly $m-2 \leq 14$ edges, those graphs have at most 12 vertices. For the rest of subgraphs presented in Figure 3, we have to investigate graphs on higher level in the game tree (they have one less edge and are usually denser) to get the same conclusion. This also implies that graphs on the last level all have at most 12 vertices. But it means that there is a strategy of the Builder to win a game on 12 vertices in $m \leq 16$ moves.

	# of initial configurations
$x_i^{10} \leq 10$	$180,147$
$11 \leq x_i^{10} \leq 17$	$110,647$
$x_i^{10} = 18$	$64,546$
$19 \leq x_i^{10} < \infty$	$641,582$
$x_i^{10} = \infty$	$1,301,381$
total	$2,298,303$

Table 4: Results for a game on 9 vertices

This contradicts Theorem 3.1 and finishes the proof. □

Let us finish this section with one more result for a game on the smallest possible number of vertices, namely, on 9 vertices. From previous results it follows that $\overline{\mathcal{R}}_n(3,4) = \overline{\mathcal{R}}(3,4) = 17$ for all $n \geq 10$ but the value of $\overline{\mathcal{R}}_9(3,4)$ is bigger (see also a discussion in the next section).

Theorem 3.3. $\overline{\mathcal{R}}_9(3,4) \geq 19$

Proof. In order to show that $\overline{\mathcal{R}}_9(3,4) \geq 19$ we reexamined $2,298,303$ initial configurations with 10 edges. Exactly $180,147$ graphs with at most 9 vertices contain a red K_3 or a blue K_4 so we put $x_i^{10} \leq 10$ for these graphs. $1,301,381$ graphs contain more than 9 vertices; we put $x_i^{10} = \infty$ for these graphs. For the rest, we run the simulation to check whether $x_i^{10} \leq 17$, $x_i^{10} = 18$ or $x_i^{10} \geq 19$. (Note that we can restrict our consideration to this interval since we know that $\overline{\mathcal{R}}_9(3,4) \geq 17$.) The results are presented below. Next we verified that the Painter has a strategy to reach one of the 'good' configurations that allow him to survive *at least* the next eight moves. □

4 Some open problems

In this section, we ask some questions for future consideration. It is clear that

$$\overline{\mathcal{R}}(k,k) \leq \binom{R(k,k)}{2},$$

since the Builder can present edges of $K_{R(k,k)}$ (in any sequence) and the Painter cannot avoid a monochromatic copy of K_k. The following intriguing conjecture was posed by Kurek and Ruciński [7].

Conjecture 4.1.

$$\lim_{k\to\infty} \frac{\binom{R(k,k)}{2}}{\overline{\mathcal{R}}(k,k)} = \infty$$

Unfortunately, we still do not know the answer to that question. In this paper, we present an upper bound for $\overline{\mathcal{R}}(k,k)$ which is roughly the same as the best known upper bound for $R(k,k)$ but since the best known lower bound for $R(k,k)$ is far away from the upper one, we cannot answer the question based on this knowledge only. (It is known that $\sqrt{2} \le \liminf R(k,k)^{1/k} \le \limsup R(k,k)^{1/k} \le 4$. Already in 1947, Erdős conjectured that $\lim_{k\to\infty} R(k,k)^{1/k}$ exists. He later offered \$100 for a proof of its existence and \$250 for its exact value [2]. Many people believe that $R(k,k) \ge (1+o(1))2^k$ but the conjecture is still open.) However, it supports our intuition that online version of the Ramsey numbers should grow slower than the classical size Ramsey numbers.

On the other hand, it seems that the generalized on-line Ramsey numbers $\overline{\mathcal{R}}_{R(k,k)}(k,k)$ grow much faster than $\overline{\mathcal{R}}(k,k)$ (see Theorem 3.3), so we conjecture the following:

Conjecture 4.2.

$$\lim_{k\to\infty} \frac{\binom{R(k,k)}{2}}{\overline{\mathcal{R}}_{R(k,k)}(k,k)} = c < \infty$$

It follows from Theorem 2.1 that $\overline{\mathcal{R}}(4,4) \le 36$ by considering the game on $n(4,4) = 2n(3,4) = 20$ vertices. But, since $9 = R(3,4) < n(3,4) = 10$, one can expect a better upper bound by considering the strategy on 18 vertices used in the proof of Lemma 2.2 (that is, present $K_{1,17}$ first (one can save two moves that way) and then use a strategy to create desired structure in additional $\overline{\mathcal{R}}_9(3,4)$ rounds). Unfortunately, it turned out that $\overline{\mathcal{R}}_9(3,4) \ge 19 = \overline{\mathcal{R}}_{10}(3,4) + 2$ so this strategy is not better than the previous one. Maybe it is even true that the strategy used to get an upper bound in Theorem 2.1 is optimal.

Conjecture 4.3.

$$\overline{\mathcal{R}}(k,l) = \tau(k,l) = \frac{3}{2}\sum_{i=0}^{k-1}\binom{2i}{i} + \binom{k+l-1}{l-1} - \binom{2k-1}{k-1} - l - k + \frac{1}{2}$$

Finally, let us mention the following variant of the game and its corresponding number $\hat{R}(k,l)$: the Builder has to force the Painter to draw a red copy of K_k or a blue copy of K_l as before, but also it is required that the final graph is connected (colours do not matter). Since the Builder has more restrictions to follow, $\hat{R}(k,l) \ge \overline{\mathcal{R}}(k,l)$. However, it seems that there is no point for the Builder to construct disconnected graphs at the end of the classical game (it might be good for her to have disconnected graphs at the very beginning) but no proof of the following conjecture is known.

Conjecture 4.4.

$$\hat{R}(k,l) = \overline{\mathcal{R}}(k,l)$$

As we already mentioned, this conjecture is important because if this is verified, then there is an easy proof of Theorem 3.2.

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In order to verify that $\overline{\mathcal{R}}_{12}(3,4) > 16$ and that $\overline{\mathcal{R}}_9(3,4) > 18$ we checked (independently) $2,298,303$ initial configurations. A running time of one serial program varied between a few seconds and 1 hour but we noticed the average running time to be around 0.2 hour for a game on 12 vertices and around 0.4 for 9 vertices. Thus, we can estimate the total computational requirements to be around $1,379,000$ CPU hours.

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