On growth rates of permutations, set partitions, ordered graphs and other objects

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Abstract

For classes $\mathcal{O}$ of structures on finite linear orders (permutations, ordered graphs etc.) endowed with containment order $\preceq$ (containment of permutations, subgraph relation etc.), we investigate restrictions on the function $f(n)$ counting objects with size n in a lower ideal in $(\mathcal{O}, \preceq)$. We present a framework of edge P-colored complete graphs $(\mathcal{C}(P), \preceq)$ which includes many of these situations, and we prove for it two such restrictions (jumps in growth): $f(n)$ is eventually constant or $f(n) \ge n$ for all $n \ge 1$; $f(n) \le n^c$ for all $n \ge 1$ for a constant $c > 0$ or $f(n) \ge F_n$ for all $n \ge 1$, F_n being the Fibonacci numbers. This generalizes a fragment of a more detailed theorem of Balogh, Bollobás and Morris on hereditary properties of ordered graphs.

1 Introduction

We aim to obtaining general results on jumps in growth of combinatorial structures, motivated by such results for permutations [19] (which were in turn motivated by results of Scheinerman and Zito [29] and Balogh, Bollobás and Weinreich [3, 4, 5] on growths of graph properties), and so we begin with them. *Pattern avoidance in permutations*, a quickly developing area of combinatorics [2, 8, 11, 12, 13, 15, 18, 22, 23, 26, 28, 30, 31, 32, 33], is primarily concerned with enumeration of sets of permutations

$$\mathrm{Forb}(F) = \{\rho \in \mathcal{S} :\ \rho \not\succeq \pi\ \forall \pi \in F\},$$

where F is a fixed finite or infinite set of forbidden permutations (patterns) and $\preceq$ is the usual containment order on the set of finite permutations $\mathcal{S} = \bigcup_{n\ge 0} \mathcal{S}_n$. Recall that

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$\pi = a_1a_2 \dots a_m \preceq \rho = b_1b_2 \dots b_n$ iff ρ has a subsequence $b_{i_1}b_{i_2} \dots b_{i_m}$, $1 \le i_1 < i_2 < \dots < i_m \le n$, such that $a_r < a_s \iff b_{i_r} < b_{i_s}$ for all $1 \le r < s \le m$.

Each set $\mathrm{Forb}(F)$ is an ideal in $(\mathcal{S}, \preceq)$ because $\pi \preceq \rho \in \mathrm{Forb}(F)$ implies $\pi \in \mathrm{Forb}(F)$ and each ideal X in $(\mathcal{S}, \preceq)$ has the form $X = \mathrm{Forb}(F)$ for some (finite or infinite) set F. For ideals of permutations X, it is therefore of interest to investigate restrictions on growth of the counting function $n \mapsto |X_n|$, where $X_n = X \cap \mathcal{S}_n$ is the set of permutations with length n lying in X. In this direction, Kaiser and Klazar [19] obtained the following results.

1. *The constant dichotomy.* Either $|X_n|$ is eventually constant or $|X_n| \ge n$ for all $n \ge 1$.

2. *Polynomial growth.* If $|X_n|$ is bounded by a polynomial in n, then there exist integers $c_0, c_1, \dots, c_r$ so that for every $n > n_0$ we have
$$|X_n| = \sum_{j=0}^{r} c_j \binom{n}{j}.$$

3. *The Fibonacci dichotomy.* Either $|X_n| \le n^c$ for all $n \ge 1$ for a constant $c > 0$ ($|X_n|$ has then the form described in 2) or $|X_n| \ge F_n$ for all $n \ge 1$, where $(F_n)_{n \ge 0} = (1, 1, 2, 3, 5, 8, 13, \dots)$ are the Fibonacci numbers.

4. *The Fibonacci hierarchy.* The main result of Kaiser and Klazar [19] states that if $|X_n| < 2^{n-1}$ for at least one $n \ge 1$ and X is infinite, then there is a unique integer $k \ge 1$ and a constant $c > 0$ such that
$$F_{n,k} \le |X_n| \le n^c F_{n,k}$$
holds for all $n \ge 1$. Here $F_{n,k}$ are the generalized Fibonacci numbers defined by $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$, and $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \dots + F_{n-k,k}$ for $n > 0$.

The dichotomy 3 is subsumed in the hierarchy 4 because $F_{n,1} = 1$ and $F_{n,k} \ge F_{n,2} = F_n$ for $k \ge 2$ and $n \ge 1$, but we state it apart as it identifies the least superpolynomial growth. Note that the restrictions 1–4 determine possible growths of ideals of permutations below 2^{n-1} but say nothing about the growths above 2^{n-1}. In fact, Klazar [21] showed that while there are only countably many ideals of permutations X satisfying $|X_n| < 2^{n-1}$ for some (hence, by 4, every sufficiently large) n, there exists an uncountable family of ideals of permutations $\mathcal{F}$ such that $|X_n| \ll (2.34)^n$ for every $X \in \mathcal{F}$.

A remarkable generalization of the restrictions 1–4 was achieved by Balogh, Bollobás and Morris [6] who extended them to ordered graphs. Their main result [6, Theorem 1.1] is as follows. Let X be a *hereditary property of ordered graphs*, that is, a set of finite simple graphs with linearly ordered vertex sets, which is closed to the order-preserving graph isomorphism and to the order-preserving induced subgraph relation. Let X_n be the set of graphs in X with the vertex set $[n] = \{1, 2, \dots, n\}$. Then, again, the counting function $n \mapsto |X_n|$ is subject to the restrictions 1–4 described above. Since ideals of permutations can be represented by particular hereditary properties of ordered graphs, this vastly generalizes the results on growth of permutations [19]. As for the proofs, for graphs they are considerably more complicated than for permutations.

In this article we present a general framework for proving restrictions of the type 1–4 on growths of other classes of structures besides permutations and ordered graphs. We shall generalize only 1 and 3, i.e., the constant dichotomy (Theorem 3.1) and the Fibonacci dichotomy (Theorem 3.8). We remark that our article overlaps in results with the work of Balogh, Bollobás and Morris [6]; we explain the overlap presently along with summarizing the content of our article. I learned about the results in [6] shortly before completing and submitting my work.

We prove in Theorems 3.1 and 3.8 that the constant dichotomy and the Fibonacci dichotomy hold for ideals of complete graphs having edges colored with l colors, where the containment is given by the order-and-color-preserving mappings between vertex sets. For $l = 2$ these structures reduce to graphs with ordered induced subgraph relation and thus our results on the two dichotomies generalize those of Balogh, Bollobás and Morris [6] for ordered graphs. To be honest, we must say that for the constant dichotomy and the Fibonacci dichotomy it is not hard to reduce the general case $l \geq 2$ to the case $l = 2$ (see Proposition 2.7 and Corollary 2.8) and so our generalization is not very different from the case of graphs. (However, this simple reduction ceases to work for the Fibonacci hierarchy 4.) Our proofs are different and shorter than the corresponding parts of the proof of Theorem 1.1 in [6] (which takes cca 24 pages).

So instead of (ordered) graphs with induced subgraph relation—which can be captured by complete graphs with edges colored in black and white—we consider here complete graphs with edges colored in finitely many colors. There is more to this generalization than it might seem, as we discuss in Section 2, and this is the main contribution of the present article. Our setting enables to capture many other classes of objects and their containments $(\mathcal{O}, \preceq)$ (which need not be directly given in graph-theoretical terms) and to show uniformly that their growths are subject to both dichotomies. For this one only has to verify (which is usually straightforward) that $(\mathcal{O}, \preceq)$ fits the framework of *binary classes of objects*. We summarize it briefly now and give details in Section 2. A binary class of objects is a partial order $(\mathcal{O}, \preceq)$ which is realized by embeddings between objects. The size of each object $K \in \mathcal{O}$ is the cardinality of its set of atoms $A(K)$, where an atom of K is an embedding of an atom of $(\mathcal{O}, \preceq)$ in K. For an ideal X in $(\mathcal{O}, \preceq)$, X_n is the subset of objects in X with size n and we are interested in the counting function $n \mapsto |X_n|$. Each set of atoms $A(K)$ carries a linear ordering $\leq_K$ and these orderings are preserved by the embeddings. The objects $K \in \mathcal{O}$ and the containment order $\preceq$ are uniquely determined by the restrictions of K to the two-element subsets of $A(K)$ (the binarity condition in Definition 2.2). Hence $(\mathcal{O}, \preceq)$ can be viewed as an ideal in the class $(\mathcal{C}(P), \preceq)$ of complete graphs which have edges colored by elements of a finite poset P and where $\preceq$ is the edgewise P-majorization ordering. For both dichotomies P can be taken without loss of generality to be the discrete poset with trivial comparisons. We conclude Section 2 with several examples of binary classes. Here we mention three of them. Permutations with the containment of permutations form a binary class. So do finite sequences over a finite alphabet A with the subsequence relation. Multigraphs (graphs with possibly repeated edges) without isolated vertices and with the ordered subgraph relation form also a binary class; note that their size is measured by the number of edges rather than vertices.

In Section 3 we prove the constant dichotomy and the Fibonacci dichotomy for binary classes of objects. In Section 4 we pose some open problems on growths of ideals of permutations and graphs and give some concluding comments.

To conclude let us review some notation. We denote $\mathbf{N} = \{1, 2, \ldots\}$, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbf{N}_0$, and $[m, n] = \{m, m+1, m+2, \ldots, n\}$ for integers $0 \le m \le n$. For $m > n$ we set $[m, n] = [0] = \emptyset$. If A, B are subsets of $\mathbf{N}_0$, $A < B$ means that $x < y$ for all $x \in A, y \in B$. In the case of one-element set we write $x < B$ instead of $\{x\} < B$. For a set X and $k \in \mathbf{N}$ we write $\binom{X}{k}$ for the set of all k-element subsets of X.

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2 Binary classes of objects and their examples

We introduce a general framework for ideals of structures and illustrate it by several examples.

Definition 2.1 *A* class of objects $\mathcal{O}$ *is given by the following data.*

1. *A countably infinite poset* $(\mathcal{O}, \preceq)$ *that has the least element* $0_{\mathcal{O}}$. *The elements of* $\mathcal{O}$ *are called* objects. *We denote the set of atoms of* $\mathcal{O}$ *(the objects* K *such that* $L \prec K$ *implies* $L = 0_{\mathcal{O}}$*) by* $\mathcal{O}_1$. $\mathcal{O}_1$ *is assumed to be finite.*

2. *Finite and mutually disjoint sets* $\mathrm{Em}(K, L)$ *that are associated with every pair of objects* K, L *and satisfy* $|\mathrm{Em}(0_{\mathcal{O}}, K)| = 1$ *for every* K *and* $\mathrm{Em}(K, L) = \emptyset \iff K \not\preceq L$. *The elements of* $\mathrm{Em}(K, L)$ *are called* embeddings of K in L.

3. *A binary operation* $\circ$ *on embeddings such that* $f \circ g$ *is defined whenever* $f \in \mathrm{Em}(K, L)$ *and* $g \in \mathrm{Em}(L, M)$ *for* $K, L, M \in \mathcal{O}$ *and the result is an embedding of* K *in* M. *This operation is associative and has unique left and right neutral elements* $\mathrm{id}_K \in \mathrm{Em}(K, K)$. *It is called a* composition of embeddings.

4. *For every object* $K \in \mathcal{O}$ *we define*
$$A(K) = \bigcup_{L \in \mathcal{O}_1} \mathrm{Em}(L, K)$$
and call the elements of $A(K)$ atoms of K. *Each set* $A(K)$ *is linearly ordered by* $\le_K$. *These linear orders are preserved by the composition: If* $f_1, f_2 \in A(K)$ *and* $g \in \mathrm{Em}(K, M)$ *for* $K, M \in \mathcal{O}$, *then* $f_1 \le_K f_2 \iff f_1 \circ g \le_M f_2 \circ g$.

Note that the set $\mathcal{O}_1$ is an antichain in $(\mathcal{O}, \preceq)$ and that the sets of atoms $A(K)$ are finite. To simplify notation, we will use just one symbol $\preceq$ to denote containments in different classes of objects. It follows from the definition that in a class of objects $\mathcal{O}$ we have $A(0_{\mathcal{O}}) = \emptyset$ and $A(K) = \{\mathrm{id}_K\}$ for every atom $K \in \mathcal{O}_1$. Every embedding $f \in \mathrm{Em}(K, L)$ induces an increasing injection I_f from $(A(K), \leq_K)$ to $(A(L), \leq_L)$: $I_f(g) = g \circ f$. For an object K we define its *size* $|K|$ to be the number $|A(K)|$ of its atoms. An *ideal* in $\mathcal{O}$ is any subset $X \subset \mathcal{O}$ that is a lower ideal in $(\mathcal{O}, \preceq)$, i.e., $K \preceq L \in X$ implies $K \in X$. For $n \in \mathbf{N}_0$ we denote

$$X_n = \{K \in X : \ |K| = |A(K)| = n\}.$$

Thus $X_0 = \{0_{\mathcal{O}}\}$. We are interested in the growth rate of the function $n \mapsto |X_n|$ for ideals X in $\mathcal{O}$.

We postulate the property of binarity.

Definition 2.2 *We call a class of objects $(\mathcal{O}, \preceq)$ given by Definition 2.1* binary *if the following three conditions are satisfied.*

1. *The set $\mathcal{O}_2 = \{K \in \mathcal{O} : \ |K| = 2\}$ of objects with size 2 is finite.*

2. *For any object K and any two-element subset $B \subset A(K)$ the set $R(K, B) = \{L \in \mathcal{O}_2 : \ \exists f \in \mathrm{Em}(L, K), I_f(A(L)) = B\}$ is nonempty and $(R(K, B), \preceq)$ has the maximum element M. We say that M is the* restriction of K to B *and write $M = K|B$.*

3. *For any object K, subset $B \subset A(K)$, and function $h : \binom{B}{2} \to \mathcal{O}_2$ such that $h(C) \preceq K|C$ for every $C \in \binom{B}{2}$, there is a* unique *object L with size $|L| = |B|$ such that $L|C = h(F(C))$ for every $C \in \binom{A(L)}{2}$ where F is the unique increasing bijection from $(A(L), \leq_L)$ to $(B, \leq_K)$. Moreover, for this unique L there is an embedding $f \in \mathrm{Em}(L, K)$ such that $I_f = F$ (in particular, $L \preceq K$).*

Condition 3 implies that every $K \in \mathcal{O}$ is uniquely determined by the restrictions to two-element sets of its atoms (set $B = A(K)$ and $h(C) = K|C$). In particular, in a binary class of objects every set $\mathcal{O}_n$ is finite. If $B \subset A(K)$ and $h(C) = K|C$ for every $C \in \binom{B}{2}$, we call the unique L a *restriction of K to B* and denote it $L = K|B$. The full strength of condition 3 for $B \subset A(K)$ and $h(C) \preceq K|C$ is used in the proofs of Propositions 2.3 and 2.5.

Proposition 2.3 *In a binary class of objects $(\mathcal{O}, \preceq)$, for any two objects K and L we have $K \preceq L$ if and only if there is an increasing injection F from $(A(K), \leq_K)$ to $(A(L), \leq_L)$ satisfying $K|B \preceq L|F(B)$ for every $B \in \binom{A(K)}{2}$.*

Proof. If $K \preceq L$, there exists an $f \in \mathrm{Em}(K, L)$ and by 2 of Definition 2.2 the mapping $F = I_f$ has the stated property. In the other way, if F is as stated, we define $h : \binom{F(A(K))}{2} \to \mathcal{O}_2$ by $h(C) = K|F^{-1}(C)$ and apply 3 of Definition 2.2 to L, $F(A(K))$, and h. The object ensured by it must be equal to K and thus $K \preceq L$. $\square$

The main and in fact the only one family of binary classes of objects is given in the following definition.

Definition 2.4 *Let $P = (P, \leq_P)$ be a finite poset. The class of* edge P-colored complete graphs $\mathcal{C}(P)$ *is the set of all pairs (n, χ), where $n \in \mathbf{N}_0$ and χ is a coloring $\chi : \binom{[n]}{2} \to P$. The containment $(\mathcal{C}(P), \preceq)$ is defined by $(m, \phi) \preceq (n, \chi)$ iff there exists an increasing mapping $f : [m] \to [n]$ such that for every $1 \leq i < j \leq m$ we have $\phi(\{i, j\}) \leq_P \chi(\{f(i), f(j)\})$.*

To show that $(\mathcal{C}(P), \preceq)$ is a binary class of objects one has to specify what are the embeddings, the composition $\circ$, and the linear orders on the sets of atoms, and one has to check that they satisfy the conditions in Definitions 2.1 and 2.2. This is easy because we modeled Definitions 2.1 and 2.2 to fit $(\mathcal{C}(P), \preceq)$. The least element $0_{\mathcal{C}(P)}$ is the pair $(0, \emptyset)$. There is just one atom $(1, \emptyset)$. The embeddings are the increasing mappings f of Definition 2.4 and $\circ$ is the usual composition of mappings. If $K = (n, \chi) \in \mathcal{C}(P)$, it is convenient to identify $A(K)$ with $[n]$. Then $\leq_K$ is the restriction of the standard ordering of integers. It is clear that the conditions of Definition 2.1 (properties of embeddings, properties of $\circ$ and the compatibility of the orders $\leq_K$ and $\circ$) are satisfied. For $K = (n, \chi) \in \mathcal{C}(P)$ and $B \subset [n] = A(K)$, $B = \{a, b\}$ with $a < b$, the restriction $K|B$ is $([2], \psi)$ where $\psi(\{1, 2\}) = \chi(\{a, b\})$. The conditions of Definition 2.2 are easily verified.

It follows from these definitions that every binary class of objects $(\mathcal{O}, \preceq)$ is isomorphic to an ideal in some $(\mathcal{C}(P), \preceq)$, up to the trivial distinction that we may have $|\mathcal{O}_1| > 1$ while always $|\mathcal{C}(P)_1| = 1$.

Proposition 2.5 *For every binary class of objects $(\mathcal{O}, \preceq)$ there is a finite poset $P = (P, \leq_P)$ and a mapping F from $(\mathcal{O}, \preceq)$ to $(\mathcal{C}(P), \preceq)$ with the following properties.*

1. *F is size-preserving.*
2. *$K \prec L \iff F(K) \prec F(L)$ for every $K, L \in \mathcal{O}$.*
3. *F sends all size 1 objects to $(1, \emptyset)$ but otherwise is injective.*
4. *$F(\mathcal{O})$ is an ideal in $(\mathcal{C}(P), \preceq)$.*

Proof. We set $(P, \leq_P) = (\mathcal{O}_2, \preceq)$; P is finite by 1 of Definition 2.2. If $K \in \mathcal{O}$ is an object with atoms $A(K) = \{a_1, a_2, \ldots, a_n\}_{\leq_K}$, we define F by $F(K) = (n, \chi)$ where $n = |K|$ and, for every $1 \leq i < j \leq n$, $\chi(\{i, j\}) = K|\{a_i, a_j\}$. F is clearly size-preserving. Also Property 3 is obvious. Property 2 was proved in Proposition 2.3. We prove Property 4. Suppose that $(m, \psi) \preceq (n, \chi) = F(K)$ for some $(m, \psi) \in \mathcal{C}(P)$ and $K \in \mathcal{O}$. Let $A(K) = \{a_1, a_2, \ldots, a_n\}_{\leq_K}$. We take an increasing injection $g : [m] \to [n]$ such that $\psi(\{i, j\}) \leq_P \chi(\{g(i), g(j)\}) = K|\{a_{g(i)}, a_{g(j)}\}$. By 3 of Definition 2.2 (applied to K, $B = g([m])$, and the h given by $h(C) = \psi(g^{-1}(C))$), there is an object L, $A(L) = \{b_1, b_2, \ldots, b_m\}_{\leq_L}$, such that $L|\{b_i, b_j\} = \psi(\{i, j\})$ for every $1 \leq i < j \leq m$. Hence $(m, \psi) = F(L) \in F(\mathcal{O})$ and Property 4 is proved. $\square$

Thus ideals in a binary class of objects are de facto ideals in $(\mathcal{C}(P), \preceq)$ for some finite poset P and it suffices to consider just the classes of objects $(\mathcal{C}(P), \preceq)$.

The next two results are useful for simplifying proofs of statements on growths of ideals in $(\mathcal{C}(P), \preceq)$. By a *discrete poset* D_P on the set P we understand $(P, =)$, i.e., the poset on P where the only comparisons are equalities.

Proposition 2.6 *Let $P = (P, \leq_P)$ be a finite poset and D_P be the discrete poset on the same set P. Then an ideal in $(\mathcal{C}(P), \preceq)$ remains an ideal in $(\mathcal{C}(D_P), \preceq)$.*

Proof. Let $X \subset \mathcal{C}(P)$ be an ideal in $(\mathcal{C}(P), \preceq)$ and let $(m, \psi) \preceq (n, \chi)$ in $(\mathcal{C}(D_P), \preceq)$ for some $(m, \psi) \in \mathcal{C}(P)$ and $(n, \chi) \in X$. By the definitions, then $(m, \psi) \preceq (n, \chi)$ in $(\mathcal{C}(P), \preceq)$. So $(m, \psi) \in X$ and X is an ideal in $(\mathcal{C}(D_P), \preceq)$ too. $\square$

Thus any general result on ideals in $(\mathcal{C}(D_P), \preceq)$ applies to ideals in $(\mathcal{C}(P), \preceq)$ and in many situations it suffices to consider only the simple discrete poset D_P.

If $P = (P, \leq_P)$ is a finite poset, $b \in P$ is a color, and $D_2 = ([2], =)$ is the two-element discrete poset, we define a mapping $R_b : \ \mathcal{C}(P) \to \mathcal{C}(D_2)$ by $R_b((n, \chi)) = (n, \psi)$ where $\psi(\{i, j\}) = 1 \iff \chi(\{i, j\}) = b$, i.e., we recolor edges colored b by 1 and to all other edges give color 2.

Proposition 2.7 *Let X be an ideal in $(\mathcal{C}(P), \preceq)$, where $P = (P, \leq_P)$ is a finite poset. Then, for every $b \in P$, the recolored complete graphs $Y^{(b)} = R_b(X)$ form an ideal in $(\mathcal{C}(D_2), \preceq)$, and for every $n \geq 1$ and every color $c \in P$ we have the estimate*

$$|Y_n^{(c)}| \leq |X_n| \leq \prod_{b \in P} |Y_n^{(b)}|.$$

Proof. Let $K^* \preceq R_b(L)$ in $(\mathcal{C}(D_2), \preceq)$, where $L \in \mathcal{C}(P)$. Returning to the original colors, we see that there is a $K \in \mathcal{C}(P)$ such that $R_b(K) = K^*$ and $K \preceq L$ (even in $(\mathcal{C}(D_P), \preceq)$). This gives the first assertion. The first inequality is trivial because the mapping R_b is size-preserving. The second inequality follows from the fact that every $K \in \mathcal{C}(P)$ is uniquely determined by the tuple of values $(R_b(K) : \ b \in P)$. $\square$

We say that a family $\mathcal{F}$ of functions from $\mathbf{N}$ to $\mathbf{N}_0$ is *product-bounded* if for any k functions $f_1, f_2, \ldots, f_k$ from $\mathcal{F}$ there is a function f in $\mathcal{F}$ such that

$$f_1(n) f_2(n) \ldots f_k(n) \leq f(n)$$

holds for all $n \geq 1$. Bounded functions, polynomially bounded functions, and exponentially bounded functions are all examples of product-bounded families. On the other hand, the family of functions which are, for example, $O(3^n)$ is not product-bounded.

Corollary 2.8 *Let $\mathcal{F}$ be a product-bounded family of functions and let $g : \ \mathbf{N} \to \mathbf{N}_0$. Suppose that for every ideal X in $(\mathcal{C}(D_2), \preceq)$, where D_2 is the two-element discrete poset, we have either $|X_n| \leq f(n)$ for all $n \geq 1$ for some $f \in \mathcal{F}$ or $|X_n| \geq g(n)$ for all $n \geq 1$. Then this dichotomy holds for ideals in every class $(\mathcal{C}(P), \preceq)$ for every finite poset P.*

Proof. If X is an ideal in $(\mathcal{C}(P), \preceq)$ and, for $b \in P$, $Y^{(b)}$ denotes $R_b(X)$, then either for some $b \in P$ we have $|X_n| \geq |Y_n^{(b)}| \geq g(n)$ for all $n \geq 1$ or for every $b \in P$ we have $|Y_n^{(b)}| \leq f_b(n)$ for all $n \geq 1$ with certain functions $f_b \in \mathcal{F}$. By the assumption on $\mathcal{F}$ and the inequality in Proposition 2.7, in the latter case we have $|X_n| \leq \prod_{b \in P} f_b(n) \leq f(n)$ for all $n \geq 1$ for a function $f \in \mathcal{F}$. $\square$

We see that to prove for $(\mathcal{C}(P), \preceq)$ an $\mathcal{F}$-g dichotomy (jump in growth) with a product bounded family $\mathcal{F}$, it suffices to prove it only in the case $P = D_2$, that is, in the case of graphs with $\preceq$ being the ordered induced subgraph relation. This is the case for the slightly weaker version of the constant dichotomy (with $|X_n| \leq c$ instead of $|X_n| = c$ for $n > n_0$) and for the Fibonacci dichotomy. On the other hand, the Fibonacci hierarchy, which is an infinite series of dichotomies, is a finer result and Corollary 2.8 does not apply to it because the corresponding families of functions are not product-bounded.

We conclude this section with several examples of binary classes of objects. Our objects are always structures with groundsets $[n]$ for n running through $\mathbf{N}_0$ and the containment $\preceq$ is defined by the existence of a structure-preserving increasing mapping. Embeddings are these mappings and the composition $\circ$ is the usual composition of mappings. With the exception of Examples 7, 8, and 9, the atoms of an object can be identified with the elements of its groundset and its size is the cardinality of the groundset. We will not repeat these features of $(\mathcal{O}, \preceq)$ in every example and we also omit verifications of the conditions of Definitions 2.1 and 2.2 which are easy. With the exception of Example 6, each set $R(K, B)$ of 2 of Definition 2.2 has only one element and condition 2 is satisfied automatically. In every example we mention what is the poset $(P, \leq_P) = (\mathcal{O}_2, \preceq)$ (see Proposition 2.5). It is the discrete ordering $D_k = ([k], =)$ for some k, with exception of Example 6 where it is the linear ordering $L_2 = ([2], \leq)$. In Example 6 the sets $R(K, B)$ have one or two elements. In Examples 7, 8, and 9 the atoms are edges rather than vertices and the size of an object is the number of its edges.

Example 1. Permutations. $\mathcal{O}$ is the set of all finite permutations, which are the bijections $\rho : [n] \to [n]$ where $n \in \mathbf{N}_0$. For two permutations $\pi : [m] \to [m]$ and $\rho : [n] \to [n]$, we define $\pi \preceq \rho$ iff there is an increasing mapping $f : [m] \to [n]$ such that $\pi(i) < \pi(j) \iff \rho(f(i)) < \rho(f(j))$; this is just a reformulation of the definition given in the beginning of Section 1. There is only one atom, the 1-permutation, and $\mathcal{O}_2$ consists of the two 2-permutations. $(P, \leq_P)$ is the discrete ordering D_2. By Proposition 2.5, permutations form an ideal in $(\mathcal{C}(D_2), \preceq)$. It is defined by the ordered transitivity of both colors: if $x < y < z$ and $\{x, y\}$ and $\{y, z\}$ are colored $c \in [2]$, then $\{x, z\}$ is colored c as well.

Example 2. Signed permutations. We enrich permutations $\rho : [n] \to [n]$ by coloring the elements of the definition domain $[n]$ white ($+$) and black ($-$), and we require that the embeddings f are in addition color-preserving. There are two atoms and $\mathcal{O}_2$ consists of eight signed 2-permutations. $(P, \leq_P)$ is the discrete ordering D_8.

Example 3. Ordered words. $\mathcal{O}$ consists of all mappings $q : [n] \to [n]$ such that the image of q is $[m]$ for some $m \leq n$. For two such mappings $p : [m] \to [m]$ and $q : [n] \to [n]$

we define $p \preceq q$ in the same way as for permutations. The elements of $(\mathcal{O}, \preceq)$ can be viewed as words $u = b_1 b_2 \dots b_n$ such that $\{b_1, b_2, \dots, b_n\} = [m]$ for some $m \le n$, and $u \preceq v$ means that v has a subsequence with the same length as u whose entries form the same pattern (with respect to $<, >, =$) as u. There is one atom and $\mathcal{O}_2$ consists of three elements (12, 21, and 11). $(P, \le_P)$ is the discrete ordering D_3.

Example 4. Set partitions. $\mathcal{O}$ consists of all partitions $([n], \sim)$ where $\sim$ is an equivalence relation on $[n]$. We set $([m], \sim_1) \preceq ([n], \sim_2)$ iff there is a subset $B = \{b_1, b_2, \dots, b_m\}_<$ of $[n]$ such that $b_i \sim_2 b_j \iff i \sim_1 j$. There is only one atom and $\mathcal{O}_2$ has two elements. $(P, \le_P)$ is the discrete ordering D_2. By Proposition 2.5, partitions form an ideal in $(\mathcal{C}(D_2), \preceq)$. It is defined by the transitivity of the color c corresponding to the partition of [2] with 1 and 2 in one block: If x, y, z are three distinct elements of $[n]$ such that $\{x, y\}$ and $\{y, z\}$ are colored c, then $\{x, z\}$ is colored c as well. To put it differently, set partitions can be represented by ordered graphs whose components are complete graphs. Pattern avoidance in set partitions was investigated by Klazar [20], for further results see Goyt [16] and Sagan [27].

Example 5. Ordered induced subgraph relation. $\mathcal{O}$ is the set of all simple graphs with vertex set $[n]$. For two graphs $G_1 = ([n_1], E_1)$ and $G_2 = ([n_2], E_2)$ we define $G_1 \preceq G_2$ iff there is an increasing mappings $f : [n_1] \to [n_2]$ such that $\{x, y\} \in E_1 \iff \{f(x), f(y)\} \in E_2$. Thus $\preceq$ is the ordered induced subgraph relation. There is only one atom and $\mathcal{O}_2$ has two elements. $(P, \le_P)$ is is the discrete ordering D_2. This class essentially coincides with $(\mathcal{C}(D_2), \preceq)$.

Example 6. Ordered subgraph relation. We take $\mathcal{O}$ as in the previous example and in the definition of $\preceq$ we change $\iff$ to $\Longrightarrow$. Thus $\preceq$ is the ordered subgraph relation. There is only one atom and $\mathcal{O}_2$ has two elements. Unlike in other examples, $(\mathcal{O}_2, \preceq)$ is not a discrete ordering but the linear ordering L_2. Every set $R(K, B)$, where K is a graph and B is a two-element set of its vertices (atoms), has one or two elements and $(R(K, B), \preceq)$ is L_1 or L_2. Thus $(P, \le_P)$ is the linear ordering L_2. This class essentially coincides with $(\mathcal{C}(L_2), \preceq)$.

Example 7. Ordered graphs counted by edges. Let $\mathcal{O}$ be the set of simple graphs with the vertex set $[n]$ and without isolated vertices, and let $\preceq$ be the ordered subgraph relation (as in the previous example). There is one atom corresponding to the single edge graph. The size of $G = ([n], E)$ is now $|E|$, the number of edges. $\mathcal{O}_2$ has six elements and $(\mathcal{O}_2, \preceq)$ is D_6. The linear ordering $\le_G$ on E, the set of atoms of $G = ([n], E)$, is the restriction of the lexicographic ordering $\le_l$ on $\binom{\mathbf{N}}{2}$: $e_1 \le_l e_2 \iff \min e_1 < \min e_2$ or $(\min e_1 = \min e_2 \;\&\; \max e_1 < \max e_2)$. It is clear that $\le_l$ is compatible with the embeddings, which are increasing mappings between vertex sets sending edges to edges, and so condition 4 of Definition 2.1 is satisfied. Let us check the conditions of Definition 2.2. Conditions 1 and 2 are clearly satisfied and we have to check condition 3.

Proposition 2.9 *Let $G = ([s], E)$ be a simple graph without isolated vertices and $B = \{e_1, e_2, \dots, e_n\}_{\le_l}$ be a subset of E. There exists a unique simple graph $H = ([r], F)$,*

$F = \{f_1, f_2, \ldots, f_n\}_{\leq_l}$, of size n without isolated vertices such that $G|\{e_i, e_j\} = F|\{f_i, f_j\}$ for every $1 \leq i < j \leq n$. Moreover, there is an increasing mapping $m : [r] \to [s]$ such that $m(f_i) = e_i$ for every $1 \leq i \leq n$.

Proof. H is obtained from B by relabeling the vertices in $V = \bigcup_{e \in B} e$, $|V| = r$, using the unique increasing mapping from V to $[r]$. To construct the mapping m, we take the unique $\leq_l$-increasing mapping $M : F \to E$ sending F to B and for a vertex $x \in [r]$ we take an arbitrary edge $f \in F$ with $x \in f$ (since x is not isolated, f exists) and define $m(x) = \min M(f)$ if $x = \min f$ and $m(x) = \max M(f)$ if $x = \max f$. Since M preserves types of pairs of edges, the value $m(x)$ does not depend on the selection of f. Also, m sends f_i to e_i and is increasing. The image of each such mapping m is $\bigcup_{e \in B} e$ and m is unique. If H' is another graph with the stated property and m' is the corresponding mapping, $m \circ (m')^{-1}$ and $m' \circ m^{-1}$ give an ordered isomorphism between H and H'. Thus H is unique. $\square$

We see that simple ordered graphs without isolated vertices, with the ordered subgraph relation and with size being measured by the number of edges, form a binary class of objects. $(P, \leq_P)$ is the discrete ordering D_6.

Example 8. Ordered multigraphs counted by edges. Let $\mathcal{O}$ be the set of multigraphs with the vertex set $[n]$ and without isolated vertices. The containment $\preceq$ is the ordered subgraph relation and size is the number of edges counted with multiplicity. More precisely, in $G = ([m], E) \in \mathcal{O}$ we interpret E as a (multiplicity) mapping $E : \binom{[m]}{2} \to \mathbf{N}_0$, and we have $G = ([m], E) \preceq H = ([n], F)$ iff there is an increasing mapping $f : [m] \to [n]$ and an $\binom{m}{2}$-tuple $\{f_e : e \in \binom{[m]}{2}\}$ of increasing mappings $f_e : [E(e)] \to \mathbf{N}$ such that, for every $e \in \binom{[m]}{2}$, the image of f_e is a subset of $[F(f(e))]$. The embeddings are the pairs $(f, \{f_e : e \in \binom{[m]}{2}\})$ and $\circ$ is composition of mappings, applied to f and to the mappings f_e. There is one atom $([2], E)$, where $E([2]) = 1$, and the size of $G = ([m], E)$ is the total multiplicity $\sum_{e \subset [m], |e|=2} E(e)$. $\mathcal{O}_2$ has seven elements. The set of atoms $A(G)$ of $G = ([m], E)$ can be identified with $\{(e, i) : e \in \binom{[m]}{2}, i \in [E(e)]\}$ and the linear ordering $(A(G), \leq_G)$ is given by $(e, i) \leq_G (e', i')$ iff $e <_l e'$ or $(e = e'$ & $i \leq i')$. The conditions in Definitions 2.1 and 2.2 are verified as in the previous example. Therefore multigraphs form a binary class of objects. $(P, \leq_P)$ is the discrete ordering D_7.

Example 9. Ordered k-uniform hypergraphs counted by edges. For $k \geq 2$, we generalize Example 7 to k-uniform simple hypergraphs $H = ([m], E)$ (so $E \subset \binom{[m]}{k}$) without isolated vertices. The containment $\preceq$ is the ordered subhypergraph relation and size is the number of edges. There is one atom $([k], \{[k]\})$. It is not hard to count that $\mathcal{O}_2$ has

$$r = r(k) = \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\binom{2k-m-1}{k-1} + \frac{1}{2} \binom{2k-m-2}{k-1} \right) - \frac{1}{2}$$

elements. $(P, \leq_P)$ is the discrete ordering D_r.

Example 10. Words with the subsequence relation. For a finite alphabet A, let $\mathcal{O}$ be the set of all words $u = a_1a_2\dots a_n$ over A and $\preceq$ be the subsequence relation, $a_1a_2\dots a_m \preceq b_1b_2\dots b_n$ iff there exists an m-tuple $1 \le j_1 < j_2 < \dots < j_m \le n$ such that $a_i = b_{j_i}$ for $1 \le i \le m$. There are $|A|$ atoms and $\mathcal{O}_2$ has $r = |A|^2$ elements. $(P, \le_P)$ is the discrete ordering D_r.

3 The constant and Fibonacci dichotomies for binary classes of objects

In this section we prove for $(\mathcal{C}(P), \preceq)$ in Theorem 3.1 the constant dichotomy and in Theorem 3.8 the Fibonacci dichotomy. Both proofs can be read independently. P denotes a finite l-element poset on $[l]$ and l is always the number of colors. We work with the class $(\mathcal{C}(P), \preceq)$ of all edge P-colored complete graphs (n, χ), $n \in \mathbf{N}_0$ and $\chi : \binom{[n]}{2} \to [l]$. Recall that

$$(m, \psi) \preceq (n, \chi) \iff \exists \text{ increasing } f : [m] \to [n],\ \psi(e) \le_P \chi(f(e))\ \forall e \in \binom{[m]}{2}.$$

Let $K = (n, \chi)$ be a coloring. The *reversal* of K is the coloring (n, ψ) where $\psi(\{i, j\}) = \chi(\{n-i+1, n-j+1\})$. If $A \subset [n]$ and $\chi|\binom{A}{2}$ is constant, we call A a (χ-) *homogeneous* or (χ-) *monochromatic* set. We denote by $R(a; l)$ the Ramsey number for pairs and l colors; $R(a; l)$ is the smallest $n \in \mathbf{N}$ such that every coloring $\chi : \binom{[n]}{2} \to [l]$ has a χ-homogeneous set $A \subset [n]$ with size $|A| = a$ (Ramsey [25], Graham, Spencer and Rothschild [17], Nešetřil [24]).

Theorem 3.1 *If X is an ideal in $(\mathcal{C}(P), \preceq)$ then either $|X_n|$ is constant for all $n > n_0$ or $|X_n| \ge n$ for all $n \ge 1$.*

By Proposition 2.6, it suffices to prove this if P is a discrete ordering D_P. We cannot use Corollary 2.8 to reduce the situation to two colors because we want to prove a result stronger than $|X_n| \ll 1$ but the argument for l colors is not too much harder than for two. We need some definitions and auxiliary results.

We say that a coloring (n, χ) is *r-rich*, where $r \ge 1$ is an integer, if $n = 2r - 1$ and one the following two conditions holds. In *type 1* r-rich coloring, in (n, χ) or in its reversal we have $\chi(\{i, i+1\}) = a$ for $1 \le i \le r-1$ and $\chi(\{r, r+1\}) = b$ for two colors $a \ne b$. In *type 2* r-rich coloring, in (n, χ) or in its reversal we have $\chi(\{1, i\}) = a$ for $2 \le i \le r$ and $\chi(\{1, r+1\}) = b$ for two colors $a \ne b$. We impose no restriction on colors of the remaining $\binom{n}{2} - r$ edges.

Lemma 3.2 *If the ideal X contains for every $r \ge 1$ an r-rich coloring then $|X_n| \ge n$ for all $n \ge 1$.*

Proof. If $K = (n, \chi) \in X$ is r-rich of type 1, the r restrictions of K to $[i, i+r-1]$ (or to $[n-i-r+2, n-i+1]$ if K is reversed) for $1 \le i \le r$ are mutually distinct and show that $|X_r| \ge r$. The argument for type 2 r-rich colorings is similar. $\square$

Note that the containment of an r-rich coloring for all $r \ge 1$ is equivalent with the containment for infinitely many r because every r-rich coloring contains an s-rich coloring for $s = 1, 2, \dots, r$.

We say that a coloring (n, χ) is r-*simple*, where $r \ge 1$ is an integer, if $[r+1, n-r]$ is χ-homogeneous and for every fixed $v \in [r] \cup [n-r+1, n]$ the $n-4r$ edges $\{v, w\}$, $w \in [2r+1, n-2r]$, have in χ the same color. By the definition, every coloring (n, χ) with $n \le 2r+2$ is r-simple. We say that a set X of colorings is r-simple if each coloring in X is r-simple.

Lemma 3.3 *If an ideal of colorings X is r-simple then there is a constant $d \in \mathbf{N}_0$ such that $|X_n| = d$ for all $n > n_0$.*

Proof. Colorings which are r-simple enjoy this property: If $n \ge 4r+2$ and (n, χ_1) and (n, χ_2) are two distinct r-simple colorings, then their restrictions to $[n] \backslash \{2r+1\}$ are also distinct. Thus for all $n \ge 4r+2$ the restrictions of the colorings in X_n to $[n] \backslash \{2r+1\}$ are mutually distinct and show that $|X_{n-1}| \ge |X_n|$, which implies the claim. $\square$

Theorem 3.1 now follows from the next proposition.

Proposition 3.4 *For every $r \in \mathbf{N}$ there is a constant $c = c(r) \in \mathbf{N}$ such that every ideal of colorings contains an r-rich coloring or is c-simple.*

Proof of Theorem 3.1. Let X be an ideal in $\mathcal{C}(P)$. If X contains an r-rich coloring for every $r \ge 1$, then $|X_n| \ge n$ for every n by Lemma 3.2. If not, then by Proposition 3.4 X is c-simple for some c and by Lemma 3.3 $|X_n|$ is constant from some n on. $\square$

For the proof of Proposition 3.4 we shall need three lemmas on situations forcing appearance of r-rich colorings.

Lemma 3.5 *Let $r \ge 1$ be an integer, (n, χ) be a coloring, $A \subset [n]$ be a χ-homogeneous set with the maximum cardinality, and $A' \subset A$ be obtained from A by deleting the first $2r-2$ and the last $2r-2$ elements. Suppose further that A' is not an interval in $[n]$. Then (n, χ) contains an r-rich coloring.*

Proof. We denote the set of the first (last) $r-1$ elements of A by B_1 (B_2) and the set of the first (last) $2r-2$ elements of A by C_1 (C_2). The assumption on $A' = A \backslash (C_1 \cup C_2)$ implies that there is an $x \in [n] \backslash A$ such that $C_1 < x < C_2$. Since $|A|$ is maximum, there is a $y \in A$ such that the color of $\{x, y\}$ is different from the color of the edges lying in A. If $y \in B_1$, then y, $C_1 \backslash B_1$, x, and the next $r-2$ elements of A after x form (with restricted χ) an r-rich coloring of type 2. If $y \notin B_1$ but $y < x$, then B_1, y, x, and the next $r-2$ elements of A after x form an r-rich coloring of type 1. The case when $y > x$ is symmetric and is treated similarly. $\square$

Lemma 3.6 *Let $r \geq 1$ be an integer, (n, χ) be a coloring, s be the maximum size of a χ-homogeneous subset of $[n]$, $A \subset [n]$ be a χ-homogeneous set with $|A| = s - (4r - 4)$, $B \subset [n]$ be a χ-homogeneous set with $A < B$, and let $|A| \geq 2r, |B| \geq 6r$. Then (n, χ) contains an r-rich coloring.*

Proof. Let a, respectively b, be the color of the edges lying in A, respectively in B. If $a \neq b$, then the last r elements of A and the first $r-1$ elements of B, or the first r elements of B and the last $r-1$ elements of A form an r-rich coloring of type 1 (depending on whether $\chi(\{\max A, \min B\})$ differs from a or from b).

Suppose that $a = b$. Since $|A| + |B| > s$, $A \cup B$ is not homogeneous and there exist $x \in A$ and $y \in B$ such that $\chi(\{x, y\}) \neq a$. Let $y - x$ be minimum, that is, if $x \leq x' \in A$, $B \ni y' \leq y$, and at least one inequality is strict, then $\chi(\{x', y'\}) = a$. We show that any position of x and y produces an r-rich coloring. We denote by C_1 (C_2) the set of the first (last) $r - 2$ elements of A (B). Suppose first that $x \notin C_1$. If y is among the last $r - 1$ elements of B, then y, the previous $r - 1$ elements of B, x, and C_1 form an r-rich coloring of type 2. If y is not among the last $r - 1$ elements of B, then these elements, y, x, and C_1 form an r-rich coloring of type 1. A symmetric argument shows that if $y \notin C_2$ then we have an r-rich coloring. The remaining case when $x \in C_1$ and $y \in C_2$ does not occur because then, by the minimality of the length of $\{x, y\}$, $(A \backslash C_1) \cup (B \backslash C_2)$ would be a homogeneous set with size $|A| + |B| - 2(r - 2) \geq |A| + 4r + 4 = s + 8$, contradicting the definition of s. □

Lemma 3.7 *Let $r \geq 1$ be an integer, (n, χ) be a coloring, $v \in [n]$, $A \subset [n]$ be a set such that $v < A$ or $v > A$, and $B \subset A$ be obtained from A by the deletion of the first $l(r-2)+1$ and the last $l(r - 2) + 1$ elements. Suppose further that not all edges $\{v, w\}$, $w \in B$, have in χ the same color. Then (n, χ) contains an r-rich coloring.*

Proof. Let $v < A$, the proof for $v > A$ is very similar. By the assumption there is a $w \in B$ such that $b = \chi(\{v, w\}) \neq a = \chi(\{v, \max B\})$. By the pigeonhole principle, some $r - 1$ edges $\{v, z_1\}, \ldots, \{v, z_{r-1}\}$, where $z_i \in A$ and $z_i < B$, have the same color c. Thus v, $z_1, \ldots, z_{r-1}$, w or $\max B$ (depending on whether $c \neq b$ or $c \neq a$), and the last $r - 2$ elements of A form an r-rich coloring of type 2. □

Proof of Proposition 3.4. We assume that X is an ideal in $\mathcal{C}(P)$ which contains no r-rich coloring for some $r \geq 2$. We show that then X must be c-simple for

$$c = \max(R(6r; l), l(r - 2) + 1)$$

where $R(\cdot; \cdot)$ is the Ramsey number. Let $(n, \chi) \in X$ be arbitrary. We may assume that $n > 2c + 2$. We take a set $A \subset [n]$ obtained from a χ-homogeneous subset with the maximum cardinality by deleting the first $2r - 2$ and the last $2r - 2$ elements. By Lemma 3.5, the Ramsey theorem, and Lemma 3.6, A is an interval in $[n]$ and $\min A < c+1$, $\max A > n-c$. Thus $[c+1, n-c]$ is χ-homogeneous. Let $v \in [c] \cup [n-c+1, n]$ be arbitrary.

By Lemma 3.7 applied to v and $[c+1, n-c]$, all edges $\{v, w\}$, $w \in [2c+1, n-2c]$, must have the same color. Thus (n, χ) is c-simple. $\square$

This completes the proof of Theorem 3.1.

Theorem 3.8 *If X is a ideal in $(\mathcal{C}(P), \preceq)$ then either $|X_n| \le n^c$ for all $n \ge 1$ for a constant $c > 0$, or $|X_n| \ge F_n$ for all $n \ge 1$ where $(F_n)_{n\ge1} = (1, 2, 3, 5, 8, 13, \ldots)$ are the Fibonacci numbers.*

As in the proof of Theorem 3.1, we define "wealthy" colorings (of four types) and "tame" colorings and show that colorings with unbounded wealth produce growth at least F_n and that bounded tameness admits only polynomially many colorings. The proof is completed by showing that the colorings in any ideal are either unboundedly wealthy or boundedly tame. By Corollary 2.8 and the following remark, it suffices to prove the theorem only for the two-element discrete poset $P = D_2$, that is, for graphs and ordered induced subgraph relation. To make explicit the symmetry between edges and nonedges in this case, we prefer to use the language of colorings. Therefore by a *coloring* we shall mean in the proof always a black-white edge coloring (n, χ) of a complete graph, $\chi : \binom{[n]}{2} \to \{black, white\}$, and if we use two distinct colors c, d, one should bear in mind that $\{c, d\} = \{black, white\}$.

Let $r \in \mathbf{N}$. A coloring $K = (r, \chi)$ is *r-wealthy of type 1* if in K or in its reversal we have $\chi(\{1, i\}) \ne \chi(\{1, i+1\})$ for all $i \in [2, r-1]$. $K = (3r, \chi)$ is *r-wealthy of type 2* if none of the r consecutive triangles $\{3i-2, 3i-1, 3i\}$, $1 \le i \le r$, is χ-homogeneous. We use two incarnations of the Fibonacci number F_n.

Lemma 3.9 *(i) F_n equals to the number of 0-1 strings $s_1s_2\ldots s_{n-1}$ with no two consecutive 1s, i.e., avoiding the pattern $s_is_{i+1} = 11$. (ii) F_n equals to the number of 0-1 strings $s_1s_2\ldots s_{n-1}$ avoiding the patterns $s_{2i-1}s_{2i} = 01$ and $s_{2i}s_{2i+1} = 10$.*

Proof. Both results are easily proved by induction on n. $\square$

We call the strings in (i) *fib1 strings* and the strings in (ii) *fib2 strings*.

Lemma 3.10 *If there is an $i \in \{1, 2\}$ so that the ideal X contains for every $r \ge 1$ an r-wealthy coloring of type i, then $|X_n| \ge F_n$ for all $n \ge 1$.*

Proof. If X contains for every $r \ge 1$ an r-wealthy coloring of type 1, it follows that for every $n \in \mathbf{N}$ and every subset $A \subset [2, n]$ there exists a coloring $K_A = (n, \chi)$ in X such that $\chi(\{1, i\}) = black \iff i \in A$, or the same holds for the reversals of K_As. Because for fixed n all 2^{n-1} colorings K_A are mutually distinct, we have $|X_n| \ge 2^{n-1} \ge F_n$.

Suppose that X contains for every $r \ge 1$ an r-wealthy coloring of type 2. Using the pigeonhole principle and the Ramsey theorem, we regularize the situation and obtain the colorings $(3, \phi)$, $(6, \psi)$, and $(3r, \chi_r)$, $r = 1, 2, \ldots$, which all lie in X and are such that in $(3r, \chi_r)$ all triangles $T_i = \{3i-2, 3i-1, 3i\}$, $1 \le i \le r$, and the edges between them are colored in the same way and independently of r: if $\{a, b\} \subset T_i$ then $\chi_r(\{a, b\}) = \phi(\{a-3(i-1), b-3(i-1)\})$ and if $a \in T_i$ and $b \in T_j$, $1 \le i < j \le r$, then $\chi_r(\{a, b\}) =$

$\psi(\{a-3(i-1), b-3(j-2)\})$. The coloring $(3, \phi)$ of the triangles is not monochromatic and thus there is an edge $\{a, b\} \subset T_1$ such that not all of the four edges connecting $\{a, b\}$ and $\{a+3, b+3\}$ have color $c = \phi(\{a, b\})$. It follows that there are colorings $(4, \kappa)$ and $(2r, \lambda_r)$, $r = 1, 2, \ldots$, which lie in X and are such that (i) the edges $\{1, 2\}, \{3, 4\}, \ldots, \{2r-1, 2r\}$ have in λ_r the same color, say black, (ii) if $a \in \{2i-1, 2i\}$ and $b \in \{2j-1, 2j\}$ with $1 \le i < j \le r$, then $\lambda_r(\{a, b\}) = \kappa(\{a-2(i-1), b-2(j-2)\})$, and (iii) at least one of the four edges $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ is in κ colored white. Suppose, for example, that $\{1, 4\}$ is white. If $\{1, 3\}$ is black, it follows that $(2r, \lambda_r)$ contains an r-wealthy coloring of type 1 and we are in the previous case. This argument shows that we may assume that in $(2r, \lambda_r)$ all edges $\{1, 2\}, \{3, 4\}, \ldots, \{2r-1, 2r\}$ are black and all other edges white. It follows that for every $n \in \mathbf{N}$ and every fib1 string $w = s_1 s_2 \ldots, s_{n-1}$ there is a coloring $K_w = (n, \chi) \in X$ such that $\chi(\{i, i+1\}) = \mathit{black} \iff s_i = 1$. Since for distinct ws the corresponding colorings K_w are distinct as well, by (i) of Lemma 3.9 we conclude that $|X_n| \ge F_n$. $\square$

Before defining wealthy colorings of types 3 and 4, we introduce notation on 0-1 matrices which we will use to represent colorings. If M is an $r \times s$ 0-1 matrix, any row and column of M consists of alternating intervals of consecutive 0s and 1s. Let $al(M)$ be the maximum number of these intervals in a row or in a column, taken over all $r + s$ rows and columns. For every $j \in [s]$ we let $C(M, j) \subset [r]$ be the row indices of the lowest entries of these intervals in the j-th column, with r omitted: $a \in C(M, j)$ iff $M(a, j) \ne M(a+1, j)$. We denote $C(M) = \bigcup_{j=1}^{s} C(M, j)$. For a coloring $K = (2r, \chi)$ we define the $r \times r$ 0-1 matrix M_K by $M_K(i, j) = 0$ iff $\chi(\{i, r+j\}) = \mathit{white}$. Similarly, if $K = (n, \chi)$ is a coloring and $I = \{x_1 < x_2 < \ldots < x_r\} < J = \{y_1 < y_2 < \ldots < y_s\}$ are two subsets of $[n]$, we define the $r \times s$ 0-1 matrix $M_{I,J}$ by $M_{I,J}(i, j) = 0$ iff $\chi(\{x_i, y_j\}) = \mathit{white}$; we suppress in notation the dependence on K which will be clear from the context.

We say that $C = (c_1, c_2, \ldots, c_k)$, with $c_i \in [r] \times [s]$ being in the (row,column) coordinates format, is a *southeast path* in an $r \times s$ 0-1 matrix M if in C alternate south and east steps and C starts with a south step: $c_{2i} - c_{2i-1} \in \mathbf{N} \times \{0\}$ and $c_{2i+1} - c_{2i} \in \{0\} \times \mathbf{N}$. If $M = M_K$ or $M = M_{I,J}$ for some coloring K then C corresponds to a path in the coloring, with the k edges

$$\{a_1, b_1\}, \{a_2, b_1\}, \{a_2, b_2\}, \{a_3, b_2\}, \{a_3, b_3\}, \ldots, \{a_p, b_q\}$$

where $p = \lceil (k+1)/2 \rceil$, $q = \lceil k/2 \rceil$, and $1 \le a_1 < a_2 < \ldots < a_p < b_1 < b_2 < \ldots < b_q$. We call such paths *back-and-forth paths.*

If M is an $r \times s$ 0-1 matrix and M' is an $r' \times s'$ 0-1 matrix, we say that M' is contained in M, $M' \preceq M$, if there are increasing injections $f:\ [r'] \to [r]$ and $g:\ [s'] \to [s]$ such that $M(f(i), g(j)) = M'(i, j)$ for all $i \in [r'], j \in [s']$. We say that M' is a *submatrix* of M. If $M' \preceq M$ and $M = M_{I,J}$ for two subsets $I < J$ in $[n]$ and a coloring (n, χ), then there are subsets $I' \subset I$ and $J' \subset J$ such that $M' = M_{I',J'}$. We denote by I_r the $r \times r$ identity matrix with 1s on the main diagonal and 0s elsewhere and by U_r the upper triangular $r \times r$ matrix with 1s above and on the main diagonal and 0s below it. We call two $r \times s$ 0-1 matrices M and M' *similar* if $M' = M$ or M' is obtained from M by the vertical mirror image and/or swapping 0 and 1.

We say that a coloring $K = (2r, \chi)$ is *r-wealthy of type 3* if M_K is similar to I_r. $K = (2r, \chi)$ is *r-wealthy of type 4* if M_K is similar to U_r. Note that for $i \in \{1, 2, 3, 4\}$ and $r \in \mathbf{N}$, every r-wealthy coloring of type i contains an s-wealthy coloring of type i for $s = 1, 2, \ldots, r$ and so for an ideal X to contain an r-wealthy coloring of type i for every $r \geq 1$ is equivalent with containing it for infinitely many r.

Lemma 3.11 *If there is an $i \in \{3, 4\}$ so that the set X contains for every $r \geq 1$ an r-wealthy coloring of type i, then $|X_n| \geq F_n$ for all $n \geq 1$.*

Proof. Let X contain for every $r \geq 1$ an r-wealthy coloring K_r of type 3. We may assume that always $M_{K_r} = I_r$. It can be seen that if $n \in \mathbf{N}$ and $w = s_1 s_2 \ldots s_{n-1}$ is any fib1 string, then for $r \geq 2n$ one can draw in I_r a southeast path $(c_1, c_2, \ldots, c_{n-1})$ such that $I_r(c_i) = s_i$. Thus for every w there is a coloring $K_w = (n, \chi) \in X$ whose unique spanning back-and-forth path is colored according to w. By (i) of Lemma 3.9 we have $|X_n| \geq F_n$.

Let X contain for every $r \geq 1$ an r-wealthy coloring K_r of type 4. We may assume that always $M_{K_r} = U_r$. It can be seen that if $n \in \mathbf{N}$ and $w = s_1 s_2 \ldots s_{n-1}$ is any fib2 string, then for $r \geq 2n$ one can draw in U_r a southeast path $(c_1, c_2, \ldots, c_{n-1})$ such that $U_r(c_i) = s_i$. Again, by (ii) of Lemma 3.9 we have $|X_n| \geq F_n$. $\square$

Lemma 3.12 *Let M be an $r \times s$ 0-1 matrix that satisfies $al(M) \leq k$ and $|C(M)| \leq l$, and a be the number of the column indices $j \in [s]$ for which the j-th column of M differs from the $(j+1)$-th one. Then*

$$a \leq (k-1)(2l+1).$$

Proof. The j-th column of M is uniquely determined by $C(M, j) \subset C(M)$ and by $M(1, j) \in \{0, 1\}$. It follows that any two different columns in M must differ in entries with row index lying in the set $D = C(M) \cup \{i+1 :\ i \in C(M)\} \cup \{1\}$, which has at most $2l+1$ elements. By the pigeonhole principle, if $a > (k-1)(2l+1)$ then there are k column indices $1 \leq j_1 < j_2 < \ldots < j_k < s$ and a row index $b \in D$ such that $M(b, j_i) \neq M(b, j_i+1)$ for all $i \in [k]$. Thus the b-th row of M consists of at least $k+1$ alternating intervals of 0s and 1s, which contradicts $al(M) \leq k$. $\square$

Lemma 3.13 *Let $(M_n)_{n \geq 1}$ be an infinite sequence of 0-1 matrices such that (i) the sequence $(al(M_n))_{n \geq 1}$ is bounded but (ii) $(|C(M_n)|)_{n \geq 1}$ is unbounded. Then either (a) for every r there is an n and a matrix I_r' similar to I_r such that $I_r' \preceq M_n$ or (b) for every r there is an n and a matrix U_r' similar to U_r such that $U_r' \preceq M_n$.*

Proof. We prove the result under the weaker assumption with $al(M_n)$ replaced by $al_c(M_n)$ that is defined by taking the maximum (of the numbers of intervals of consecutive 0s and 1s) only over the columns of M_n. Using the pigeonhole principle and replacing $(M_n)_{n \geq 1}$ by an appropriate subsequence of submatrices, we may assume in addition to (ii) that there is an $s \geq 1$ and a $c \in \{0, 1\}$ such that the first row of every M_n contains only cs and

$|C(M_n,j)| = s$ for every $n \geq 1$ and j. We set $C(M_n,j) = \{r_{n,j,1} < r_{n,j,2} < \ldots < r_{n,j,s}\}$ and denote c_n the number of columns in M_n. We proceed by induction on s. It is clear that if $s = 1$ then the sequence $S = (|\{r_{n,j,1} : 1 \leq j \leq c_n\}|)_{n\geq 1}$ is unbounded. Suppose that $s \geq 2$ and S is bounded. Taking a subsequence of submatrices, we may then assume in addition to (ii) that $r_{n,j,1} = r_n$ for $1 \leq j \leq c_n$ and all n. We take from every M_n only rows $r_n+1, r_n+2, \ldots$ and obtain a sequence of matrices $(N_n)_{n\geq 1}$ satisfying $|C(N_n,j)| = s-1$ for every n, j and (ii), which means that we are done by induction. Thus we may assume that S is unbounded even for $s \geq 2$. We take a subsequence of submatrices once more and may assume that $(M_n)_{n\geq 1}$ satisfies: $|C(M_n,j)| = s$ for all n and j, $(c_n)_{n\geq 1}$ is unbounded, and for every n the c_n row indices $r_{n,j,1}$, $j \in [c_n]$, are mutually distinct.

Suppose that $s = 1$. Using the Erdős-Szekeres lemma, we may assume that moreover for every n the sequence $(r_{n,j,1} : j = 1, 2, \ldots, c_n)$ is strictly increasing or that for every n it is strictly decreasing. Keeping from M_n only the rows $r_{n,1,1}, r_{n,2,1}, \ldots, r_{n,c_n,1}$ (and all columns), we obtain a matrix similar to U_{c_n}. We see that (b) holds. In the case that $s \geq 2$ we denote by $I_{n,j}$ the interval $[r_{n,j,1}+1, r_{n,j,s}]$ and by $i(n)$ (resp. $d(n)$) the maximum number of intervals among $I_{n,1}, I_{n,2}, \ldots, I_{n,c_n}$ which share one point (resp. which are mutually disjoint). It follows that $(i(n))_{n\geq 1}$ or $(d(n))_{n\geq 1}$ is unbounded. In the former case we may assume, turning to a subsequence of submatrices, that $r_n \in \bigcap_{j=1}^{c_n} I_{n,j}$ for every $n \geq 1$ for some row indices r_n. Keeping from M_n only the rows $1, 2, \ldots, r_n$, we obtain a sequence of matrices $(N_n)_{n\geq 1}$ which satisfies $al_c(N_n) \leq s-1$ for every n and (ii), which means that we are done by induction. In the latter case we may assume, using the Erdős-Szekeres lemma and turning to a subsequence of submatrices, that for every n we have $I_{n,1} < I_{n,2} < \ldots < I_{n,c_n}$ or that for every n we have $I_{n,1} > I_{n,2} > \ldots > I_{n,c_n}$. We select row indices $t_{n,j} \in I_{n,j}$ such that, for all n and $j \in [c_n]$, $M_n(t_{n,j},j) \neq M_n(r_{n,j,1},j) = M_n(r_{n,j,s}+1,j)$ if s is even and $M_n(t_{n,j},j) = M_n(r_{n,j,1},j) \neq M_n(r_{n,j,s}+1,j)$ if s is odd. Keeping from M_n only the c_n rows $t_{n,1}, t_{n,2}, \ldots, t_{n,c_n}$, we obtain a matrix similar to I_{c_n} (for even s) or to U_{c_n} (for odd s). Thus (a) or (b) holds. □

A coloring (n,χ) is *m-tame*, where $m \in \mathbf{N}$, if the following three conditions are satisfied.

1. There is an interval partition $I_1 < I_2 < \ldots < I_s$ of $[n]$ such that $s \leq m$ and every I_i is χ-monochromatic.

2. For every two subintervals $I < J$ in $[n]$ we have $al(M_{I,J}) \leq m$.

3. For every two subintervals $I < J$ in $[n]$ we have $|C(M_{I,J})| \leq m$.

A set of colorings X is m-tame if every coloring $(n,\chi) \in X$ is m-tame.

Lemma 3.14 *For every $m \in \mathbf{N}$ there is a constant $c = c(m)$ such that the number of m-tame colorings (n,χ) is bounded by n^c.*

Proof. The partition $I_1 < I_2 < \ldots < I_s$ of $[n]$ satisfying condition 1 and the s colors $\chi|\binom{I_i}{2}$ can be selected in $c_1 = \sum_{s=1}^{m}\binom{n-1}{s-1}2^s \leq (2n)^m$ ways. The colors of the remaining edges in

(n,χ) are determined by the 0-1 matrices $M = M_{I_u,I_v}$, $1 \le u < v \le s \le m$. Let us bound, for fixed u, v, the number of matrices M satisfying conditions 2 and 3. The number of possibilities for one column of M is by condition 2 at most $c_2 = 2\sum_{i=1}^{m}\binom{p-1}{i-1} \le (2n)^m$, where $p = |I_u| \le n$, and all q columns of M, $q = |I_v| \le n$, can be selected by Lemma 3.12 in at most

$$c_3 = \sum_{i=1}^{2m^2}\binom{q-1}{i-1}c_2^i \le 2m^2 \cdot (n-1)^{2m^2-1} \cdot (2n)^{2m^3} \le (2n)^{4m^3}$$

ways. The total number of m-tame colorings (n,χ) is therefore at most

$$c_1 c_3^{\binom{s}{2}} \le c_1 c_3^{\binom{m}{2}} \le (2n)^{2m^5+m} \le n^c,\ n \ge 2.$$

$\square$

Let $K = (n,\chi)$ be a coloring. The *interval decomposition* of K is the unique partition of $[n]$ in nonempty intervals $I_1 < I_2 < \ldots < I_s$ defined as follows. I_1 is the longest initial interval such that I_1 is χ-monochromatic, I_2 is the longest following interval such that I_2 is χ-monochromatic, and so on. Clearly $|I_i| \ge 2$ for all $i < s$. We let $I(K) = s$ denote the number of intervals in the decomposition.

Proposition 3.15 *If X is an ideal in $(\mathcal{C}(D_2), \preceq)$ that is not m-tame for any m, then there is an $i \in \{1,2,3,4\}$ such that X contains for every $r \ge 1$ an r-wealthy coloring of type i.*

Proof. Suppose there is no $m \in \mathbf{N}$ such that X is m-tame. Thus one of the three conditions in the definition of tameness is violated for infinitely many m on some colorings in X. If it is condition 1, the quantity $I(K)$, $K \in X$, is unbounded and for every $r \ge 1$ there is a coloring $(n,\chi) \in X$ whose interval decomposition $I_1 < I_2 < \ldots < I_s$ satisfies $s \ge r$. By the definition, for every i, $1 < i \le s$, there is an $x_{i-1} \in I_{i-1}$ such that $\chi(\{x_{i-1}, \min I_i\})$ differs from the color $\chi|\binom{I_{i-1}}{2}$. Hence the triangles $\{x_{2i-1}, y_{2i-1}, \min I_{2i}\}$, where $1 \le i \le r/2$ and $y_{2i-1} \in I_{2i-1}\backslash\{x_{2i-1}\}$ is selected arbitrarily, are not monochromatic in (n,χ) and X contains for every $r \ge 1$ an r-wealthy coloring of type 2. If condition 2 is violated infinitely many times, it is easy to see that X contains for every $r \ge 1$ an r-wealthy coloring of type 1.

We are left with the case when conditions 1 and 2 of tameness are satisfied for all colorings in X with a constant m_0 but condition 3 is violated for all m. This implies that for $n = 1, 2, \ldots$ there are colorings $(n, \chi_n) \in X$ and subintervals $I_n < J_n$ in $[n]$ such that the sequence of 0-1 matrices $(M_{I_n,J_n})_{n\ge1}$ satisfies the hypothesis of Lemma 3.13. By the conclusion of the lemma, there is an $i \in \{3,4\}$ such that X contains for every $r \ge 1$ an r-wealthy coloring of type i. $\square$

Proof of Theorem 3.8. If X is an ideal in $\mathcal{C}(D_2)$ that is m-tame for an m, then by Lemma 3.14 we have $|X_n| \le n^c$ for all $n \ge 1$ with a constant $c > 0$. If X is not m-tame for any m, by Proposition 3.15 there is an $i \in \{1, 2, 3, 4\}$ so that X contains for every $r \geq 1$ an r-wealthy coloring of type i. By Lemmas 3.10 and 3.11 this means that for all $n \geq 1$ we have $|X_n| \geq F_n$. □

4 Concluding remarks

We conclude with mentioning a few open problems on growths of ideals of permutations and graphs.

The Stanley-Wilf conjecture (Bóna [9, 10, 11]) asserted that for every permutation π the number of n-permutations not containing π is exponentially bounded. Equivalently stated, for every ideal of permutations X different from the set of all permutations $\mathcal{S}$ we have $|X_n| < c^n$ for all $n \geq 1$. The conjecture was proved by Marcus and Tardos in [23] and therefore now we know that

$$c(X) = \limsup_{n \to \infty} |X_n|^{1/n} < \infty$$

for every ideal $X \neq \mathcal{S}$. However, many interesting and challenging problems on growth of ideals of permutations remain open. The following problem was posed by V. Vatter [14].

Problem 1. Is it true that $\lim_{n \to \infty} |X_n|^{1/n}$ always exists?

It was proved by Arratia [1] in the case $X = \mathrm{Forb}(\{\pi\})$. By the Fibonacci hierarchy 4 (Introduction), it is also true when $c(X) \leq 2$.

Problem 2. What are the constants of growth

$$C = \{c(X) : \ X \text{ is an ideal of permutations}\}?$$

Are all of them algebraic?

Similar problem was posed [6, Conjecture 8.9] for hereditary properties of ordered graphs.

It is easy to find ideals of permutations X such that the function $n \mapsto |X_n|$ is, respectively, constant 0, constant 1, $n \mapsto F_{n,k}$ for any fixed $k \geq 2$, and $n \mapsto 2^{n-1}$. Thus, by the Fibonacci hierarchy 4,

$$C \cap [0, 2] = \{0, 1, \alpha_2, \alpha_3, \alpha_4, \ldots, 2\}$$

where $\alpha_2 \approx 1.61803$, $\alpha_3 \approx 1.83928$, $\alpha_4 \approx 1.92756, \ldots$ are the limits $\alpha_k = \lim F_{n,k}^{1/n}$. By the standard results from asymptotic enumeration, α_k is the largest positive real root of $x^k - x^{k-1} - x^{k-2} - \cdots - 1$. It follows that $\alpha_k \uparrow 2$. It would be interesting to determine further elements of C lying above the first limit point 2.

Problem 3. Show that $\min(C \cap (2, \infty))$ exists. What is this number?

A natural question arises if also the remaining two restrictions on growth of hereditary properties of ordered graphs proved by Balogh, Bollobás and Morris [6], the polynomial growth 3 and the Fibonacci hierarchy 4, can be extended to edge-colored complete graphs with $l \geq 2$ colors. It is not too hard to achieve this for the polynomial growth by elaborating the final "tame" part of our proof of Theorem 3.8; we hope to say more on this elsewhere. It is plausible to conjecture that the proof of the Fibonacci hierarchy in [6] also can be "upgraded" from $l = 2$ to $l \geq 2$ but this would require more effort.

Finally, we present an interesting problem on an exponential-factorial jump in growth due to Balogh, Bollobás and Morris [7, Conjecture 2].

Problem 4. Let X be a hereditary property of ordered graphs. Prove that either $|X_n| < c^n$ for all $n \geq 1$ with a constant $c > 1$ or

$$|X_n| \geq \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$$

for all $n \geq 1$.

They proved [7, Theorem 4] this jump for the smaller family of monotone properties of ordered graphs (and in fact more generally for hypergraphs).

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