Construction of Minimal Bracketing Covers for Rectangles

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Abstract

We construct explicit δ-bracketing covers with minimal cardinality for the set system of (anchored) rectangles in the two dimensional unit cube. More precisely, the cardinality of these δ-bracketing covers are bounded from above by $\delta^{-2}+o(\delta^{-2})$. A lower bound for the cardinality of arbitrary δ-bracketing covers for d-dimensional anchored boxes from [M. Gnewuch, Bracketing numbers for axis-parallel boxes and applications to geometric discrepancy, J. Complexity 24 (2008) 154-172] implies the lower bound $\delta^{-2}+O(\delta^{-1})$ in dimension $d=2$, showing that our constructed covers are (essentially) optimal.

We study also other δ-bracketing covers for the set system of rectangles, deduce the coefficient of the most significant term δ^{-2} in the asymptotic expansion of their cardinality, and compute their cardinality for explicit values of δ.

1 Introduction

Entropy numbers are measures of the size of a given class $\mathcal{F}$ of functions or sets and they are frequently used in fields like density estimation, empirical processes or machine learning. Good bounds for these entropy numbers, in particular the covering or the bracketing numbers, can, e.g., be used to prove bounds on the expectations of suprema of empirical processes (as, e.g., Dudley's metric entropy bound), concentration of measure results for these suprema, or to verify that a class $\mathcal{F}$ of functions or sets is a Glivenko-Cantelli or Donsker Class, i.e., that the corresponding $\mathcal{F}$-indexed empirical process $\mathbb{G}_n$ exhibits a certain convergence behavior as n tends to infinity (cf. [4, 20, 23]).

They are also useful in geometric discrepancy theory, i.e., in the theory of uniform distribution. (Different facets of this theory are nicely described in the monographs [2, 3, 9, 16, 18].) In geometric discrepancy theory one tries to distribute n points in a way to minimize the "discrepancy" between a given (probability) measure and the measure induced by the points (each point has mass $1/n$) with respect to some class of measurable sets $\mathcal{C}$. If one takes, e.g., the class $\mathcal{C}_d := \{\prod_{i=1}^d [0, x_i) \mid x_1, \dots, x_d \in [0,1]\}$ of anchored d-dimensional axis-parallel boxes, the Lebesgue measure λ^d on $[0,1]^d$, and an n-point set $P \subset [0,1]^d$, then the so-called star discrepancy of P

$$d_\infty^*(P) := \sup_{C \in \mathcal{C}_d} \left| \lambda^d(C) - \frac{1}{n}|P \cap C| \right|$$

is a measure of how uniform the points of P are distributed in $[0,1]^d$; here $|P \cap C|$ denotes the cardinality of the set $P \cap S$. If one substitutes the set system $\mathcal{C}_d$ by, e.g., the system of all d-dimensional axis-parallel boxes $\mathcal{R}_d := \{\prod_{i=1}^d [x_i, y_i) \mid x_1, y_1, \dots, x_d, y_d \in [0,1]\}$, one gets another measure of uniformity, the so-called extreme discrepancy

$$d_\infty^{\mathrm{e}}(P) := \sup_{C \in \mathcal{R}_d} \left| \lambda^d(C) - \frac{1}{n}|P \cap C| \right|.$$

Certain types of discrepancy are intimately related to multivariate numerical integration of certain function classes (see, e.g., [3, 9, 14, 16, 18, 19]); a well-known result in this direction is the Koksma-Hlawka inequality which, written as an equality, reads

$$\sup_{f \in B} \left| \int_{[0,1]^d} f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{i=1}^n f(t_i) \right| = d_\infty^*(t_1, \dots, t_n),$$

where B is the unit ball in some particular Sobolev space of functions (see, e.g., [14]).

Thus for multivariate numerical integration it is desirable to be able to calculate the star discrepancy of a given point configuration $\{t_1, \dots, t_n\}$, to have (useful) bounds on the smallest possible discrepancy of any n-point set, and to be able to construct sets satisfying such bounds.

Algorithms approximating the star discrepancy of a given n-point set up to some admissible error δ with the help of bracketing covers have been provided in [21, 22] (see also the discussion in [11]). The more efficient algorithm from [22] generates δ-bracketing covers of $\mathcal{C}_d$ (for a rigorous definition see Sect. 2) and uses those to test the discrepancy of a given point set. The last step raises the task of orthogonal range counting. Depending whether the orthogonal range counting is done in a naive way or (in small dimensions) by employing data structures based on range trees, the running time of the algorithm is of order

$$O(dn|\mathcal{B}_\delta|) \quad \text{or} \quad O\left((d + (\log n)^d)|\mathcal{B}_\delta| + C^d n (\log n)^d\right),$$

where $\mathcal{B}_\delta$ is the generated δ-bracketing cover and $C > 1$ some constant. The cost of generating the δ-bracketing cover $\mathcal{B}_\delta$ is obviously a lower bound for the running time of the algorithm and is of order $\Omega(d|\mathcal{B}_\delta|)$. Thus the running time of the algorithm from [22] depends linear on the size of the generated bracketing covers.

Bounds on the smallest possible star discrepancy with essentially optimal asymptotic behavior for fixed dimension d have been known for a long time (see, e.g, [2, 3, 9, 16, 18]). Nevertheless, they are nearly useless for high-dimensional numerical integration, because one needs exponentially many sample points in d to reach the asymptotic range. Starting with the paper [13] probabilistic approaches have been used to prove bounds for the star, the extreme, and other types of discrepancy that are useful for samples of moderate size [5, 6, 7, 8, 10, 11, 14, 17]. In particular, these investigations focused on the explicit dependence on the number of points n *and* on the dimension d. (Of course, probabilistic approaches had been used in discrepancy theory before [13], see, e.g., [1, 2]. But these studies had not explored the explicit dependence on the dimension d.)

Let us describe these results in more detail: We denote the smallest possible star discrepancy of any n-point configuration in $[0,1]^d$ by

$$d^*_\infty(n,d) = \inf_{P\subset[0,1]^d;|P|=n} d^*_\infty(P)$$

and the so-called inverse of the star discrepancy by

$$n^*_\infty(\varepsilon,d) = \min\{n\in\mathbb{N}\,|\,d^*_\infty(n,d)\le\varepsilon\}\,.$$

In [13] Heinrich, Novak, Wasilkowski, and Woźniakowski proved the bounds

$$d^*_\infty(n,d) \le C\sqrt{\frac{d}{n}} \quad\text{and}\quad n^*_\infty(\varepsilon,d)\le\lceil C^2d\varepsilon^{-2}\rceil\,, \tag{1}$$

where C is a universal constant. The proof uses a theorem of Talagrand on empirical processes [20, Thm. 6.6] combined with an upper bound of Haussler on so-called covering numbers of Vapnik-Červonenkis classes [12]. (Since the theorem of Talagrand holds not only under a condition on the covering number of the set system $\mathcal{S}$ under consideration, but also under the alternative condition that the δ-bracketing number of $\mathcal{S}$ is bounded from above by $(C\delta^{-1})^d$, C some constant [20, Thm. 1.1], one can reprove (1) by using the bracketing result [11, Thm. 1.15] instead of the result of Haussler.)

An advantage of (1) is that the dependence of the inverse of the discrepancy on d is optimal. This was verified in [13] by a lower bound for the inverse, which was improved by Hinrichs [15] to $n^*_\infty(d,\varepsilon)\ge c_0d\varepsilon^{-1}$. A disadvantage of (1) is that so far no good estimate for the constant C has been published.

An alternative approach via using bracketing covers and large deviation inequalities of Chernov-Hoeffding type leads to slightly worse bounds with explicitly given small constants [5, 6, 7, 8, 11, 13]. Let $N_{[\,]}(\mathcal{C}_d,\delta)$ denote the bracketing number, i.e., the cardinality of a minimal δ-bracketing cover of $\mathcal{C}_d$. Then

$$n^*_\infty(\varepsilon,d)\le\left\lceil\frac{2}{\varepsilon^2}\left(\ln N_{[\,]}(\mathcal{C}_d,\varepsilon/2)+\ln 2\right)\right\rceil\,, \tag{2}$$

see [8, Proof of Thm. 3.2]. Thus improved bounds of the bracketing entropy $\ln N_{[\,]}(\mathcal{C}_d,\delta)$ would lead directly to improved bounds on the inverse of the star discrepancy and of the star disprepancy as well (although its dependence on the entropy cannot be expressed by an explicit formula like (2), since the corresponding parameter δ should be chosen to be of the order of the star discrepancy; see again [8, Proof of Thm. 3.2]).

Attempts have been made to provide deterministic algorithms constructing point sets whose star discrepancy satisfies the probabilistic bounds resulting from this alternative approach [6, 7, 8]. The running times of the algorithms depend on the cardinality of suitable δ-bracketing covers; smaller covers would reduce the running times.

These examples show that for discrepancy theory and its application to multivariate numerical integration it is of interest to be able to construct minimal bracketing covers.

In [8, Thm. 2.7] we derived for fixed dimension d the upper bound

$$N_{[]}(\mathcal{C}_d, \delta) \le \frac{d^d}{d!}\delta^{-d} + O(\delta^{-d+1}) \tag{3}$$

for the bracketing number of the set system $\mathcal{C}_d$. In [11] the bounds

$$\delta^{-d}(1 - c_d\delta) \le N_{[]}(\mathcal{C}_d, \delta) \le 2^{d-1}\frac{d^d}{d!}(\delta+1)^{-d}, \tag{4}$$

where c_d depends only on the dimension d, where proved. Obviously there is a gap between the upper bounds and the lower bound. In this paper we prove that in dimension $d = 2$ the lower bound is sharp. More precisely, we construct explicit δ-bracketing covers $\mathcal{R}_\delta$ whose cardinality is bounded from above by $\delta^{-2} + o(\delta^{-2})$; thus 1 is the correct coefficient in front of the most significant term in the expansion of the bracketing number $N_{[]}(\mathcal{C}_d, \delta)$ with respect to δ^{-1}. Furthermore, we discuss other constructions in dimension $d = 2$ (e.g., the cover from [22]) and compare them. We conjecture that the lower bound in (4) is sharp in the sense that $N_{[]}(\mathcal{C}_d, \delta) = \delta^{-d} + o_d(\delta^{-d})$ holds for all d; here o_d should emphasize that the implicit constants in the o-notation may depend on d. We are convinced that this upper bound can be proved constructively by extending the ideas we used to generate $\mathcal{R}_\delta$ to higher dimensions.

2 Preliminaries

Let $d \in \mathbb{N}$ and put $[d] := \{1, \ldots, d\}$. For $x, y \in [0,1]^d$ we write $x \le y$ if $x_i \le y_i$ holds for all $i \in [d]$. We write $[x, y] := \prod_{i\in[d]}[x_i, y_i]$ and use corresponding notation for open and half-open intervals. We put $V_x := \lambda^{\mathrm{d}}([0, x])$ and $V_{x,y} := \lambda^{\mathrm{d}}([x, y])$, where λ^{d} is the d-dimensional Lebesgue measure. Similarly, we put $V_A := \lambda^{\mathrm{d}}(A)$ for any measurable subsets A of $[0,1]^d$. In this paper we consider the classes

$$\mathcal{C}_d = \{[0, x) \,|\, x \in [0,1]^d\} \quad \text{and} \quad \mathcal{R}_d = \{[x, y) \,|\, x, y \in [0,1]^d\}$$

of subsets of $[0,1]^d$. The elements of $\mathcal{C}_d$ are called *anchored (axis-parallel) boxes* or simply *corners*. The elements of $\mathcal{R}_d$ are called *unanchored (axis-parallel) boxes*. (Here the word "unanchored" is of course meant in the sense of "not necessarily anchored".)

Let $\mathcal{F} \in \{\mathcal{C}_d, \mathcal{R}_d\}$. For a given $\delta \in (0,1]$ and $A, B \in \mathcal{F}$ with $A \subseteq B$ we call the set

$$[A, B]_{\mathcal{F}} := \{C \in \mathcal{F} \mid A \subseteq C \subseteq B\}$$

a *δ-bracket* of $\mathcal{F}$ if its *weight* $W([A, B])$ defined by

$$W([A, B]) := V_B - V_A$$

does not exceed δ. A *δ-bracketing cover* of $\mathcal{F}$ is a set of δ-brackets whose union is $\mathcal{F}$. By $N_{[\,]}(\mathcal{F}, \delta)$ we denote the *bracketing number* of $\mathcal{F}$, i.e., the smallest number of δ-brackets whose union is $\mathcal{F}$. The quantity $\ln N_{[\,]}(\mathcal{F}, \delta)$ is called the *bracketing entropy* of $\mathcal{F}$. In [11] we showed in particular that

$$N_{[\,]}(\mathcal{C}_d, \delta) \leq N_{[\,]}(\mathcal{R}_d, \delta) \leq (N(\mathcal{C}_d, \delta/2))^2. \tag{5}$$

The second inequality was verified by using arbitrary $\delta/2$-bracketing covers of $\mathcal{C}_d$ of cardinality Λ to construct δ-bracketing covers of $\mathcal{R}_d$ of cardinality at most Λ^2 (cf. [11, Lemma 1.18]); that is why we can restrict ourselves to the construction of bracketing covers of $\mathcal{C}_d$.

Let us identify the boxes $[0, x)$ in $\mathcal{C}_d$ with their right upper corners $x \in [0,1]^d$. According to this convention, we identify the bracket $[[0, x), [0, y)]_{\mathcal{C}_d}$ with the d-dimensional box $[x, y]$.

If we are interested in δ-bracketing covers of $\mathcal{C}_d$ with small cardinality it is clear that we should try to maximize the volume of the δ-brackets used. The following lemma states how δ-brackets of $\mathcal{C}_d$ with maximum volume look like.

Lemma 2.1. *Let $d \geq 2$, $\delta \in (0,1)$, and let $z \in [0,1]^d$ with $V_z > \delta$. Put*

$$x = x(z, \delta) := \left(1 - \frac{\delta}{V_z}\right)^{1/d} z.$$

Then $[x, z]$ is the uniquely determined δ-bracket having maximum volume of all δ-brackets of $\mathcal{C}_d$ that contain z. Its volume is

$$V_{x,z} = \left(1 - \left(1 - \frac{\delta}{V_z}\right)^{1/d}\right)^d V_z.$$

(In the case where $V_z \leq \delta$ it is easy to see that z is always contained in some δ-bracket $[0, \zeta)$ with maximum volume $V_\zeta = \delta$.) For a proof of the lemma see [11, Lemma 1.1].

Now we state a "scaling lemma" which we shall use frequently throughout the paper.

Lemma 2.2. *Let $\delta \in (0,1)$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in (0, \infty)^d$. Let*

$$\Phi(\lambda) : \mathbb{R}^d \to \mathbb{R}^d \,,\ (x_1, \ldots, x_d) \mapsto (\lambda_1 x_1, \ldots, \lambda_d x_d).$$

Furthermore, let $S \subseteq [0,1]^d$ such that $\Phi(\lambda)S \subseteq [0,1]^d$. Then the smallest number of δ-brackets whose union covers S is the smallest number of $((\prod_{i=1}^d \lambda_i)\delta)$-brackets whose union covers $\Phi(\lambda)S$.

The proof is obvious since scaling a bracket by applying $\Phi(\lambda)$ implies that its weight is scaled by the multiplicative factor $\prod_{i=1}^d \lambda_i$.

Let us briefly recapitulate the construction of a δ-bracketing cover $\mathcal{G}_\delta$ from [8] in which the δ-brackets are the cells in a non-equidistant grid. We do so for two reasons: We want to compare the cardinality of $\mathcal{G}_\delta$ with the (more sophisticated) bracketing covers we present later, and, what is more important, the construction of $\mathcal{G}_\delta$ can be viewed as a "building block" of all these bracketing covers.

We construct the non-equidistant grid

$$\Gamma_\delta = (\{x_0, x_1, ..., x_{\kappa(\delta,d)}\} \cup \{0\})^d, \tag{6}$$

where $x_0, x_1, ..., x_{\kappa(\delta,d)}$ is a decreasing sequence in $(0,1]$. We calculate this sequence recursively in the following way: Put $x_0 := 1$ and $x_1 := (1-\delta)^{1/d}$. If $x_i > \delta$, then define $x_{i+1} := (x_i - \delta)x_1^{1-d}$. If $x_{i+1} \le \delta$, then put $\kappa(\delta, d) := i+1$, otherwise proceed by calculating x_{i+2}.

Since $\mathcal{G}_\delta$ consists of the cells of Γ_δ, i.e., of all closed d-dimensional boxes B whose intersection with Γ_δ consists exactly of the 2^d corners of B, we have

$$|\mathcal{G}_\delta| = (\kappa(\delta, d) + 1)^d. \tag{7}$$

It was shown in [8], that $\mathcal{G}_\delta$ is a bracketing cover (without explicitly using this notion) and that

$$\kappa(\delta, d) = \left\lceil \frac{d}{d-1} \frac{\ln(1-(1-\delta)^{1/d}) - \ln(\delta)}{\ln(1-\delta)} \right\rceil. \tag{8}$$

Furthermore, it was shown that the inequality $\kappa(\delta, d) \le \lceil \frac{d}{d-1} \frac{\ln(d)}{\delta} \rceil$ holds, and that the quotient of the left and the right hand side of this inequality converges to 1 as δ approaches 0. But to make proofs shorter in what follows, it is better to use the more precise estimate

$$\kappa(\delta, d) = \frac{d}{d-1} \ln(d)\delta^{-1} + O(1) \quad \text{as } \delta \text{ approaches zero.} \tag{9}$$

It follows directly from the following identities which are easy to check:

$$(\ln(1-\delta))^{-1} = -\delta^{-1} + O(1) \tag{10}$$

and

$$\ln(1-(1-\delta)^{1/d}) = \ln(\delta) - \ln(d) + O(\delta) \tag{11}$$

as δ tends to zero.

Let us now confine ourselves to dimension $d = 2$ and use the shorthand $\kappa(\delta)$ for $\kappa(\delta, 2)$. Put $a_i(\delta) := (1 - i\delta)^{1/2}$ for $i = 0, 1, ..., \lceil \delta^{-1} \rceil - 1$. Then in fact, $\kappa(\delta) + 1$ is the minimal number of δ-brackets of heights $1 - a_1(\delta)$ whose union covers the stripe $[(0, a_1(\delta)), (1, 1)]$; the δ-brackets covering the stripe are the rectangles $[(x_1, a_1(\delta)), (x_0, 1)]$, $[(x_2, a_1(\delta)), (x_1, 1)]$,..., $[(0, a_1(\delta)), (x_{\kappa(\delta)}, 1)]$.

Let us more generally define $\omega(\delta, t)$ to be the minimal number of δ-brackets of heights $1 - a_1(\delta)$ whose union covers the stripe $[(t, a_1(\delta)), (1, 1)]$ for some $t \in [0, 1]$. We calculate

$x_0, x_1, \ldots$ as above and determine $\omega(\delta, t)$ such that $x_{\omega(\delta,t)-1} > t$ and $[(x_1, a_1(\delta)), (x_0, 1)]$, $[(x_2, a_1(\delta)), (x_1, 1)]$,..., $[(t, a_1(\delta)), (x_{\omega(\delta,t)-1}, 1)]$ are δ-brackets whose union covers the stripe $[(t, a_1(\delta)), (1, 1)]$. From the construction of the x_i we see that

$$x_i = (1-\delta)^{-i/2} - \delta(1-\delta)^{-1/2} \frac{1-(1-\delta)^{-i/2}}{1-(1-\delta)^{-1/2}}$$

and that $x_{i+1} \leq t$ is satisfied if and only if

$$i + 1 \geq 2 \, \frac{\ln\left(1-(1-\delta)^{1/2}\right) - \ln\left(t(1-(1-\delta)^{-1/2}) + \delta(1-\delta)^{-1/2}\right)}{\ln(1-\delta)}.$$

Thus

$$\omega(\delta, t) = \left\lceil 2 \, \frac{\ln\left(1-(1-\delta)^{1/2}\right) - \ln\left(t(1-(1-\delta)^{-1/2}) + \delta(1-\delta)^{-1/2}\right)}{\ln(1-\delta)} \right\rceil. \tag{12}$$

Observe that for $t = 0$ we have indeed $\omega(\delta, 0) = \kappa(\delta) + 1$. We shall use the numbers $\omega(\delta, t)$ for different δ and t to show that the last bracketing cover we present in this paper exhibits the (asymptotically) optimal cardinality.

In the following three sections we present δ-bracketing covers with reasonably smaller cardinality than $\mathcal{G}_\delta$.

3 The Construction of Thiémard

Before stating the algorithm of Thiémard to construct a δ-bracketing cover $\mathcal{T}_\delta$, we want to explain its main idea in dimension $d = 2$. (In [22] the algorithm is discussed for arbitrary d.)

It covers $[0,1]^2$ successively with δ-brackets by decomposing all rectangles P with weight $W(P) > \delta$ into smaller rectangles starting with the rectangle $[0,1]^2$. More precisely, if P is of the form $P = [\alpha, \beta]$ for some $\alpha = \alpha^P$, $\beta = \beta^P \in [0,1]^2$, then it calculates parameters $\gamma_1 = \gamma_1^P$, $\gamma_2 = \gamma_2^P$ satisfying $\alpha_1 \leq \gamma_1 \leq \beta_1$ and $\alpha_2 \leq \gamma_2 \leq \beta_2$ and decomposes P into

$$Q_1^P = [(\alpha_1, \alpha_2), (\gamma_1, \beta_2)] \quad \text{and} \quad P_1^P = [(\gamma_1, \alpha_2), (\beta_1, \beta_2)].$$

Afterwards it decomposes P_1^P into

$$Q_2^P = [(\gamma_1, \alpha_2), (\beta_1, \gamma_2)] \quad \text{and} \quad P_2^P = [(\gamma_1, \gamma_2), (\beta_1, \beta_2)],$$

resulting in the (almost disjoint) decomposition

$$P = Q_1^P \cup Q_2^P \cup P_2^P.$$

The right choice of $\gamma = (\gamma_1, \gamma_2)$ ensures $W(P_2^P) = \delta$ and P_2^P is chosen to become an element of the final δ-bracketing cover $\mathcal{T}_\delta$.

The rectangle Q_1^P is of "type 1", the rectangle Q_2^P of "type 2": if the algorithm decomposes them, then it chooses $\gamma_1^{Q_1^P} \in (\alpha_1^P, \gamma_1^P)$ and $\gamma_1^{Q_2^P} = \gamma_1^P$ implying that Q_1^P will be decomposed into three, but Q_2^P only into two non-trivial rectangles.

That is why in the algorithm a rectangle P is described by the triple $(P, i, W(P))$, where $i \in \{1, 2\}$ denotes the type of the rectangle.

Denoted in pseudo-code, the algorithm looks as follows:

Algorithm THIEMARD

Input: $\delta \in (0, 1)$.

Output: A δ-bracketing cover $\mathcal{T}_\delta$.

Main

$\mathcal{T}_\delta := \emptyset$

Decompose $([0,1]^2, 1, 1)$

Procedure DECOMPOSE (P, j, v)

Compute δ^P according to (13)

Compute γ^P according to (14)

If $\delta^P v > \delta$

For i from j to 2

Decompose $(Q_i^P, i, \delta^P v)$

Else

For i from j to 2

$\mathcal{T}_\delta := \mathcal{T}_\delta \cup \{Q_i^P\}$

$\mathcal{T}_\delta := \mathcal{T}_\delta \cup \{[\gamma^P, \beta^P]\}$

For each triple (P, j, v) we calculate $\delta^P \in (0, 1)$ and $\gamma^P \in [0, 1]^2$ as follows:

$$\delta^P = \left(\frac{\beta_1^P \beta_2^P - \delta}{\beta_1^P \beta_2^P} \right)^{1/2} \quad \text{if } j = 1, \quad \delta^P = \frac{\beta_1^P \beta_2^P - \delta}{\alpha_1^P \beta_2^P} \quad \text{if } j = 2, \tag{13}$$

and

$$\gamma_i^P = \begin{cases} \alpha_i^P & \text{if } i < j, \\ \delta^P \beta_i^P & \text{if } i \geq j. \end{cases} \tag{14}$$

That the resulting set $\mathcal{T}_\delta$ is indeed a δ-bracketing cover was proved in [22]. In Figure 1 and 2 one can see the resulting cover $\mathcal{T}_\delta$ for $\delta = 0.25$ and $\delta = 0.05$.

Let us now determine the asymptotic behavior of $|\mathcal{T}_\delta|$ for δ tending to zero. In [22, Theorem 3.4] Thiémard proved the bound

$$|\mathcal{T}_\delta| \leq \binom{2+h}{2}, \quad \text{where} \quad h = \left\lceil \frac{2 \ln(\delta)}{\ln(1-\delta)} \right\rceil.$$

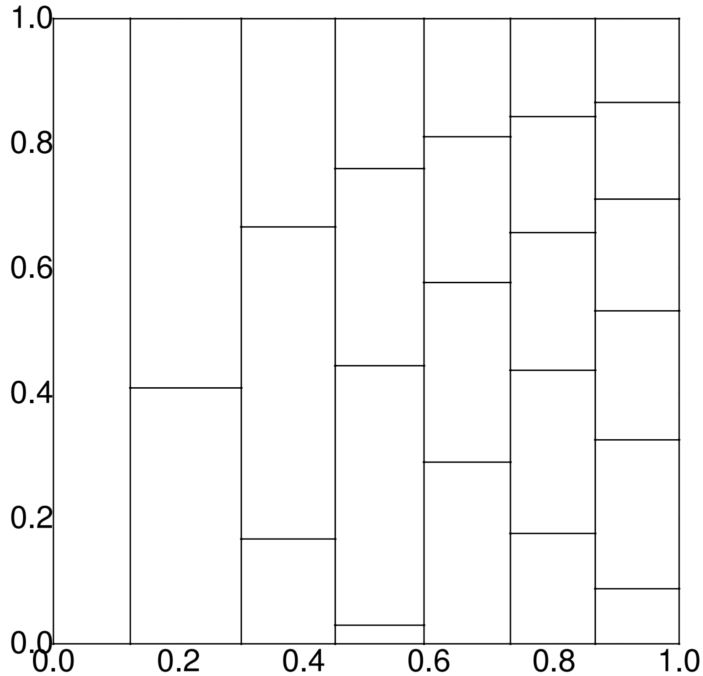

Figure 1: $\mathcal{T}_\delta$ for $\delta = 0.25$.

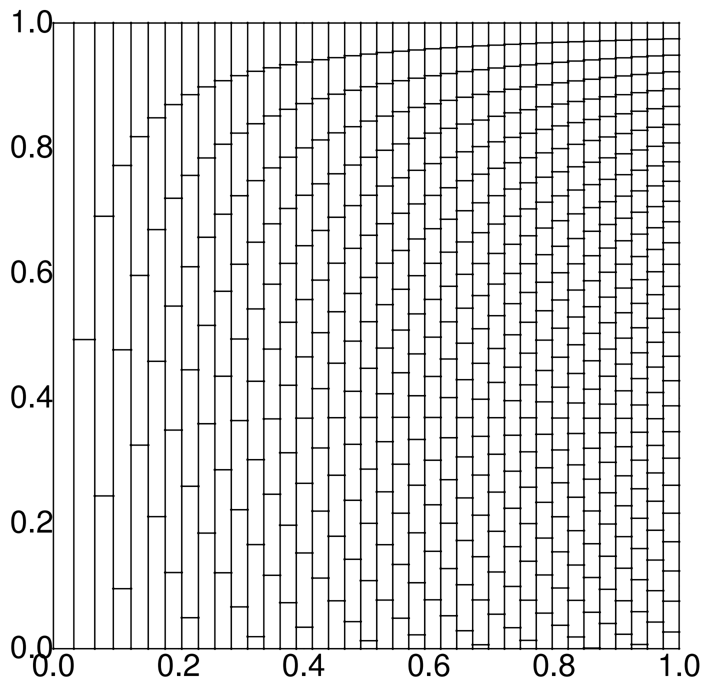

Figure 2: $\mathcal{T}_\delta$ for $\delta = 0.05$.

This implies $|\mathcal{T}_\delta| \le 2(\ln(\delta^{-1}))^2\,\delta^{-2} + o(\delta^{-2})$. We improve this estimate in the following Proposition by deducing the correct asymptotic behavior in terms of δ^{-1} and the exact coefficient in front of the most significant term δ^{-2}.

Proposition 3.1. *For a given $\delta \in (0,1)$ we get*

$$|\mathcal{T}_\delta| = 2\ln(2)\delta^{-2} + O(\delta^{-1}).$$

Proof. From the discussion above (and also from Figure 1 and 2) we see that Thiémard's algorithm decomposes the unit rectangle $[0,1]^2$ into stripes

$$S_\delta^{(i)} := [(t_{i+1}, 0), (t_i, 1)], \quad i = 0, \ldots, \tau(\delta),$$

and these stripes again into δ-brackets; here the numbers t_i are the x-coordinates of the corners of all rectangles of type 1 that appear in the course of the algorithm. More precisely, we have $t_0 = 1$, $t_{\tau(\delta)+1} = 0$,

$$t_{i+1} = \left(1 - \frac{\delta}{t_i}\right)^{1/2} t_i = \left(\left(t_i - \frac{\delta}{2}\right)^2 - \frac{\delta^2}{4}\right)^{1/2}, \tag{15}$$

and $\tau(\delta)$ is uniquely determined by the relation

$$0 < t_{\tau(\delta)} \le \delta. \tag{16}$$

We have

$$t_i - \delta \le t_{i+1} < t_i - \frac{\delta}{2}; \tag{17}$$

both inequalities follow easily from (15). From (16) and (17) we get

$$\lceil \delta^{-1} \rceil - 1 \le \tau(\delta) \le \lceil 2\delta^{-1} \rceil - 1. \tag{18}$$

Furthermore, we get by simple induction

$$t_{i+1}^2 = 1 - \delta \sum_{k=0}^{i} t_k,$$

which, together with (16), results in

$$\delta^{-1} - \delta \le \sum_{k=0}^{\tau(\delta)-1} t_k < \delta^{-1}. \tag{19}$$

Let us now calculate the number $s_\delta^{(i)}$ of δ-brackets of widths $t_i - t_{i+1}$ that cover the stripe $S_\delta^{(i)}$. Since the bracketing problem is symmetric in the x- and y-coordinate, we get from the discussion in the previous section

$$s_\delta^{(0)} = \kappa(\delta) + 1, \quad \text{where} \quad \kappa(\delta) = \kappa(\delta, 2) \quad \text{as defined in (8).}$$

(Note that $t_1 = (1-\delta)^{1/2}$.) From this we can derive $s_\delta^{(i)}$ for all $i \in \{0,\ldots,\tau(\delta)-1\}$ via "scaling": Lemma 2.2 gives us with the choice $\lambda = (t_i^{-1}, 1)$

$$s_\delta^{(i)} = \kappa(\delta/t_i) + 1 \quad \text{for all } i \in \{0,\ldots,\tau(\delta)-1\}.$$

(Observe that $t_{i+1}/t_i = a_1(\delta/t_i)$.) Furthermore, we have trivially $s_\delta^{(\tau(\delta))} = 1$.

Then identity (9) and the inequalities (18) and (19) result in

$$|\mathcal{T}_\delta| = \sum_{i=0}^{\tau(\delta)} s_\delta^{(i)} = \sum_{i=0}^{\tau(\delta)-1} (\kappa(\delta/t_i)+1)+1 = 2\ln(2)\delta^{-1} \sum_{i=0}^{\tau(\delta)-1} t_i + O(\delta^{-1}) = 2\ln(2)\delta^{-2} + O(\delta^{-1}).$$

□

4 Another Construction

Let us consider another algorithm constructing δ-bracketing covers $\mathcal{Z}_\delta$ for anchored rectangles:

Let again $a_i = a_i(\delta) = (1-i\delta)^{1/2}$ for $i = 0,\ldots,\zeta(\delta) := \lceil \delta^{-1} \rceil - 1$, and $a_{\zeta(\delta)+1} = 0$. Put $\overline{a}_i := (a_i(\delta), a_i(\delta))$ for all i. We first decompose $[0,1]^2$ into layers

$$L^{(i)}(\delta) := [0, \overline{a}_i] \setminus [0, \overline{a}_{i+1}).$$

Then, starting with $L^{(0)}(\delta)$, we will cover each layer $L^{(i)}(\delta)$ separately with δ-brackets. To this purpose we cover for fixed $i \in \{0,\ldots,\zeta(\delta)-1\}$ the stripe $[(0, a_{i+1}), (a_i, a_i)]$ recursively by the following procedure:

Put $\mathcal{S}_i(\delta) := \{[\overline{a}_{i+1}, \overline{a}_i]\}$ and $x_1 := a_{i+1}(\delta)$.
If $x_j a_i(\delta) > 0$, then define

$$x_{j+1} := \max\{0, (x_j a_i(\delta) - \delta)/a_{i+1}(\delta)\}$$

and put

$$\mathcal{S}_i(\delta) := \mathcal{S}_i(\delta) \cup \{[(x_{j+1}, a_{i+1}(\delta)), (x_j, a_i(\delta))]\}.$$

If $x_j a_i(\delta) \leq \delta$, then stop the covering procedure.

It is easy to see that for each i the resulting set $\mathcal{S}_i(\delta)$ consists of δ-brackets whose union is $[(0, a_{i+1}), (a_i, a_i)]$. In fact, we see that for $i = 0$ the x_j, $j = 0, 1, \ldots$, we get from the procedure above form the projection of Γ_δ (defined as in (6)), i.e., the set $\{x_0, x_1, \ldots, x_{\kappa(\delta)}, x_{\kappa(\delta)+1}\}$, where $x_{\kappa(\delta)} \leq \delta$ and $x_{\kappa(\delta)+1} = 0$. Thus $|\mathcal{S}_0(\delta)| = \kappa(\delta)+1$. Using the scaling Lemma 2.2 with $\lambda = (a_i^{-1}, a_i^{-1})$ we deduce that consequently $|\mathcal{S}_i(\delta)| = \kappa(\delta_i)+1$, where $\delta_i := \delta/(1-i\delta)$. (Observe that $a_{i+1}(\delta)/a_i(\delta) = a_1(\delta_i)$ for $i < \zeta(\delta)$.) By symmetry, we can cover $L^{(i)}(j)$ by $2\kappa(\delta_i)+1$ δ-brackets. (Observe that $L^{\zeta(\delta)}(\delta)$ is already a δ-bracket.) More precisely, using the mapping $\mathrm{ref} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (y,x)$ we have

$$\mathcal{Z}_\delta = \bigcup_{i=0}^{\zeta(\delta)} \big(\mathcal{S}_i(\delta) \cup \mathrm{ref}(\mathcal{S}_i(\delta))\big).$$

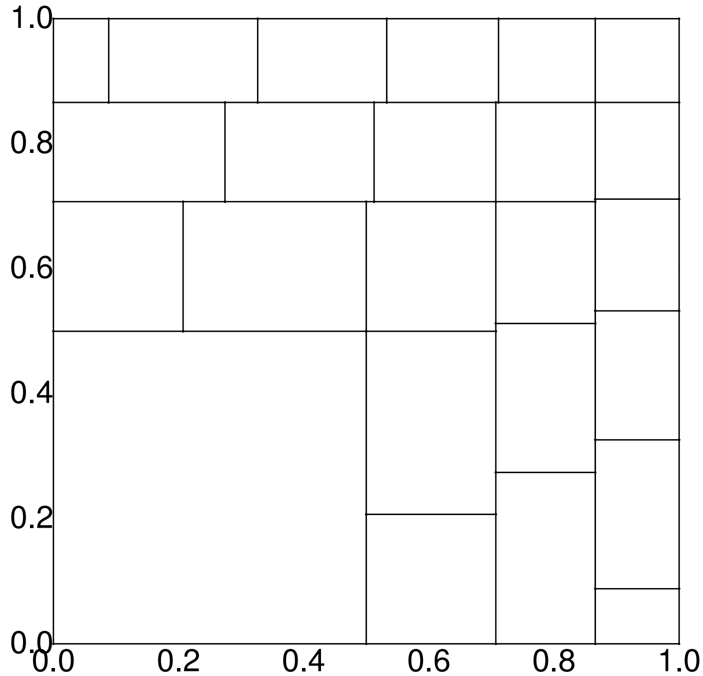

Figure 3: $\mathcal{Z}_\delta$ for $\delta = 0.25$.

The set $\mathcal{Z}_\delta$ is a δ-bracketing cover of $[0,1]^2$ with

$$|\mathcal{Z}_\delta| = \left(\sum_{i=0}^{\zeta(\delta)-1} (2\kappa(\delta_i) + 1) \right) + 1. \tag{20}$$

In Figure 3 and 4 we see $\mathcal{Z}_\delta$ for $\delta = 0.25$ and 0.05.

From the identity (20), the definition of $\zeta(\delta)$, and (9) we get

$$|\mathcal{Z}_\delta| = 2 \left(\sum_{i=0}^{\zeta(\delta)-1} 2\ln(2)\delta_i^{-1} \right) + O(\delta^{-1}).$$

Now $\delta_i^{-1} = \delta^{-1} - i$, hence

$$\sum_{i=0}^{\zeta(\delta)-1} \delta_i^{-1} = \zeta(\delta)\delta^{-1} - \frac{\zeta(\delta)(\zeta(\delta)-1)}{2} = \frac{1}{2}\delta^{-2} + O(\delta^{-1}).$$

Thus

$$|\mathcal{Z}_\delta| = 2\ln(2)\delta^{-2} + O(\delta^{-1}).$$

Altogether, we proved the following proposition.

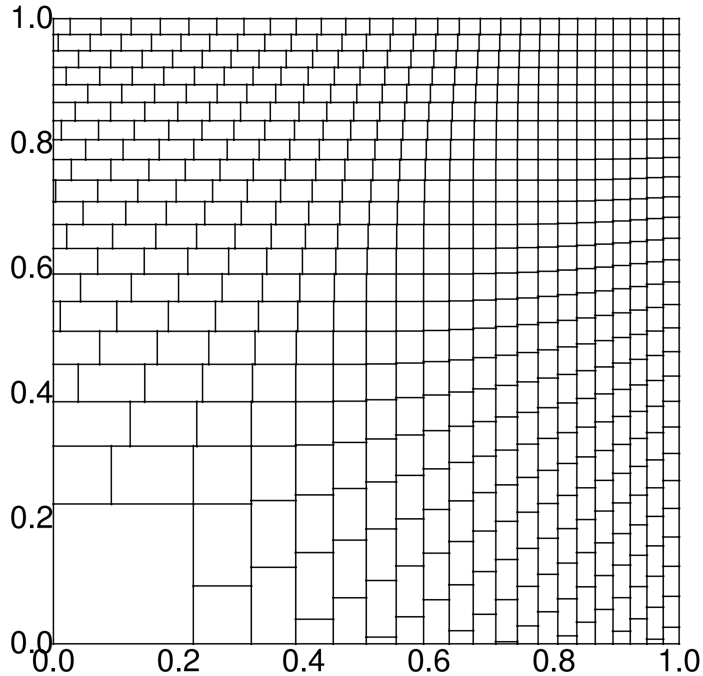

Figure 4: $\mathcal{Z}_\delta$ for $\delta = 0.05$.

Proposition 4.1. *For $\delta \in (0,1)$ the set of rectangles $\mathcal{Z}_\delta$ constructed above is a δ-bracketing cover of $\mathcal{C}_2$. Its cardinality is given by*

$$|\mathcal{Z}_\delta| = \left(\sum_{i=0}^{\zeta(\delta)-1} (2\kappa(\delta_i)+1) \right) + 1 = 2\ln(2)\delta^{-2} + O(\delta^{-1}), \tag{21}$$

where $\zeta(\delta) = \lceil \delta^{-1} \rceil - 1$, $\delta_i = \delta/(1-i\delta)$, and $\kappa(\delta_i) = \kappa(\delta_i, 2)$ as defined in (8).

5 Re-Orientation of the Brackets

A positive aspect of the two previous constructions is that (essentially) all brackets in the resulting δ-bracketing covers have largest possible weight δ and overlap only on sets of Lebesgue measure zero. But if we look at the brackets in Thiémard's construction which have some distance to the upper edge of the unit rectangle $[0,1]^2$, then these boxes do certainly not satisfy the "maximum area criterion" stated in Lemma 2.1. The same holds for the brackets in Z_δ which are close to the x- or the y-axis and away from the main diagonal. The idea of our next construction is to generate a bracketing cover similarly as in the previous section, but to "re-orientate" the brackets from time to time in the course of the algorithm to enlarge the area which is covered by a single bracket. Of course the algorithm should still be simple and avoid to much overlap of the generated brackets.

Before stating the technical details, we want to present the underlying geometrical idea in a simplified way:

Like the construction in the previous section, our new bracketing cover should be symmetric with respect to both coordinate axes. Thus we only have to state explicitly how to cover the subset

$$H := \{(x,y) \in [0,1]^2 \,|\, x \leq y\}$$

of $[0,1]^2$. For a certain number $p = p(\delta)$ we then subdivide H into sectors

$$T^{(h)} := \left\{ (x,y) \in H \setminus \{(0,0)\} \,\middle|\, \frac{h-1}{2^p} \leq \frac{x}{y} \leq \frac{h}{2^p} \right\} \cup \{(0,0)\}, \quad h = 1, \dots, 2^p.$$

In the same manner as we decomposed in the previous construction the set H into stripes $[(0,a_{i+1}),(a_i,a_i)]$, we now decompose $T^{(2^p)}$ into stripes $[(0,a_{i+1}),(a_i,a_i)] \cap T^{(2^p)}$. We do it similarly with the sectors $T^{(1)}, \dots, T^{(2^p-1)}$, but we use thicker (and therefore less) stripes there. Covering each of these stripes by brackets whose height is exactly the height of the corresponding stripe, we see that each bracket has almost the maximum possible area. Provided we can avoid to much overlap at the boundaries of the sectors, we thus need only a very small number of these brackets to cover $[0,1]^2$.

Let us now state the generating algorithm precisely. We define "discretized" versions $T^{(h)}_{\mathrm{dis}}$ of the sectors $T^{(h)}$, composed of stripes. To this purpose we define for each $h \in \{1, \dots, 2^p\}$

$$\rho^{(h)}(\delta) := \lceil h 2^{-p} \delta^{-1} \rceil - 1.$$

(Note that $\rho^{(2^p)}(\delta)$ is precisely $\zeta(\delta)$ as defined in the previous section.) For $i = 0, \dots, \rho^{(h)}(\delta)$ let

$$T^{(h)}_i(\delta) := [(t^{(h)}_{i+1}(\delta), a^{(h)}_{i+1}(\delta)), (h 2^{-p} a^{(h)}_i(\delta), a^{(h)}_i(\delta))],$$

where

$$t^{(h)}_{i+1}(\delta) := \frac{h-1}{2^p} \left(1 - \left\lceil \frac{h-1}{h} i - \frac{1}{h} \right\rceil \frac{2^p}{h-1} \delta \right)_+^{1/2}$$

and

$$a^{(h)}_i(\delta) := \left(1 - i \frac{2^p}{h} \delta \right)_+^{1/2}.$$

(Here we use the convention to denote for a general function f by f_+ the function $f 1_{f^{-1}([0,\infty))}$, where 1_A is the characteristic function of a set A. In particular we have $t^{(1)}_{i+1}(\delta) = 0$ for all i and $a^{(h)}_{\rho^{(h)}(\delta)+1}(\delta) = 0$ for all h.) We put $\overline{a}^{(h)}_i(\delta) := (h 2^{-p} a^{(h)}_i(\delta), a^{(h)}_i(\delta))$. Then

$$T^{(h)}_{\mathrm{dis}} := \bigcup_{i=0}^{\rho^{(h)}(\delta)} T^{(h)}_i(\delta)$$

can be viewed as a discretized version (discretized with respect to a decomposition into stripes) of $T^{(h)}$.

Now for $h = 2^p, 2^p - 1, \ldots, 1$ we cover each stripe $T_i^{(h)}(\delta)$, $i = 0, 1, \ldots, \rho^{(h)}(\delta)$, of the "discretized" sectors $T_{\text{dis}}^{(h)}$ by brackets having exactly the height of the stripe $T_i^{(h)}(\delta)$ in the following manner:

Algorithm RE-ORIENTED BRACKETS

Input: $\delta \in (0,1)$, $p \in [0, \infty)$.

Output: A δ-bracketing cover $\mathcal{R}_\delta$.

Main

$\mathcal{R}_\delta := \emptyset$

For $h = 2^p$ to 1

For $i = 0, ..., \rho^{(h)}(\delta)$

$$x_1 := \left(\frac{h}{2^p}(a_i^{(h)}(\delta))^2 - \delta\right)\left(a_{i+1}^{(h)}(\delta)\right)^{-1} = \frac{h}{2^p} a_{i+1}^{(h)}(\delta)$$

$$\mathcal{R}_\delta := \mathcal{R}_\delta \cup \{[\overline{a}_{i+1}^{(h)}(\delta), \overline{a}_i^{(h)}(\delta)]\}$$

For $j = 1, 2, \ldots$

If $x_j > t_{i+1}^{(h)}(\delta)$

$$x_{j+1} := \left(x_j a_i^{(h)}(\delta) - \delta\right)_+ \left(a_{i+1}^{(h)}(\delta)\right)^{-1}$$

$$\mathcal{R}_\delta := \mathcal{R}_\delta \cup \{[(x_{j+1}, a_{i+1}^{(h)}(\delta)), (x_j, a_i^{(h)}(\delta))]\}$$

Else next i

$\mathcal{R}_\delta := \mathcal{R}_\delta \cup \mathrm{ref}(\mathcal{R}_\delta)$

The output set $\mathcal{R}_\delta$ is visualized in Figure 5 and 6 for $\delta = 0.05$ and $\delta = 0.02$; there we have chosen $p = p(\delta)$ to be

$$p(\delta) = \max\left\{\left\lfloor \frac{\ln(\delta^{-1}) - k}{c} \right\rfloor, 0\right\} \tag{22}$$

with $k = 0$ and $c = 1.7$. With this choice we get for $\delta = 0.25$ that $p = 0$ and consequently $\mathcal{Z}_\delta = \mathcal{R}_\delta$; thus Figure 3 shows $\mathcal{R}_\delta$ for $\delta = 0.25$.

Let us now prove the following proposition.

Proposition 5.1. *The output set $\mathcal{R}_\delta$ of the algorithm stated above is a δ-bracketing cover. If $p = p(\delta)$ is a decreasing function on $(0,1)$ with $\lim_{\delta \to 0} p(\delta) = \infty$ and $2^p = o(\delta^{-1})$ as δ tends to zero, then the bracketing cover $\mathcal{R}_\delta$ satisfies*

$$|\mathcal{R}_\delta| = \delta^{-2} + o(\delta^{-2}).$$

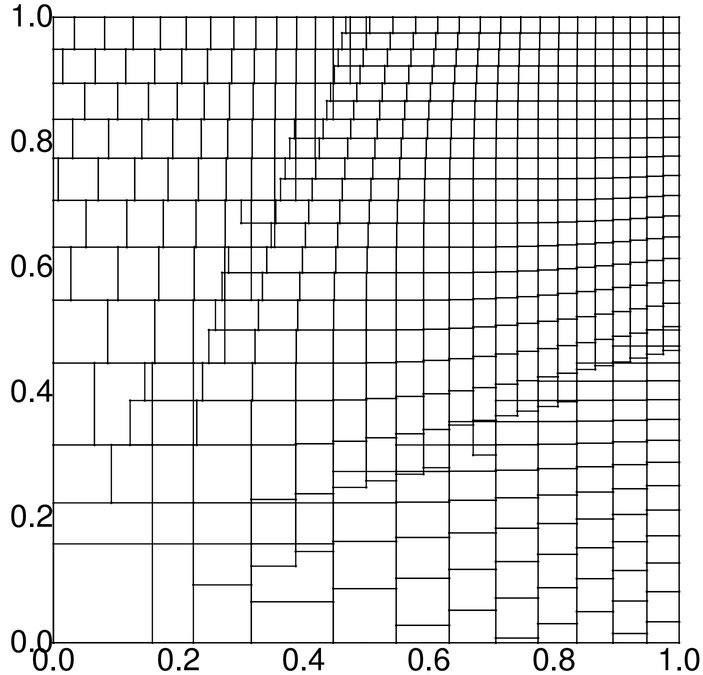

Figure 5: $\mathcal{R}_\delta$ for $\delta = 0.05$.

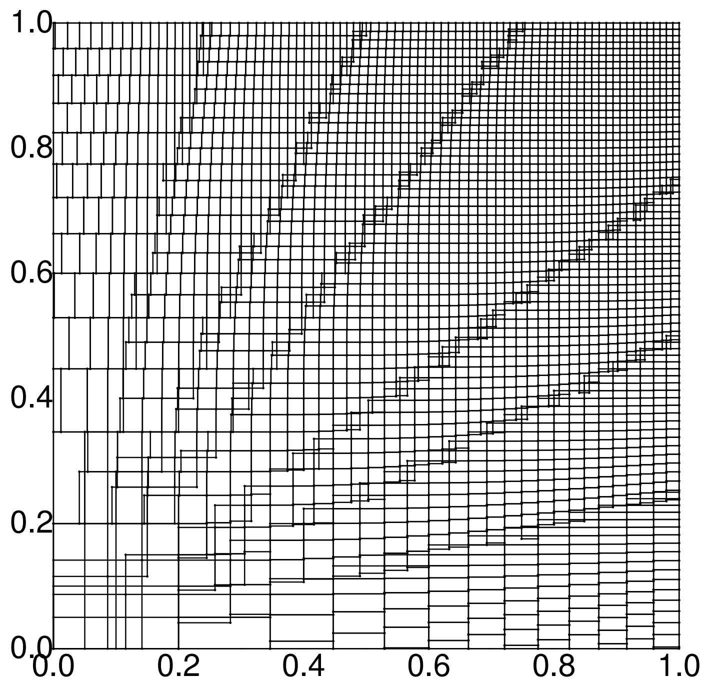

Figure 6: $\mathcal{R}_\delta$ for $\delta = 0.02$.

Proof. One can check by direct calculation that all rectangles that are added to $\mathcal{R}_\delta$ are in fact δ-brackets. The points $\overline{a}_i^{(h)}(\delta)$ are lying on the lines $\frac{x}{y} \equiv \frac{h}{2^p}$ and the x-coordinates $t_{i+1}^{(h)}(\delta)$ of the left corners of the stripes $T_i^{(h)}(\delta)$ are chosen in such a way that $H \subseteq \cup_{h=1}^{2^p} T_{\mathrm{dis}}^{(h)}$: For given $h \geq 2$ and a given i the index

$$j_i := \left\lceil \frac{h-1}{h} i - \frac{1}{h} \right\rceil$$

is uniquely determined by

$$a_{j_i}^{(h-1)}(\delta) > a_{i+1}^{(h)}(\delta) \geq a_{j_i+1}^{(h-1)}(\delta),$$

and we have

$$t_{i+1}^{(h)}(\delta) = \frac{h-1}{2^p} a_{j_i}^{(h-1)}(\delta).$$

Let now $\omega(\delta, h, i)$ be the minimal number of δ-brackets of heights $a_i^{(h)}(\delta) - a_{i+1}^{(h)}(\delta)$ that we need to cover $T_i^{(h)}(\delta)$. Using the scaling Lemma 2.2 with $\lambda = (2^p(ha_i(\delta))^{-1}, (a_i(\delta))^{-1})$ we see that we have $\omega(\delta, h, i) = \omega(\delta_i^{(h)}, t(\delta, h, i))$ as defined in Section 2, where $\delta^{(h)} = \frac{2^p}{h}\delta$ and, coinciding with the convention from the previous section, $\delta_i^{(h)} = \delta^{(h)}/(1 - i\delta^{(h)})$, and

$$t(\delta, h, i) = \frac{h-1}{h}\left(1 - \left(\left\lceil \frac{h-1}{h} i - \frac{1}{h} \right\rceil \frac{h}{h-1} - i\right)\delta_i^{(h)}\right)_+^{1/2}.$$

Due to (12) we get

$$\omega(\delta, h, i) = \\ \left\lceil 2\, \frac{\ln\left(1 - (1-\delta_i^{(h)})^{1/2}\right) - \ln\left(t(\delta, h, i)(1 - (1-\delta_i^{(h)})^{-1/2}) + \delta_i^{(h)}(1-\delta_i^{(h)})^{-1/2}\right)}{\ln(1 - \delta_i^{(h)})} \right\rceil.$$

We claim that

$$\omega(\delta, h, i) = 2\ln\left(1 + \frac{1}{h}\right)\left(\delta_i^{(h)}\right)^{-1} + O(1) \quad \text{as } \delta_i^{(h)} \text{ tends to zero.} \tag{23}$$

According to (9) this is true for $h = 1$. In general it follows from the inequalities (10), (11) and

$$\ln\left(t(\delta, h, i)(1-(1-\delta_i^{(h)})^{-1/2}) + \delta_i^{(h)}(1-\delta_i^{(h)})^{-1/2}\right) = \ln\left(1 + \frac{1}{h}\right) - \ln(2) + \ln(\delta_i^{(h)}) + O(\delta_i^{(h)}).$$

We have

$$|\mathcal{R}_\delta| = \sum_{h=1}^{2^p}\left(\sum_{i=0}^{\rho^{(h)}(\delta)} 2\omega(\delta, h, i)\right) - (\rho^{(2^p)}(\delta) + 1); \tag{24}$$

here we have to subtract the last term to avoid double-counting of the δ-brackets on the main diagonal of $[0,1]^2$. According to (23) we get

$$\begin{aligned}
|\mathcal{R}_\delta| &= 4\sum_{h=1}^{2^p}\sum_{i=0}^{\rho^{(h)}(\delta)} \ln\left(1+\frac{1}{h}\right)\left(\frac{h}{2^p}\delta^{-1}-i\right)+o(\delta^{-2}) \\
&= 4\sum_{h=1}^{2^p}\ln\left(1+\frac{1}{h}\right)\left(\frac{1}{2}\left(\frac{h}{2^p}\delta^{-1}\right)^2+O(\delta^{-1})\right)+o(\delta^{-2}) \\
&= 2\left(\sum_{h=1}^{2^p}\ln\left(1+\frac{1}{h}\right)\left(\frac{h}{2^p}\right)^2\right)\delta^{-2}+o(\delta^{-2}).
\end{aligned}$$

It remains to show that the sum in parentheses is of the form $\frac{1}{2}+o(1)$ as δ tends to zero (and thus p tends to infinity). But this follows easily from the identity

$$\ln\left(1+\frac{1}{h}\right) = h^{-1} - h^{-2}\sum_{k=0}^{\infty}(-1)^k\frac{h^{-k}}{k+2}.$$

$\square$

6 Numerical Comparison and Conclusion

Let us now compare the cardinalities of the different constructions of δ-bracketing covers for some values of δ, see the table below. For the construction of $\mathcal{R}_\delta$ we have chosen $p = p(\delta)$ exactly as in (22). Thus

$$2^p \approx \delta^{-\frac{\ln(2)}{c}} \approx \delta^{-0.4} \approx o(\delta^{-1}),$$

and the conditions of Proposition 5.1 are clearly satisfied. Note that $2\ln(2) = 1.386294...$ and $(2\ln(2))^2 = 1.921812....$ Thus the table underlines the dominance of the leading terms in the expansion of the cardinalities of the δ-bracketing covers with respect to δ^{-1}.

δ	0.25	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
$\lvert\mathcal{G}_\delta\rvert$	36	196	784	19321	77284	1923769	7689529	192182769
$\lvert\mathcal{T}_\delta\rvert$	25	142	565	13922	55575	1386908	5546403	138635574
$\lvert\mathcal{Z}_\delta\rvert$	24	146	572	13962	55650	1387292	5547174	138639434
$\lvert\mathcal{R}_\delta\rvert$	24	128	490	10888	42162	1021122	4055986	100514774
δ^{-2}	16	100	400	10000	40000	1000000	4000000	100000000

Altogether, we provided in this paper an explicit construction of a δ-bracketing cover $\mathcal{R}_\delta$ of $\mathcal{C}_2$ which is optimal in the sense that the coefficient in front of the most significant term δ^{-2} in the expansion of $|\mathcal{R}_\delta|$ with respect to δ^{-1} is optimal.

We compared $\mathcal{R}_\delta$ to its simplified version $\mathcal{Z}_\delta$ (which does not "re-orientate" the brackets) and known bracketing covers from [8] and [22].

We conjecture that extending the idea of construction of $\mathcal{R}_\delta$ to arbitrary dimension d, one can generate δ-bracketing covers $\mathcal{R}_\delta^{(d)}$ of $\mathcal{C}_d$ whose cardinality satisfies

$$|\mathcal{R}_\delta^{(d)}| = \delta^{-d} + o_d(\delta^{-d}) \quad \text{as } \delta \text{ approaches zero}$$

(here o_d should emphasize that the implicit constants in the o-notation may depend on d), i.e., has the best possible coefficient in front of the most significant term δ^{-d} in the expansion with respect to δ^{-1}.

We suspect that a rigorous proof of the conjecture might be rather technical and tedious. That is why we would find even a rigorous analysis for $d = 3$ or computational experiments for higher dimension quite interesting.

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